# MODE-SHELL CORRESPONDENCE: A UNIFYING THEORY IN TOPOLOGY

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# Mode-shell correspondance in short

Topological property of low-energy *modes*

Correspondence

Invariant on the *shell* enclosing the modes in phase space

**Phase space (x,k)**
position + wavenumber → Wigner-Weyl transform

$$H(x,k) = W(\hat{H}) = \int dx' \left\langle x + \frac{x'}{2} \middle| \hat{H} \middle| x - \frac{x'}{2} \right\rangle e^{-ikx'}$$

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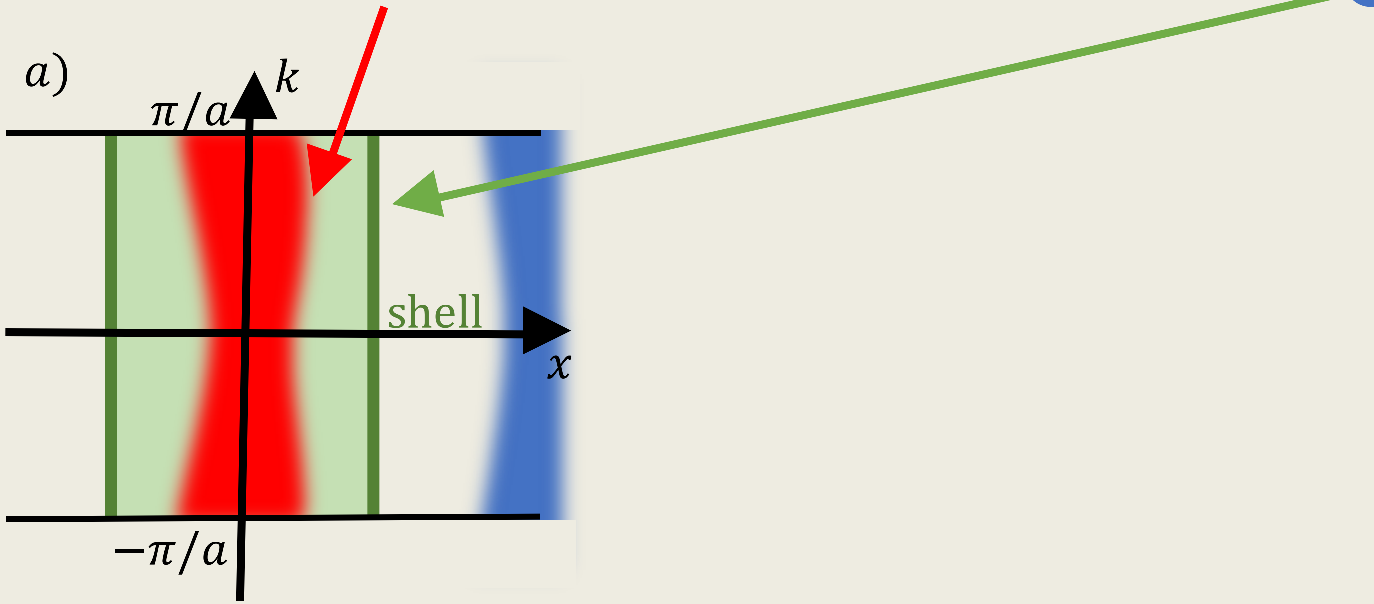


a) bulk-edge correspondance in position
b) Low-high wavenumber correspondance
c) Mixed correspondence in position/wavenumber
d) Higher-order correspondence

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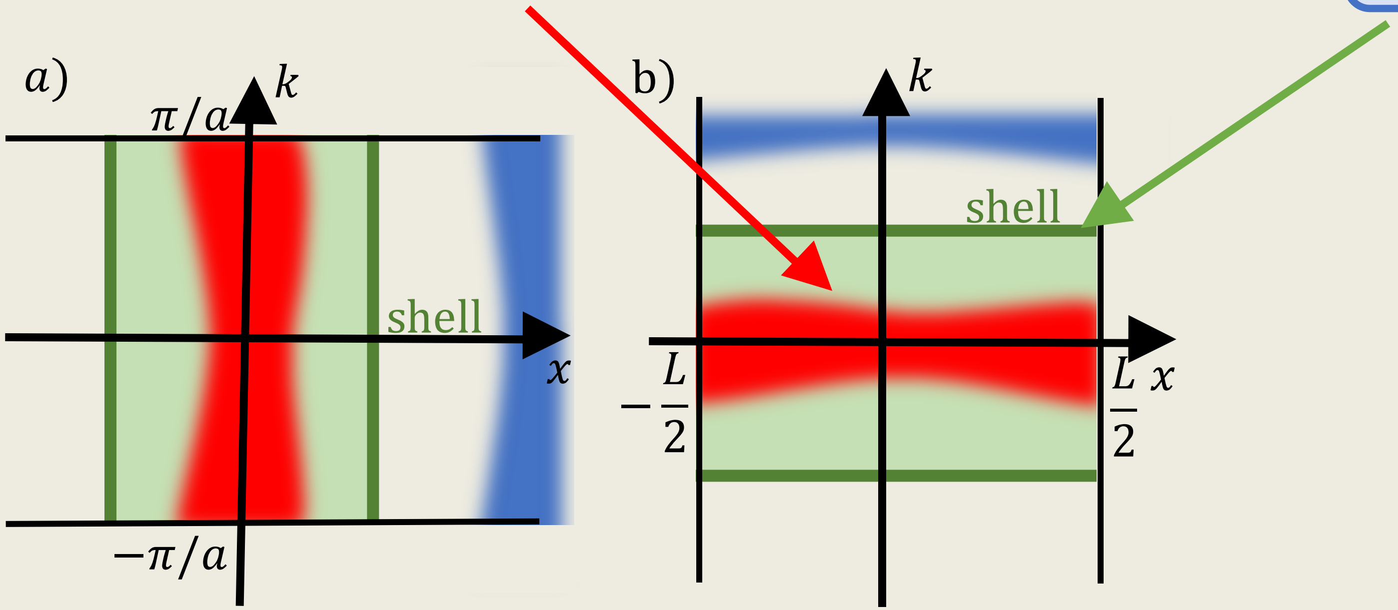


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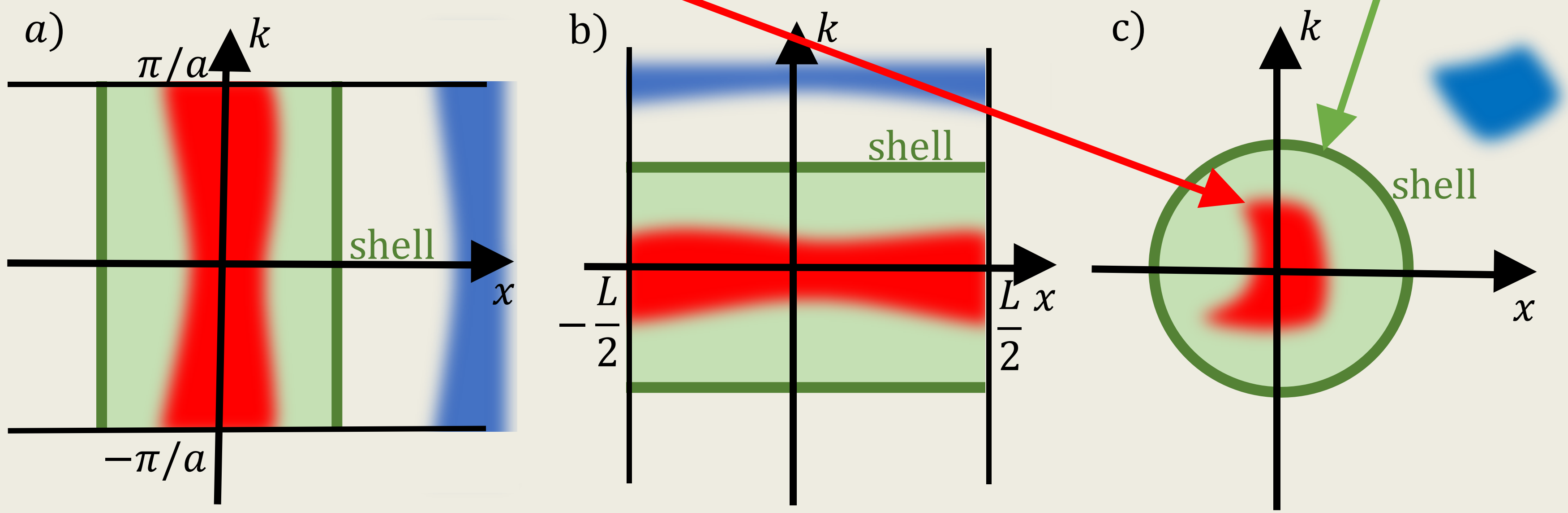


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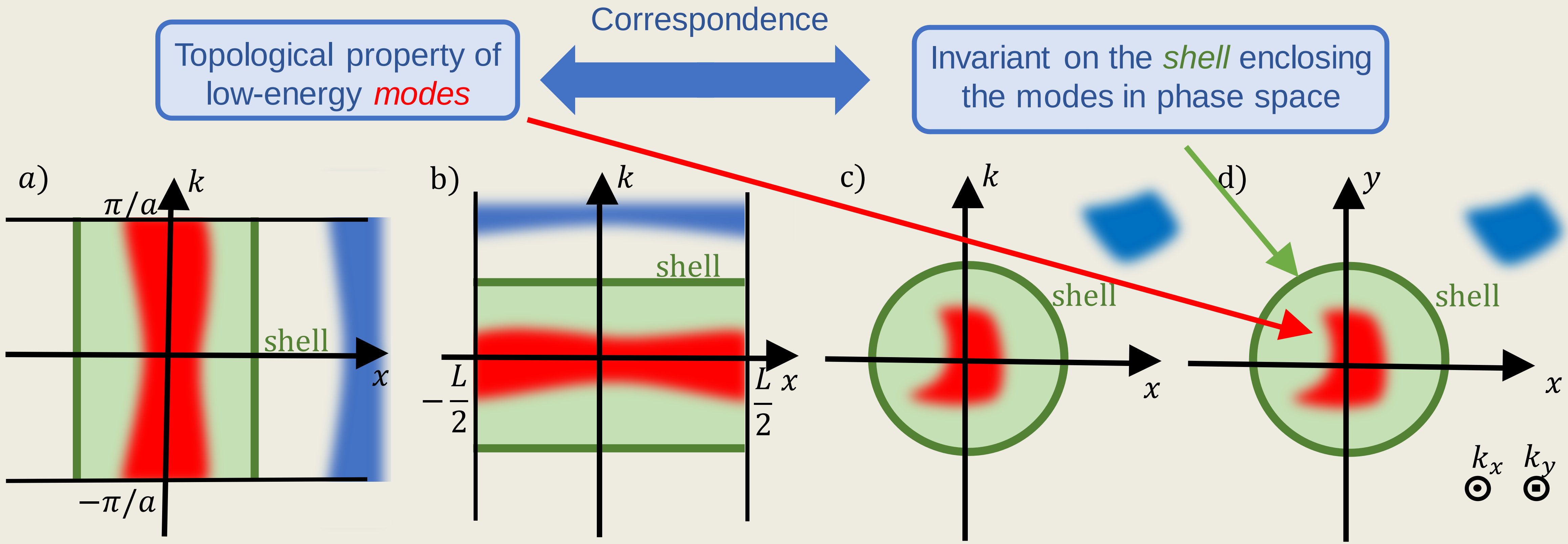


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# The chiral number of zero modes

We restrict to the case of hamiltonian with chiral symmetry

$$\hat{H} = \begin{pmatrix} 0 & \hat{h}^{\dagger} \\ \hat{h} & 0 \end{pmatrix} \quad \hat{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{C}\hat{H} + \hat{H}\hat{C} = 0$$

Zero modes has positive/negative chirality

A topological invariant:
the chiral number of zero modes

Zero modes of positive chirality

$$I_{modes} = dim \ker(\hat{h}) - dim \ker(\hat{h}^{\dagger}) = Ind(\hat{h})$$

Zero modes of negative chirality

# The chiral number of zero modes

Generalisation of the index:

- More robust in finite system
- Coincide for infinite system
- Selection in phase space

$$I_{modes} = Tr(\hat{C}\left(1 - \hat{H}_F^2\right)\hat{\theta}_{\Gamma})$$

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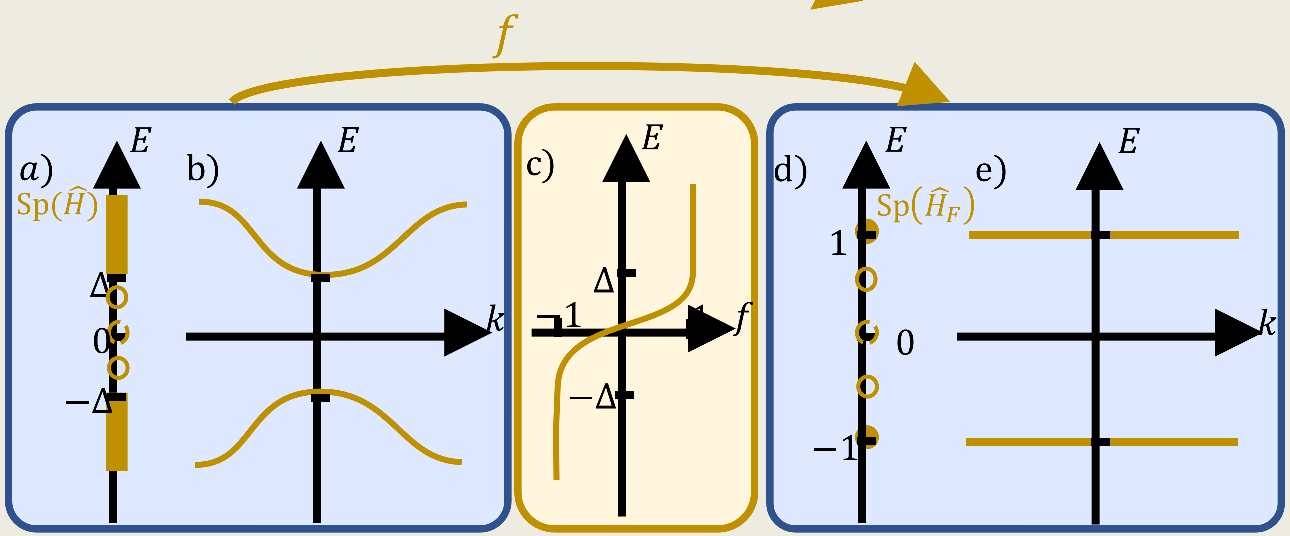
Energy filter by smooth flattening of the gapped bands



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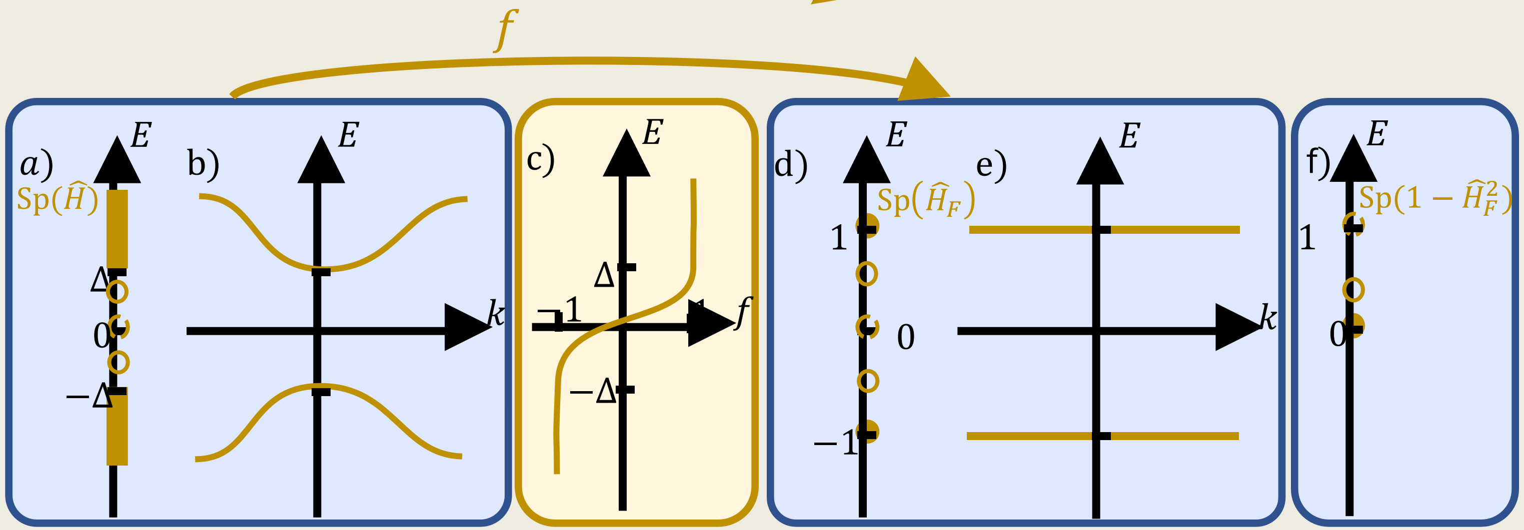

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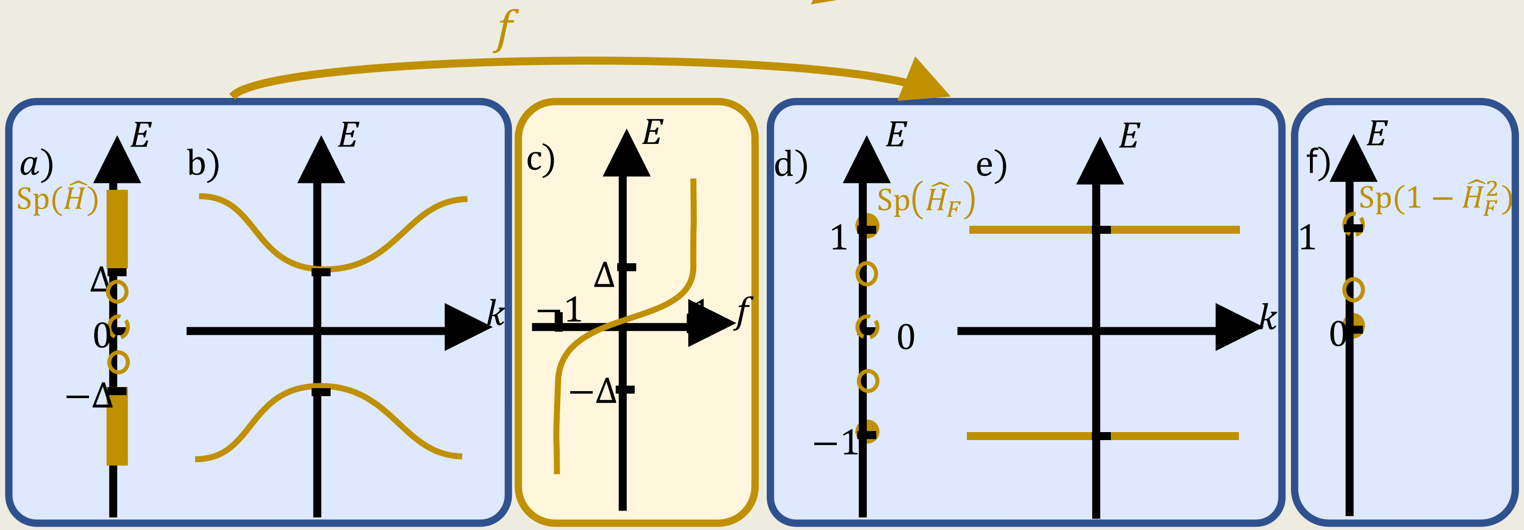
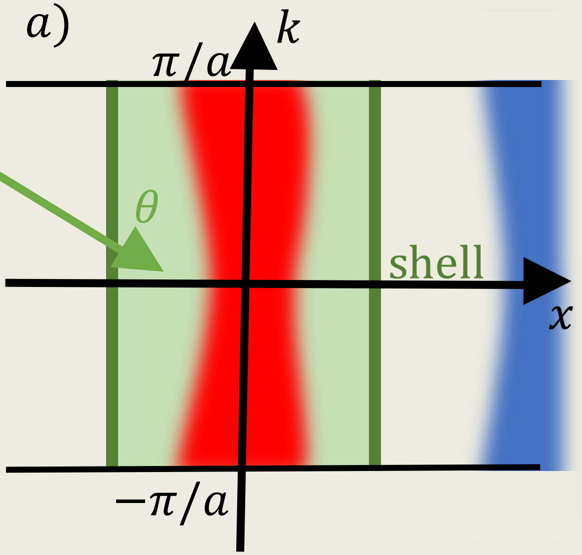
Selection in phase space by a cut-off

$$\hat{\theta}_\Gamma = e^{-\frac{x^2}{\Gamma^2}} \qquad \Gamma \to \infty$$



# The cut-off can be anything

$$I_{modes} = Tr\left(\hat{C}\left(1 - \widehat{H}_F^2\right)\hat{\theta}_\Gamma\right)$$

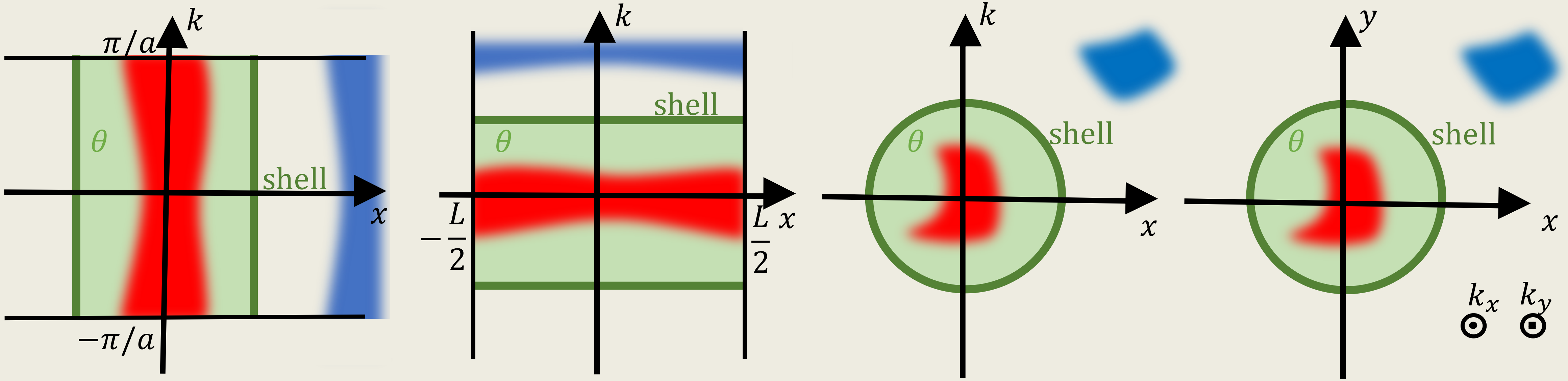
Shell: Transition region of the cut-off.

$$\hat{\theta}_\Gamma = e^{-\frac{x^2}{\Gamma^2}}$$

$$\hat{\theta}_\Gamma = e^{\frac{\partial_x{}^2}{\Gamma^2}} \approx e^{-\frac{k^2}{\Gamma^2}}$$

$$\hat{\theta}_\Gamma = e^{\frac{-x^2+\partial_x{}^2}{\Gamma^2}} \approx e^{-\frac{x^2+k^2}{\Gamma^2}}$$

$$\hat{\theta}_\Gamma = e^{-\frac{x^2+y^2}{\Gamma^2}}$$

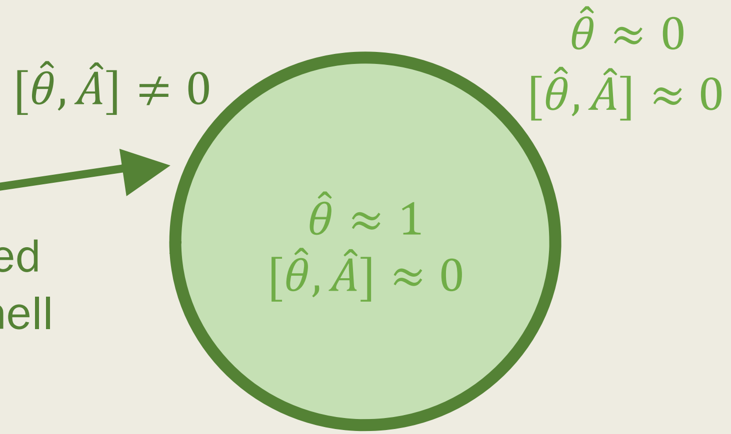


# The mode-shell correspondance

Up to a rearrangement of the terms, the index can be re-expressed as a quantity on the shell

$$I_{modes} = Tr\big(\hat{C}(1-\hat{H}_F^2)\hat{\theta}_\Gamma\big)$$
$$= Tr\big(\hat{C}\hat{\theta}_\Gamma\big) + \frac{1}{2}Tr(\hat{C}\hat{H}_F[\hat{\theta}_\Gamma, \hat{H}_F]) \overset{\text{def}}{=} I_{shell}$$

Supported on the shell




summed chirality of site/d.o.f weighted by the cut-off. Often zero

$$\sum_{\lambda \in d.o.f} C_\lambda \langle\lambda|\hat{\theta}_\Gamma|\lambda\rangle$$

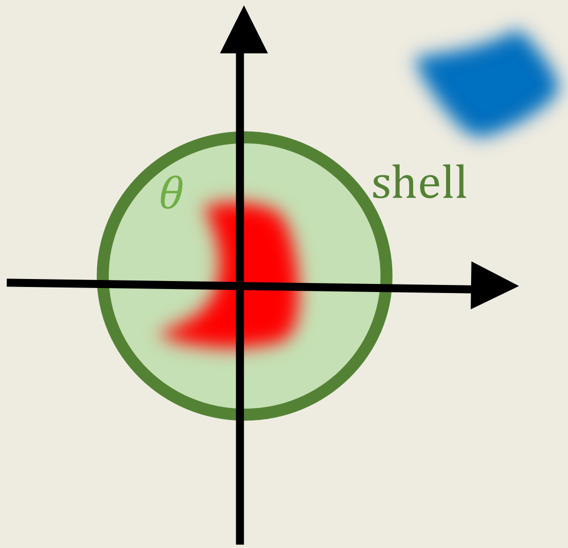
# Semi-classical approximation

Semi-classical approximation:

$$W\left(\hat{A}\hat{B}\right)(x,k) \approx A(x,k)B(x,k)$$

$$W\left([\hat{A},\hat{B}]\right)(x,k) \approx \{A(x,k),B(x,k)\}$$

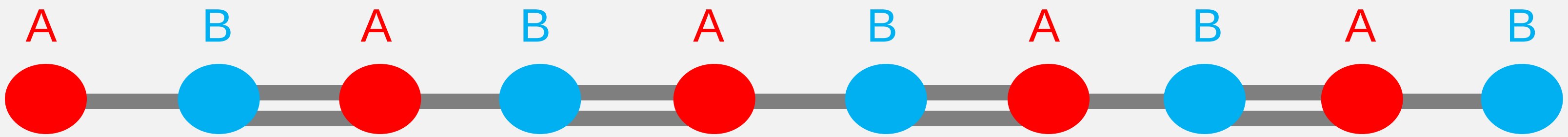
When $H(x,k)$ varies slowly in $x$ or in $k$ in the shell
→ shell invariant reduced to a (higher) winding number



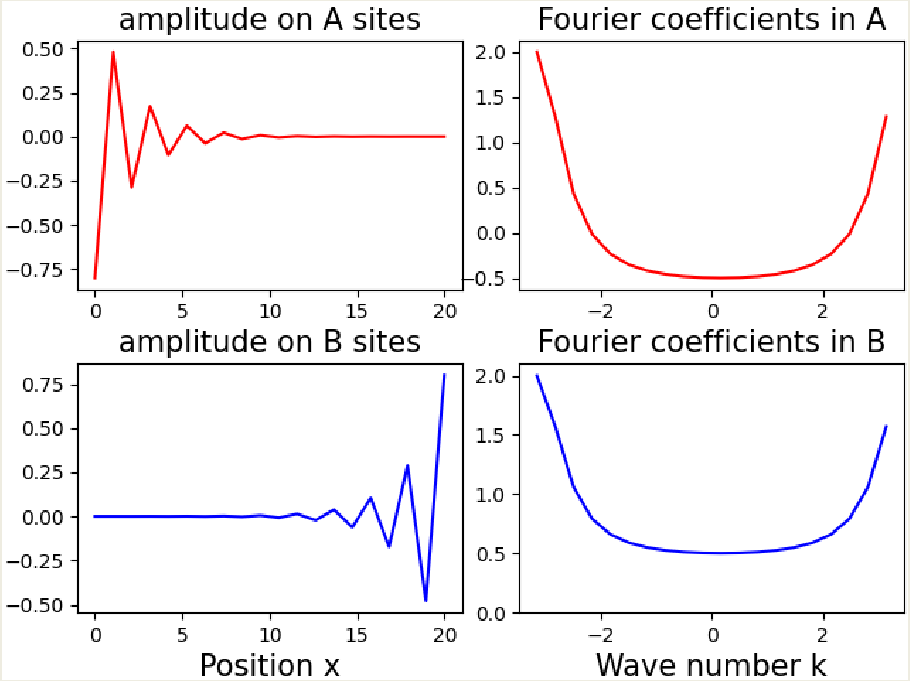

$$I_{shell} \xrightarrow[sc\ lim]{} \frac{2(D)!}{(2D)!\,(2i\pi)^D} \int_{shell} Tr^{int}((U^\dagger dU)^{2D-1}) = w_{2D-1}$$

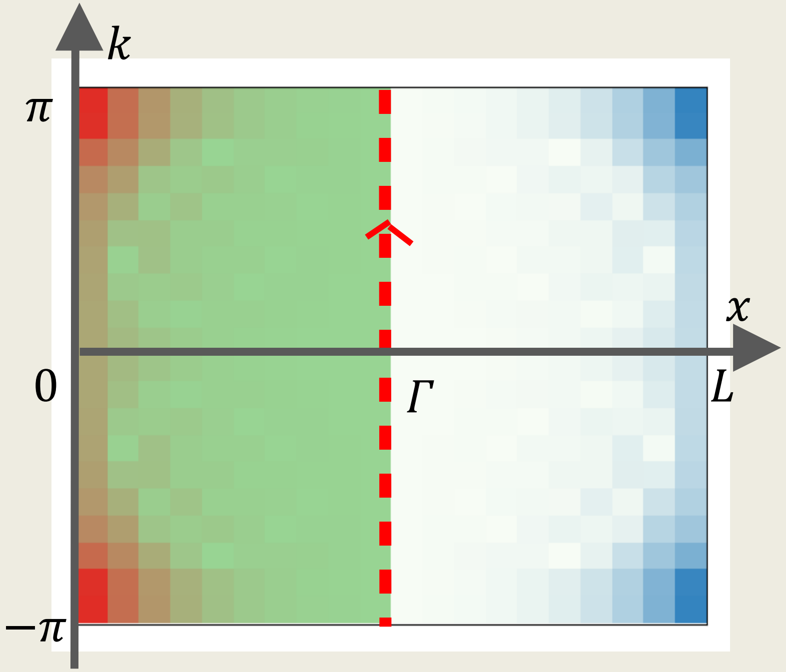
Where $U$ is defined as $\quad H_F(x,k) = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix}(x,k)$

# Bulk-edge correspondance




Dimer (SSH) model



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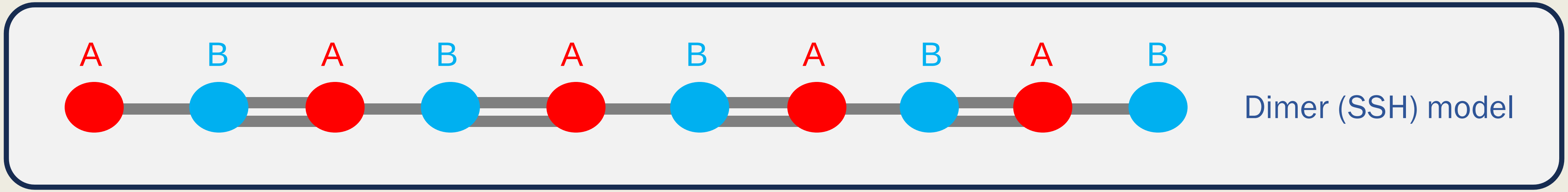


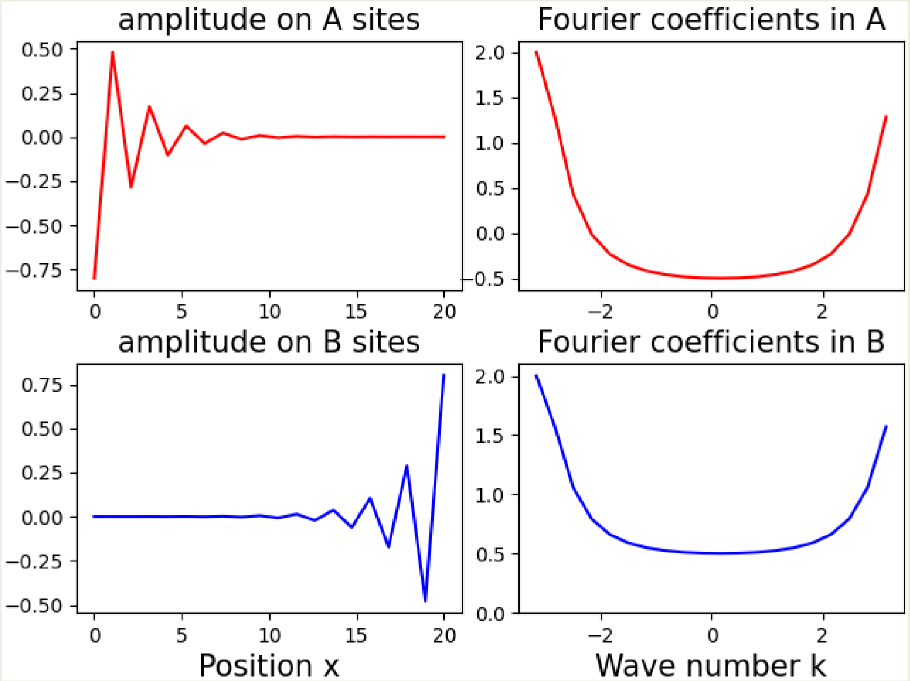


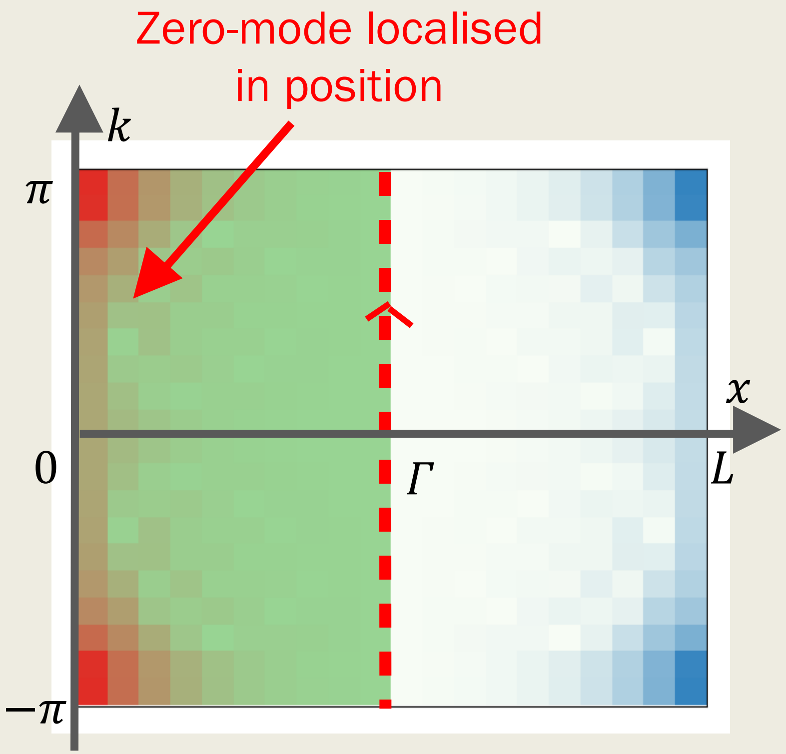

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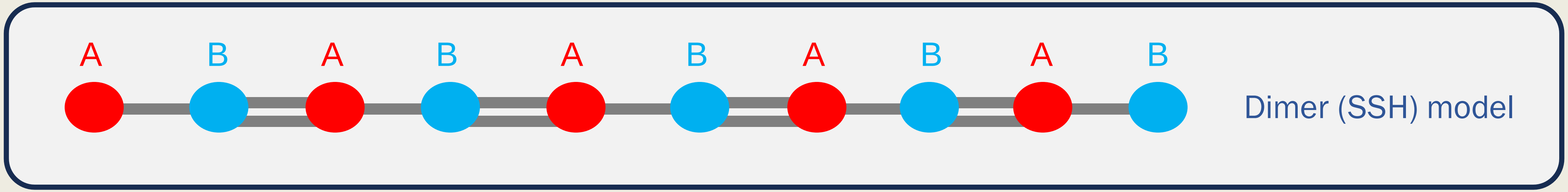


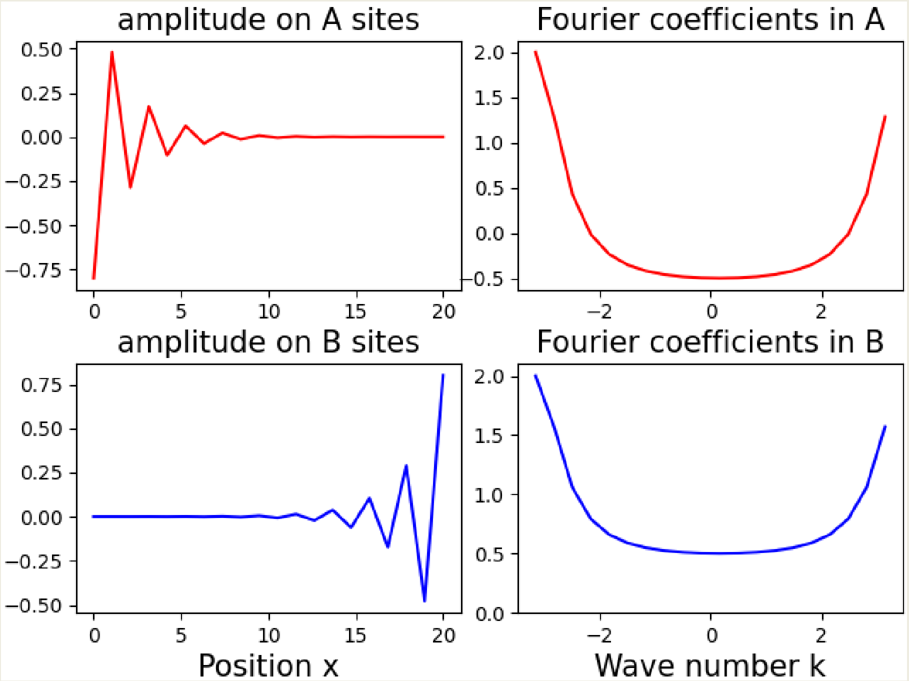


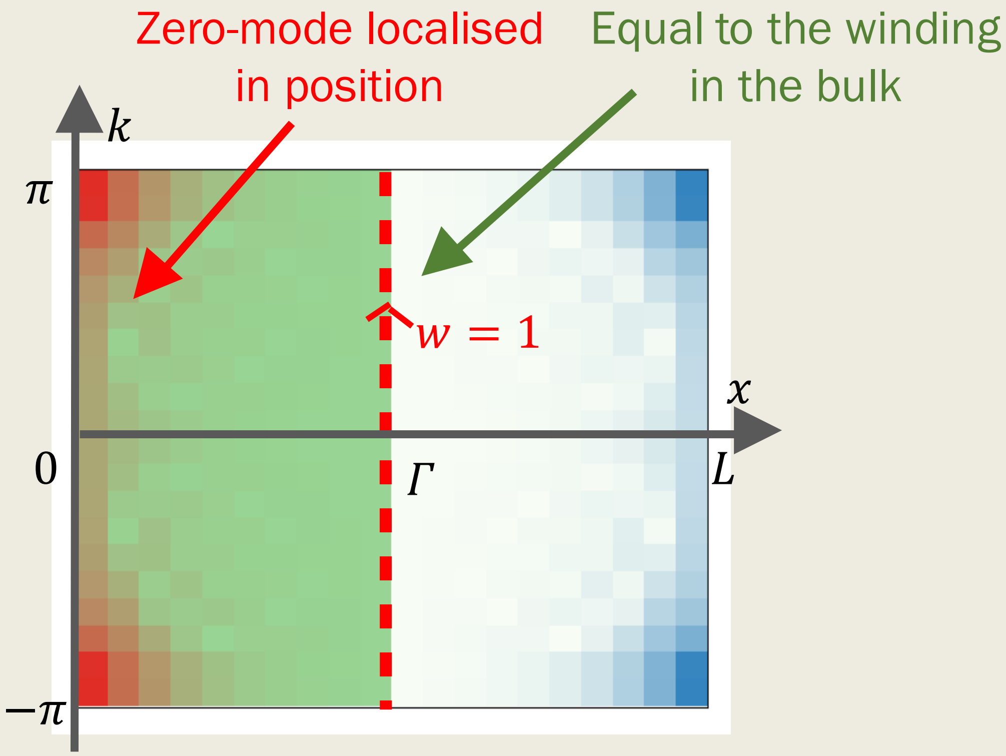

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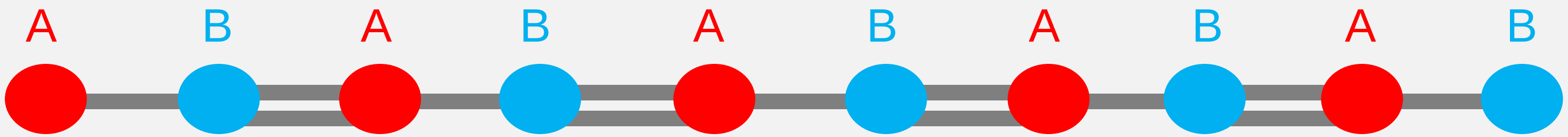


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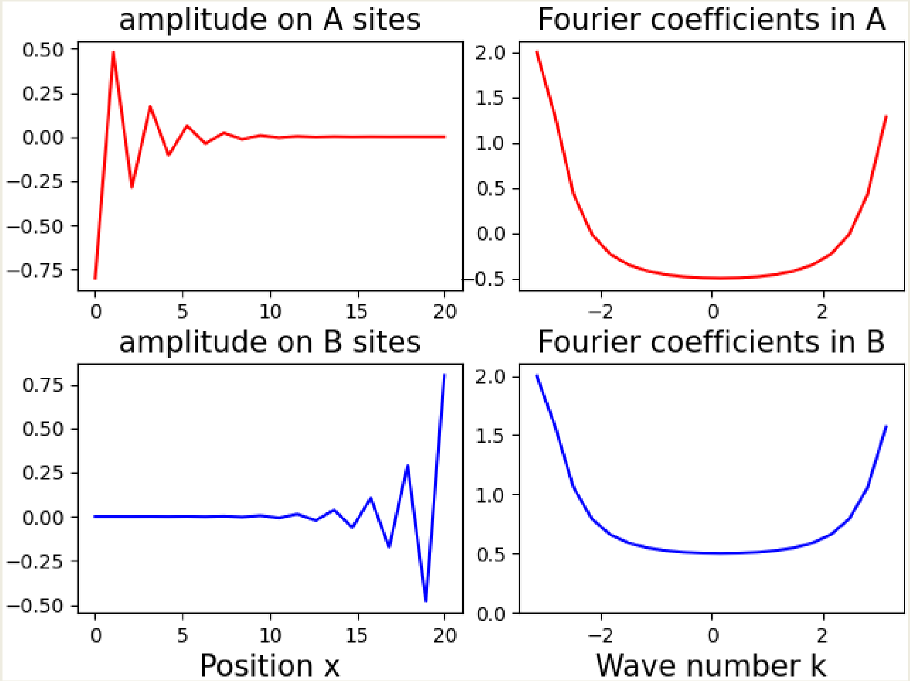
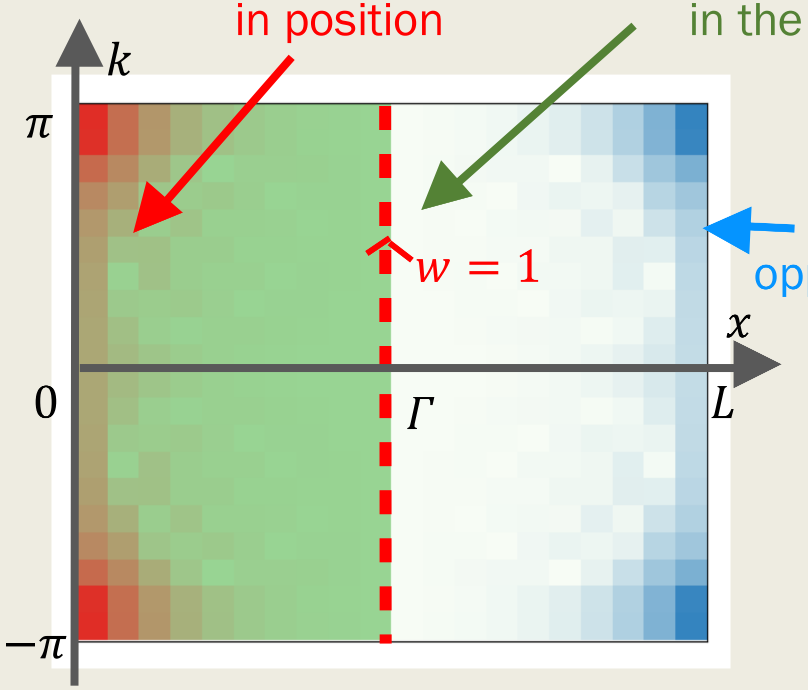
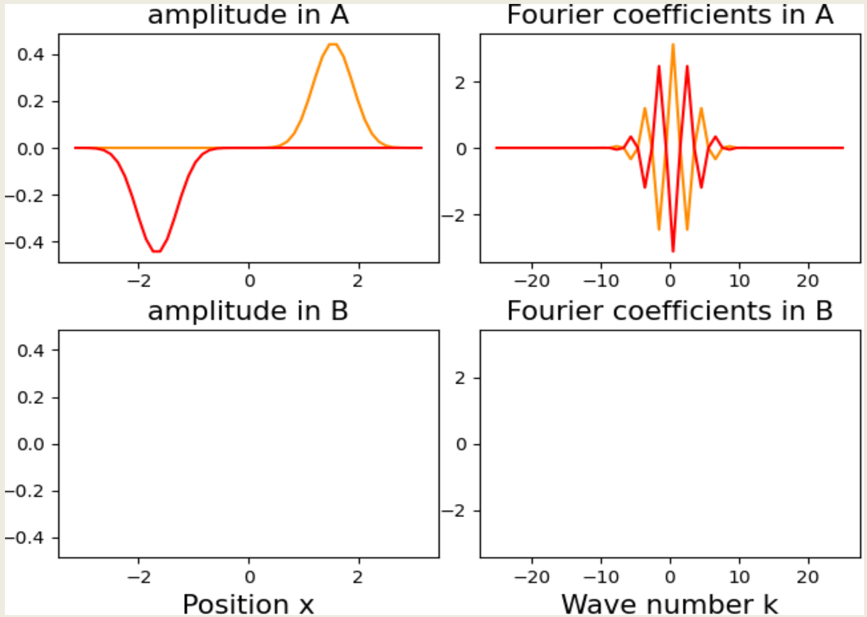


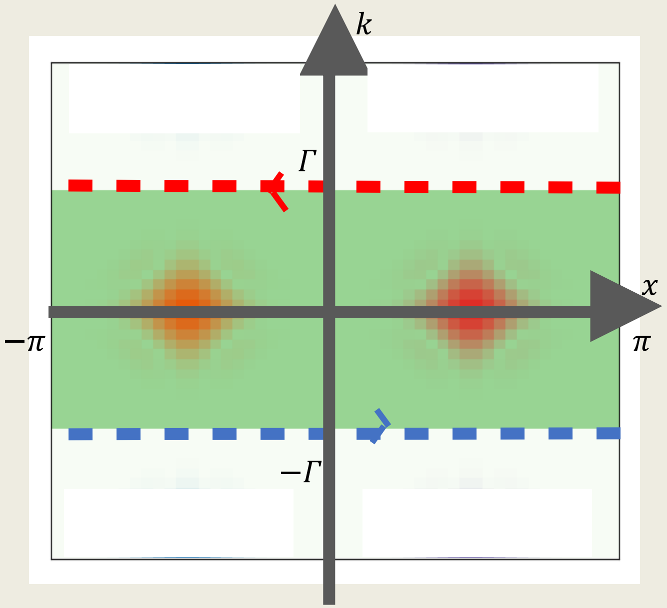



# Low-high wavenumber correspondance

$$\hat{H} = \begin{pmatrix} 0 & V(x) + \varepsilon c(x)\partial_x \\ V(x) - \varepsilon\partial_x\, c(x) & 0 \end{pmatrix}$$

Wave system with a periodic local wave velocity $c(x) = \sin x$ and potential $V(x) = \cos x$



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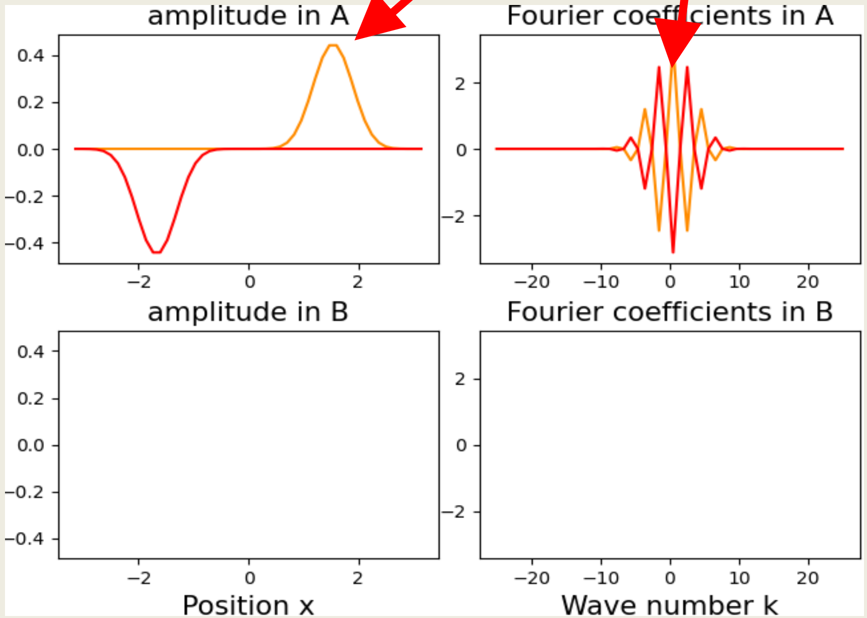
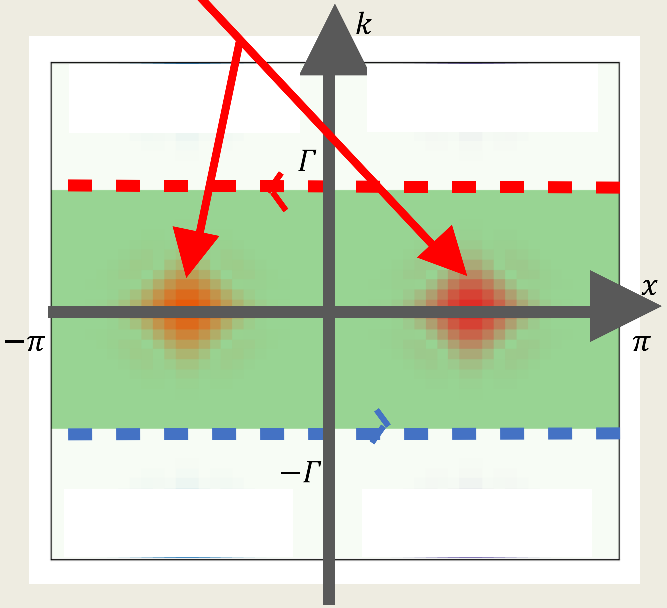
Zero modes of positive chirality at low-wave number



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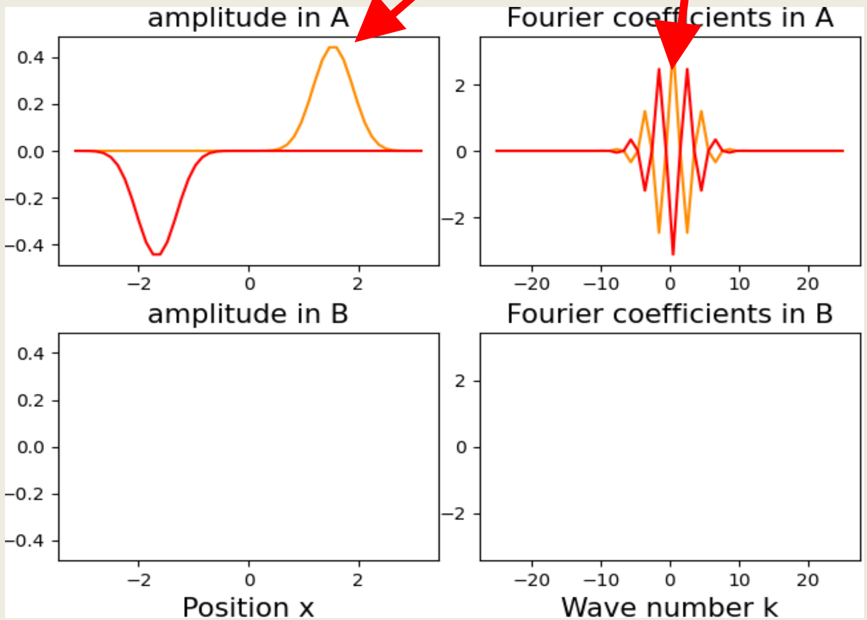


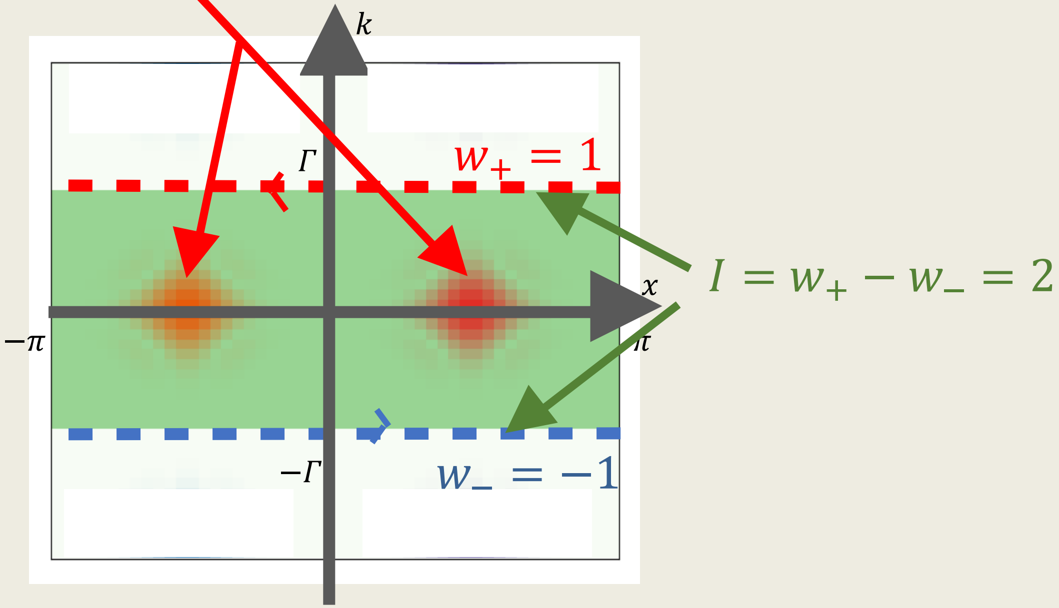

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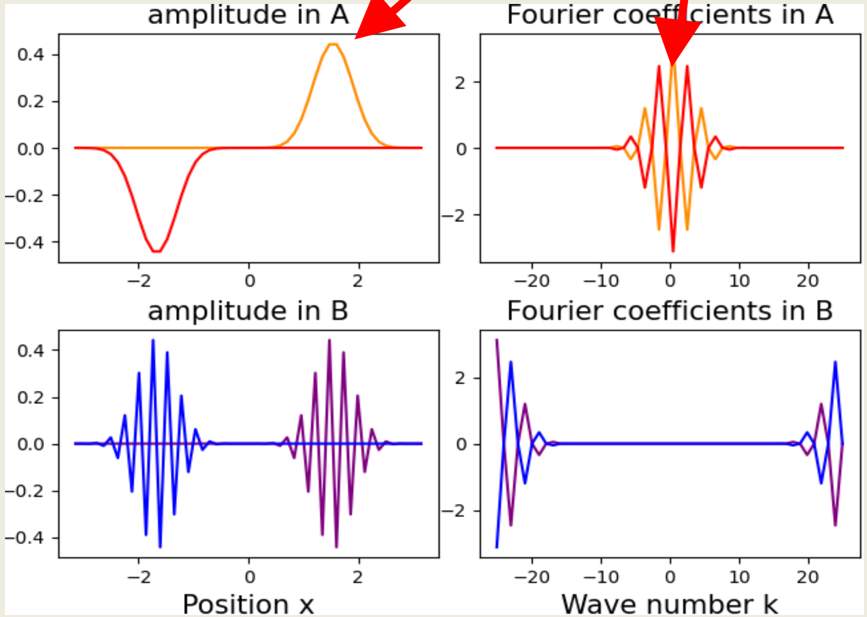
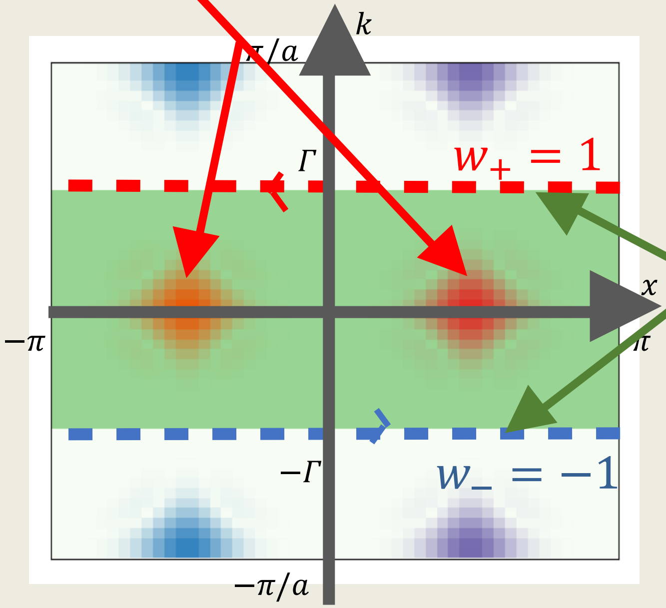
Zero modes of opposite chirality

Protection from separation in wavenumber




$I = w_+ - w_- = 2$

# Mixed correspondance in position/wavenumber

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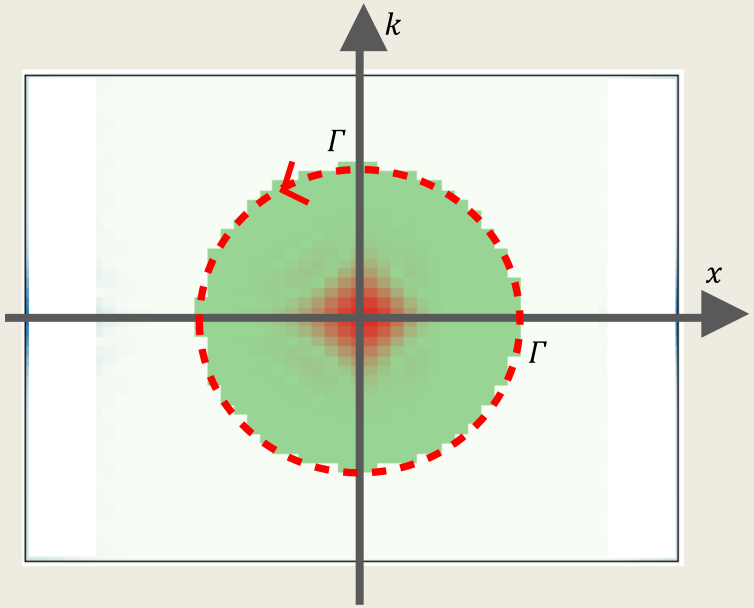
Jackiw-Rebbi model
→ Continuous system with constant local wave velocity but unbounded in position



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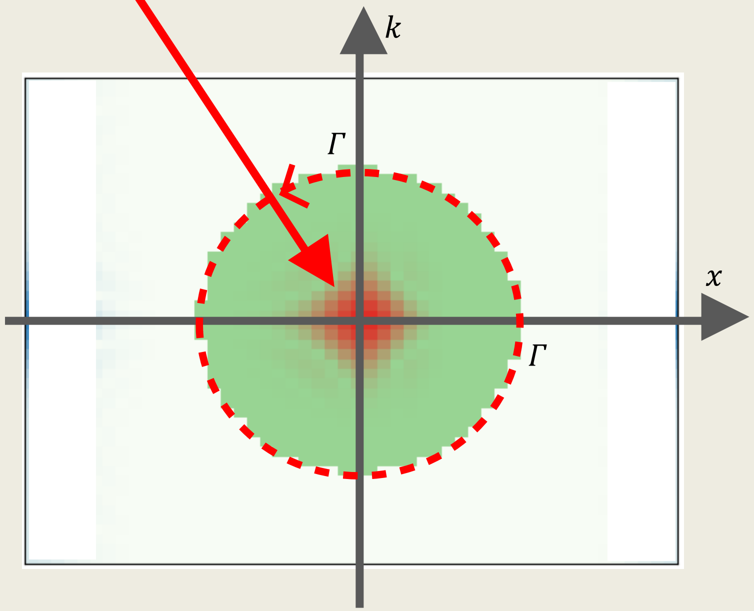

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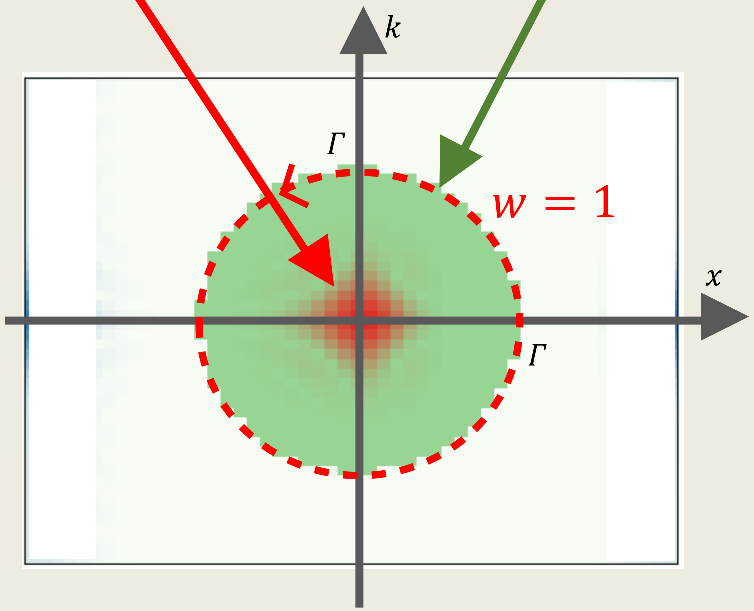
Equal to the winding on the phase space circle



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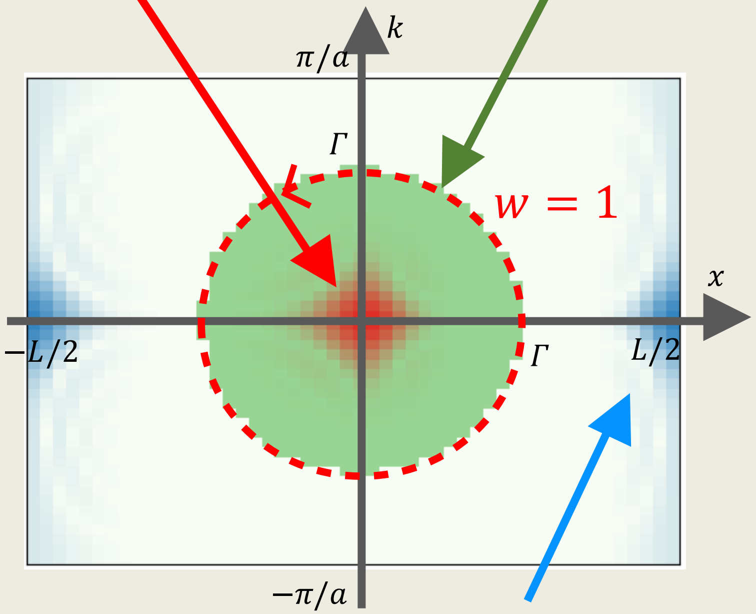


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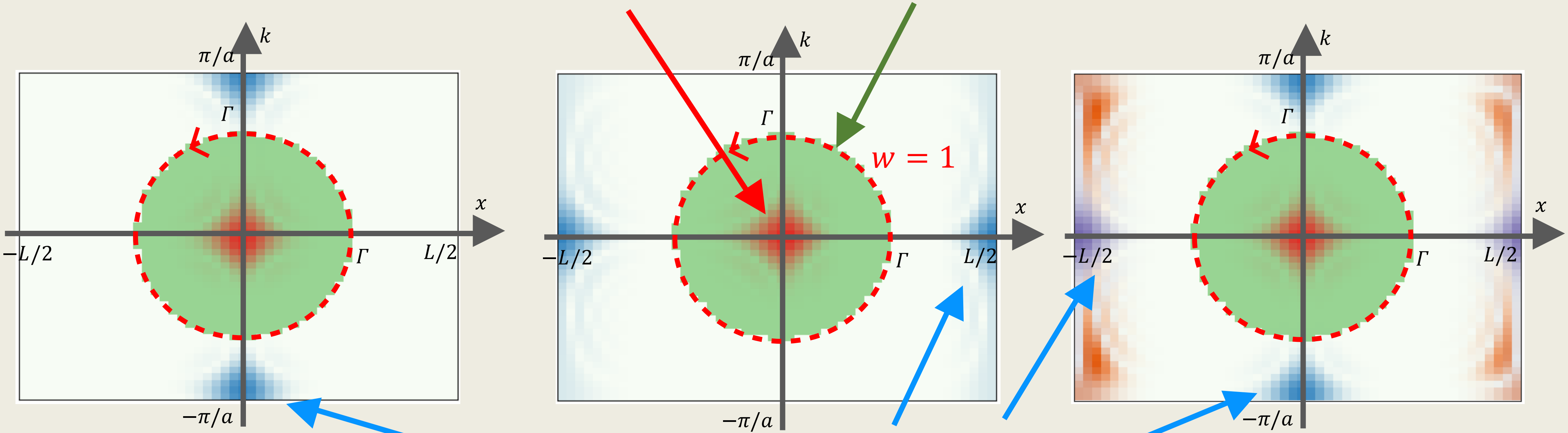


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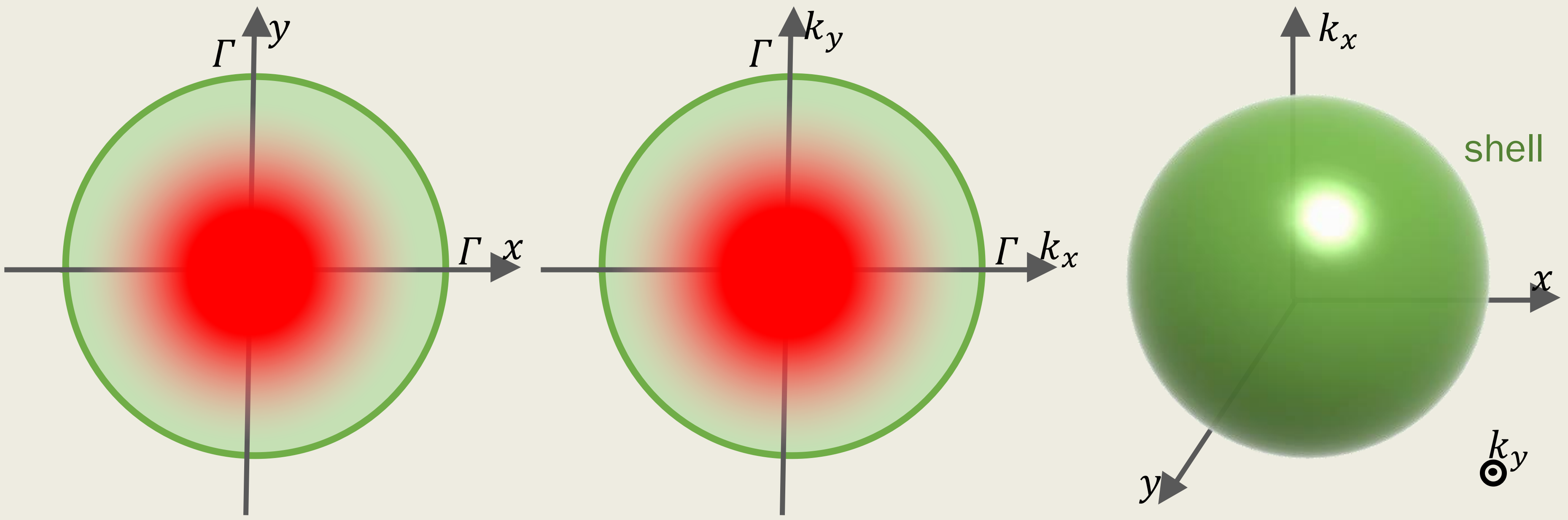
# A higher-order correspondance

$$\hat{H} = \begin{pmatrix} \mathbf{0} & \begin{matrix} x - \partial_x & y - \partial_y \\ -(y + \partial_y) & x + \partial_x \end{matrix} \\ \begin{matrix} x + \partial_x & -(y - \partial_y) \\ y + \partial_y & x - \partial_x \end{matrix} & \mathbf{0} \end{pmatrix}$$

2D model: zero-mode localised in position *and* wavenumber in 4D phase space



# A higher-order correspondance

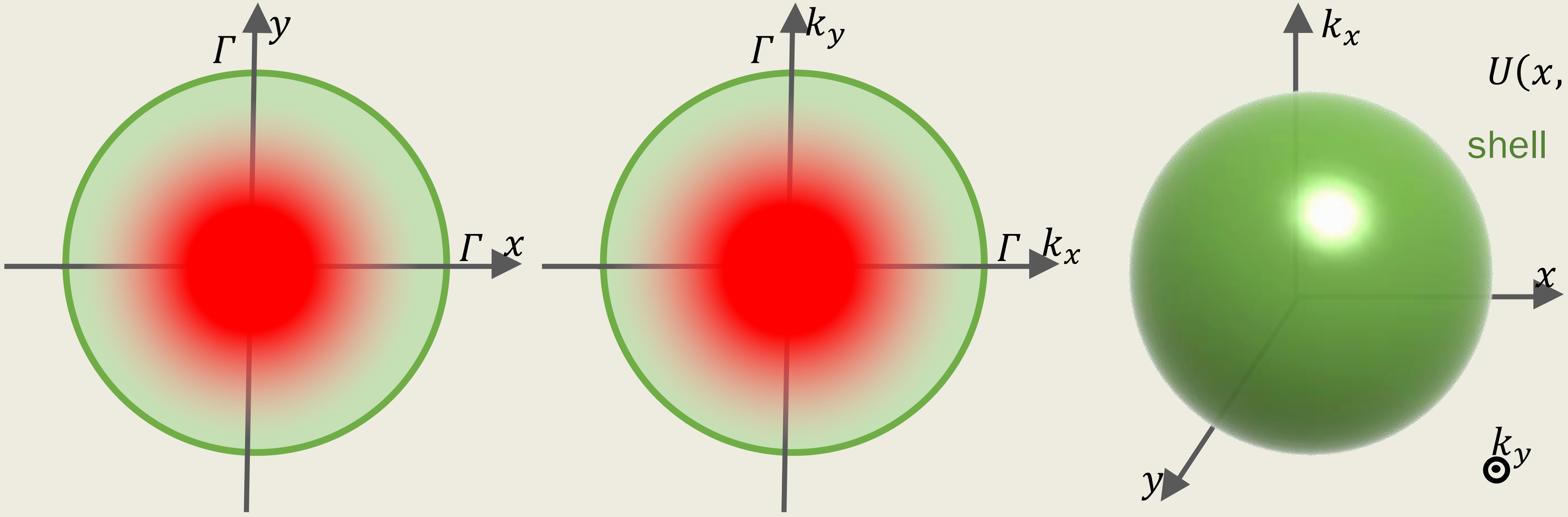
$$\hat{H} = \begin{pmatrix} 0 & & x-\partial_x & y-\partial_y \\ & & -(y+\partial_y) & x+\partial_x \\ x+\partial_x & -(y-\partial_y) & 0 & \\ y+\partial_y & x-\partial_x & & \end{pmatrix}$$

2D model: zero-mode localised in position *and* wavenumber in 4D phase space

→higher winding number on the 3D sphere in phase space

$$W_{2D-1} = \frac{-1}{24\pi^2}\int_{S^3} Tr^{int}((U^\dagger dU)^3)$$

$$U(x,k) = \begin{pmatrix} x-ik_x & y-ik_y \\ y+ik_y & x+ik_x \end{pmatrix}$$

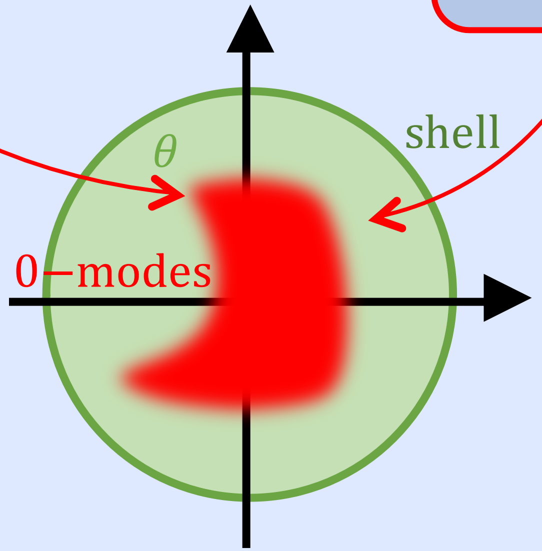


# Conclusion

$I_{modes} = \mathrm{Tr}(\hat{C}(1 - \hat{H}_F^2)\hat{\theta}_\Gamma)$

Infinite size limit

$Ind(\hat{h}) = dim\ Ker(\hat{h}) - dim\ Ker(\hat{h}^\dagger)$



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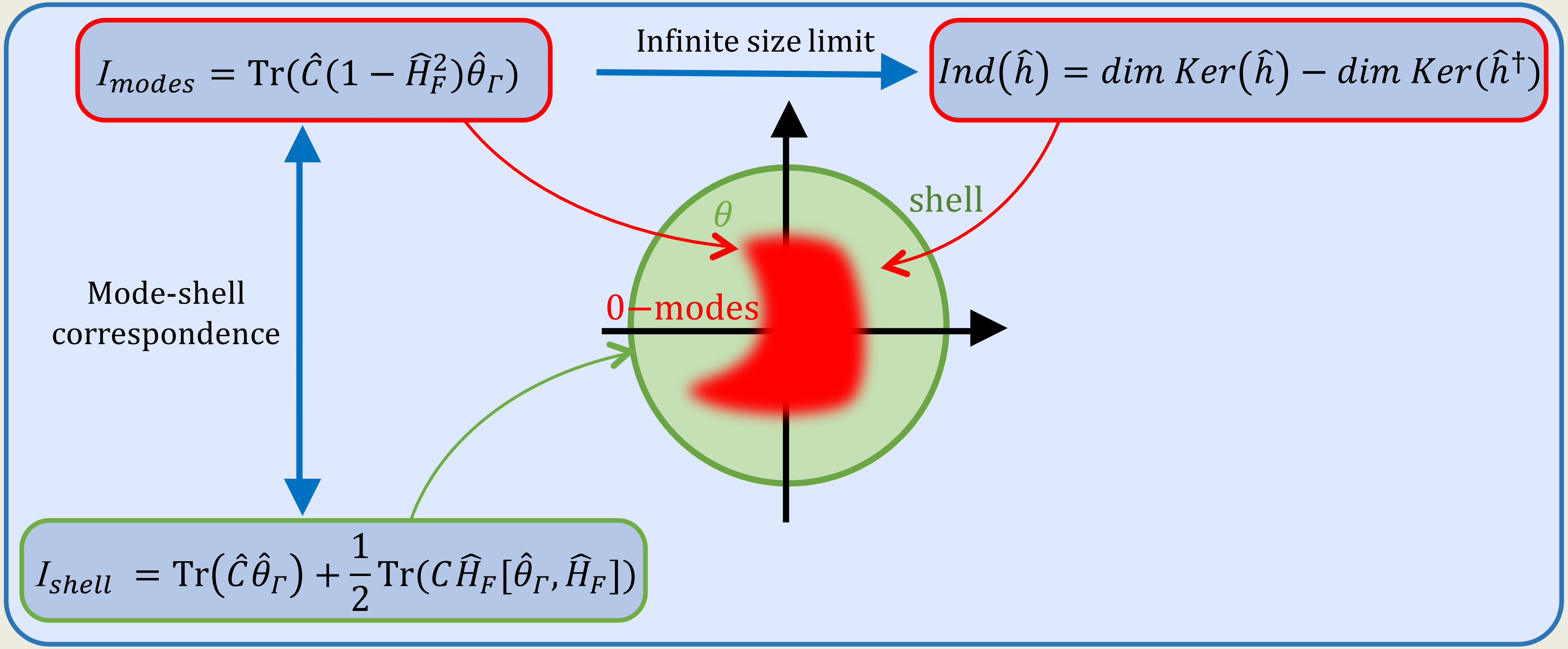

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shell
Mode-shell correspondence
0−modes
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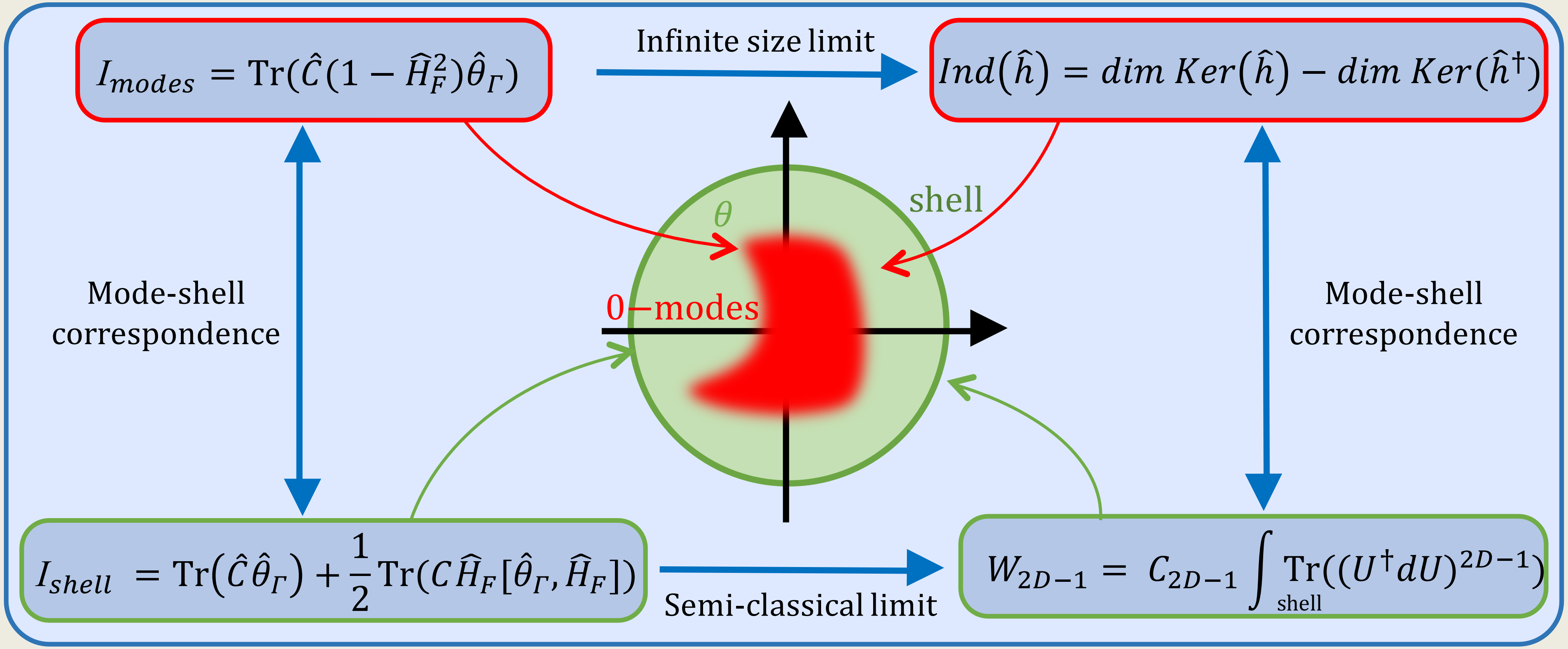

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$I_{shell} = \mathrm{Tr}(\hat{C}\hat{\theta}_\Gamma) + \frac{1}{2}\mathrm{Tr}(C\hat{H}_F[\hat{\theta}_\Gamma,\hat{H}_F])$
Semi-classical limit
$W_{2D-1} = C_{2D-1}\int_{\text{shell}} \mathrm{Tr}((U^\dagger dU)^{2D-1})$

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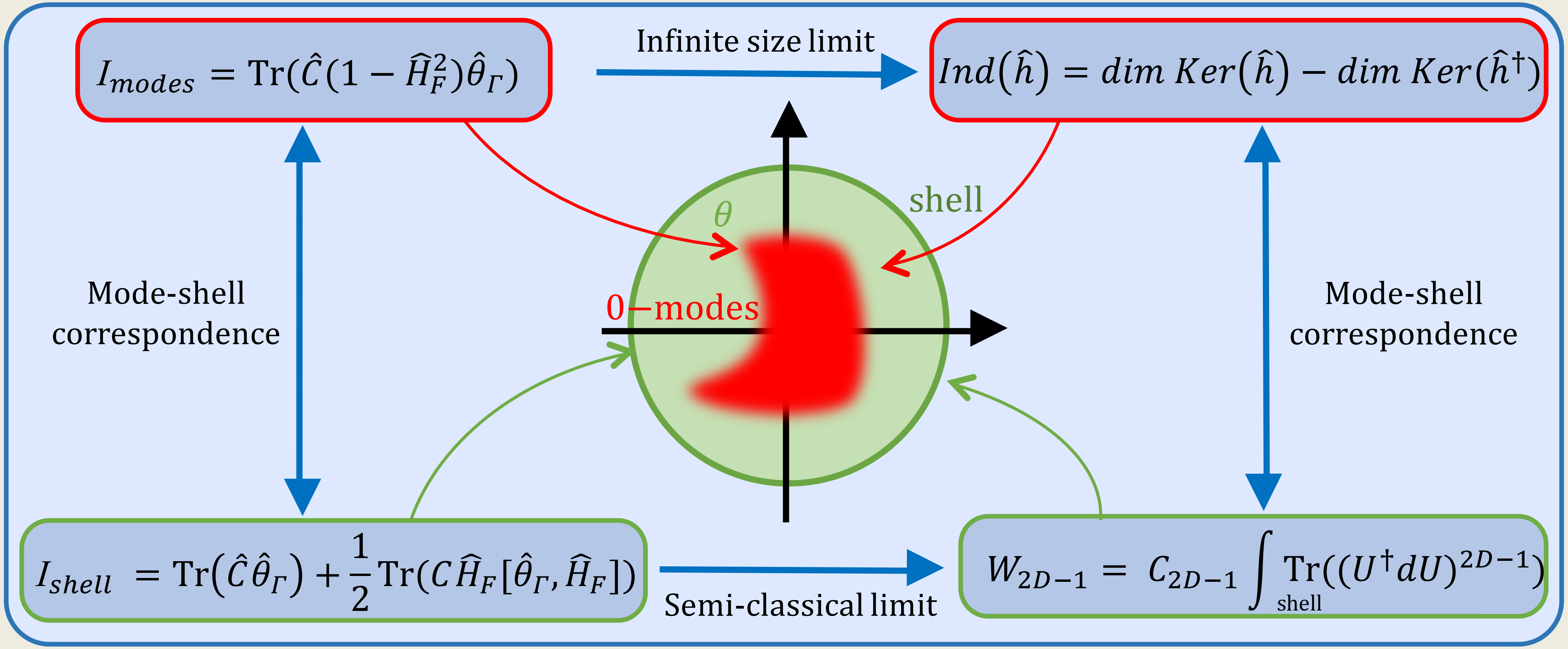


Other mode-shell correspondance:
-spectral flow (1D) → Pierre Delplace's talk (Wed)
-number of Dirac-Weyl point (2D & 3D)