Revisiting Hyperelliptic Feynman Integrals

Andrew McLeod

Elliptics and Beyond 2023, ETH Zürich
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arXiv:2307.11497 [hep-th]
with R. Marzucca, B. Page, S. Pögel, and S. Weinzierl

arXiv:23nn.nnnnn [hep-th]
with S. Abreu, A. Behring, and B. Page

Guiding Question

Can we classify the full set of integral geometries that appear in four-dimensional quantum field theories at a given loop order?

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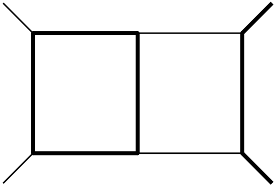
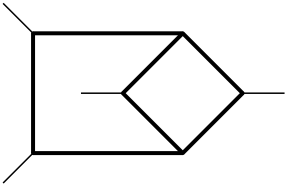


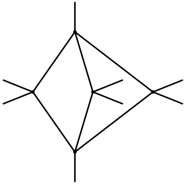
multiple elliptic curves

[Adams, Chaubey, Weinzierl, 2018]



genus 3 curve

[Georgoudis, Zhang, 2015]



K3 surface

[Bourjaily, AJM, von Hippel, Wilhelm, 2018]

. . .

???

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simplest complete
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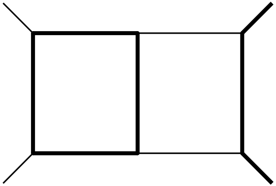

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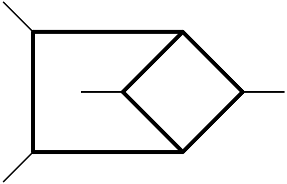

genus 2 curve
~~genus 3 curve~~

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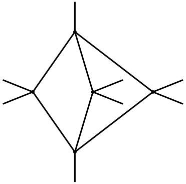

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$\cdots$

???

From Elliptic to Hyperelliptic Curves

Today, I will focus on just hyperelliptic Feynman integrals, which still hold interesting new surprises compared to the elliptic case

- Hyperelliptic curves can be defined by an equation of the form

  $$y^2 = \prod_{i=1}^{n} (z - r_i)$$

  for some set of distinct roots r_i

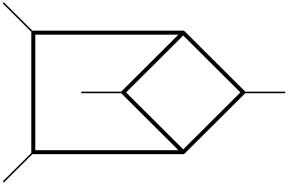
$$n = 3, 4 \quad \Rightarrow \quad \text{elliptic curve}$$

$$n \geq 5 \quad \Rightarrow \quad \text{hyperelliptic curve of genus } g = \left\lceil \frac{n-2}{2} \right\rceil$$

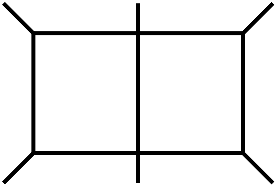
Hyperelliptic Feynman Integrals

A handful of Feynman integrals are already known to give rise to hyperelliptic curves

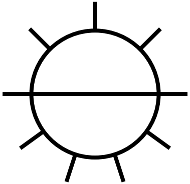
[Huang, Zhang, 2013] [Georgoudis, Zhang, 2015] [Doran, Harder, Vanhove, 2023]



$D = 4$



$D = 6$



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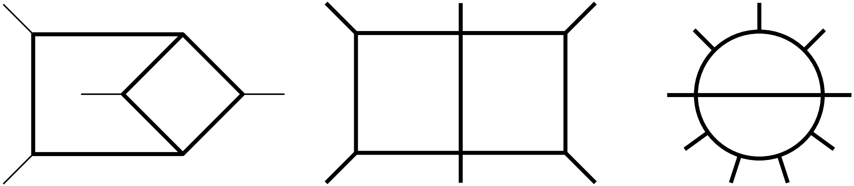

$D = 4$ $\quad$ $D = 6$ $\quad$ $D = 8$

However. . .

- **Fewer than 5 papers** written on hyperelliptic Feynman integrals
- Compare this to **more than 40 papers** written on Calabi-Yau Feynman integrals

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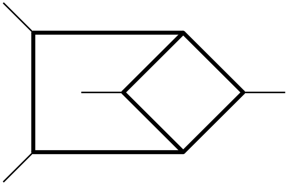

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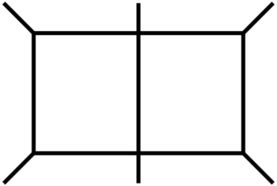

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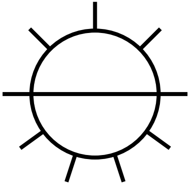

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However. . .

- **Fewer than 5 papers** written on hyperelliptic Feynman integrals
- Compare this to **more than 40 papers** written on Calabi-Yau Feynman integrals

$\Rightarrow$ Much remains to be learned about this class of Feynman integrals

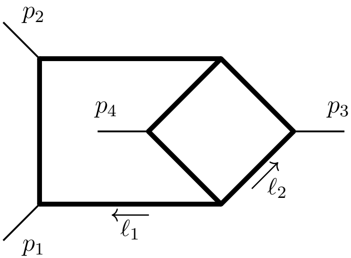
Genus Drop in Hyperelliptic Feynman Integrals

arXiv:2307.11497 [hep-th]

with R. Marzucca, B. Page, S. Pögel, and S. Weinzierl

The Nonplanar Crossed Box

We focus on the example of the nonplanar crossed box diagram:



$$= \int d^4\ell_1 d^4\ell_2 \frac{1}{\prod_{i=1}^{7} D_i}$$

$$D_i = q_i^2 - m^2$$

(massless external particles, all internal propagators have mass m)

- Function of $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, and m^2
- Over ten years ago, shown to give rise to an integral over a genus-three curve
  [Huang, Zhang, 2013]

Momentum Space

- More specifically, it was shown cutting all seven propagators in momentum space resulted in an integral [Huang, Zhang, 2013]

$$\sim \int \frac{dz\ z}{\sqrt{P_8(z)}}$$

where $P_8(z)$ is a degree-eight polynomial whose coefficients depend on s, t, and m^2

$$\begin{aligned} P_8(z) = {} & (s+t)^2 \left(t^2 m^2 + s^2 z(sz+t)\right) \left(m^2 (s+t)^2 + s^2 z(sz+s+t)\right) \times \\ & \left(s^2 z m^2 \left(-3s^3 z + s^2(2tz+t) + st^2(2z+3) + 2t^3\right) + t^2 \left(m^2\right)^2 (s+t)^2 + s^4 z^2 (sz+t)(sz+s+t)\right) \end{aligned}$$

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$\Rightarrow$ We expect this Feynman integral to depend on iterated integrals involving one-forms that can be defined on the curve

$$y^2 = P_8(z) = \prod_{i=1}^{8} (z - r_i)$$

Baikov Representation

- However, we can also compute the maximal cut after changing to a Baikov parametrization. In this case, one finds an integral

$$\sim \int \frac{dz}{\sqrt{P_6(z)}}$$

where now $P_6(z)$ is just a degree-six polynomial

$$\boxed{P_6(z) = s\left(2z(s+2z) - 3m^2 s\right)\left(m^2 s + 2z(s+2z)\right)\left(s(s+t+2z)^2 - 4m^2 t(s+t)\right)}$$

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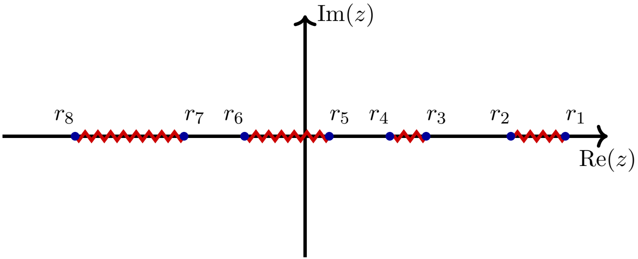
Does the nonplanar crossed box integral evaluate to iterated integrals that involve one-forms related to a **genus-two** or a **genus-three** curve?

Period Matrix

To explore this apparent tension, we can study the period matrix associated with each curve

- The branch cut structure of the genus-three curve takes the form



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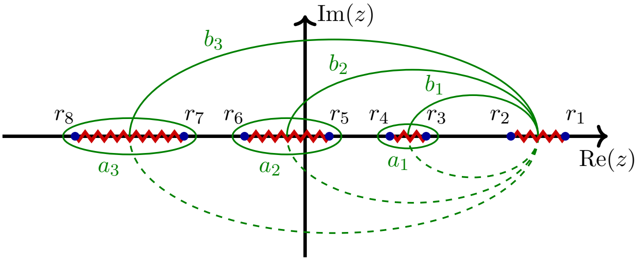

- We can thus find a basis of six independent integration contours
- We can also define three independent holomorphic differentials

$$\frac{z^i dz}{\sqrt{P_8(z)}}, \qquad i \in \{0, 1, 2\}$$

Extra Period Matrix Relations

- It is simple to numerically evaluate this period matrix for generic values of s, t, and m^2

  $\Rightarrow$ Doing this for a number of kinematic points, we find that the entries of this matrix satisfy simple **unexpected linear relations**

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  $\Rightarrow$ Doing this for a number of kinematic points, we find that the entries of this matrix satisfy simple **unexpected linear relations**

- This motivates looking for some kind of hidden symmetry or constraint that might explain these relations

  $\Rightarrow$ For instance, it is possible that $P_8(z)$ has a symmetry that is only made manifest if one makes the right change of coordinates

- To search for such a symmetry, we apply a general $SL_2(\mathbb{C})$ transformation

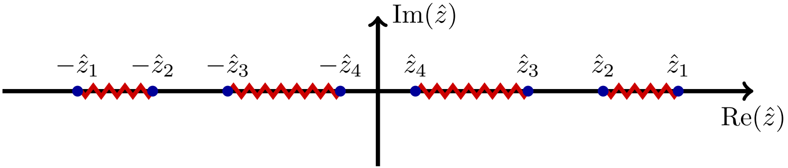
$$z \mapsto \frac{a\hat{z} + b}{c\hat{z} + d}$$

  and ask whether anything special happens for particular values of a, b, c, and d

A Hidden Symmetry

- Surprisingly, this change of variables can be chosen such that all eight roots pair up:

$$P_8(z) \mapsto \hat{P}_4(\hat{z}^2) = \prod_{i=1}^{4} (\hat{z}^2 - \hat{r}_i^2)$$



$\Rightarrow$ In this representation, it's clear why relations exist between different periods

A Hidden Symmetry

Now there's only one thing to do... search for this type of symmetry in the math literature!

Curves with an Extra Involution

- Hyperelliptic curves with this symmetry are described as respecting an **extra involution**

  $$e_1 : \hat{z} \mapsto -\hat{z}$$

  above and beyond the involution that all hyperelliptic curves respect

  $$e_0 : y \mapsto -y$$

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- There are then two hyperelliptic curves that can be associated with $P_8(z)$:

$$v_1^2 = \hat{P}_4(w) \qquad \text{(genus 1)}$$
$$v_2^2 = w\hat{P}_4(w) \qquad \text{(genus 2)}$$

- These curves can be mapped back to $P_4(\hat{z}^2)$ by the e_1-invariant map $(v_1, w) \mapsto (y, \hat{z}^2)$ and the $e_1 \circ e_0$-invariant map $(v_2, w) \mapsto (y\hat{z}, \hat{z}^2)$, respectively

Curves with an Extra Involution

Let's see how this pair of curves arises in a more pedestrian way:

- Consider a hyperelliptic curve $P_{2g+2}(z)$ of genus g that respects an extra involution, which can be made manifest by the change of variables $z \mapsto \frac{a\hat{z}+b}{c\hat{z}+d}$
- Like before, we define

$$\hat{P}_{g+1}(w) = \hat{P}_{g+1}(\hat{z}^2) = (c\hat{z}+d)^{2g+2}\ P_{2g+2}\left(\frac{a\hat{z}+b}{c\hat{z}+d}\right)$$

- Finally, using the fact that

$$dz = \pm\frac{ad-bc}{2(d\pm c\sqrt{w})^2\sqrt{w}}dw$$

we compute the entries of the period matrix of P_{2g+2} in terms of w to be

$$\int_{\gamma_j}\frac{dz\ z^i}{\sqrt{P_{2g+2}(z)}} = \pm\frac{(ad-bc)}{2}\int_{\gamma_j} dw\frac{(\pm a\sqrt{w}+b)^i(\pm c\sqrt{w}+d)^{g-1-i}}{\sqrt{w\hat{P}_{g+1}(w)}}$$

Curves with an Extra Involution

$$\int_{\gamma_j} dw \frac{(\pm a\sqrt{w}+b)^i(\pm c\sqrt{w}+d)^{g-1-i}}{\sqrt{w\hat{P}_{g+1}(w)}}$$

Two types of terms appear in this integral, when the numerator is expanded out

- Terms with integer powers of w evaluate to periods of the curve $w\hat{P}_{g+1}(w)$
- Terms with half-integer powers of w evaluate to periods of the curve $\hat{P}_{g+1}(w)$

Curves with an Extra Involution

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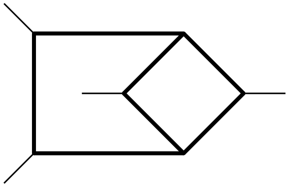
In the case of the nonplanar crossed box, we only get integer powers of w

- The original periods can be expressed as linear combinations of genus-two periods
- A further change of variables maps $w\hat{P}_4(w)$ to the genus-two Baikov curve

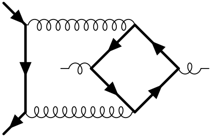
A Few Comments



- There is nothing incorrect about the cut computation in momentum space—the curve one finds this way is genuinely genus three, so we expect the nonplanar crossed box *could* be evaluated in terms of iterated integrals involving $P_8(z)$
- The point is that this genus-three curve has an extra symmetry that allows it to be algebraically mapped to a curve of lower genus without losing any information
- This corresponds to a massive simplification of the types of iterated integrals needed to evaluate this Feynman integral

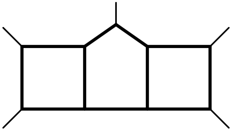
Further Examples

- $gg \to t\bar{t}$ with a top quark loop



genus 3 drops to 2

- In the kinematic limit where $s = -2t$, the equal-mass nonplanar crossed box drops further in genus from 2 to 1
- Beyond hyperelliptic curves (... is this due to an extra involution?)



genus 5 drops to 3

(equal internal masses, massless external particles)

Feynman Integrals of All Genus

arXiv:23nn.nnnnn [hep-th]

with S. Abreu, A. Behring, and B. Page

Hyperelliptic Feynman Integrals

We've established that interesting new mathematical phenomena can appear when curves of genus greater than one appear in Feynman integrals

$\Rightarrow$ However, it's reasonable to ask how commonly these types of curves appear—are the examples we're looking at exceptional, or just the simplest of a large class of examples?

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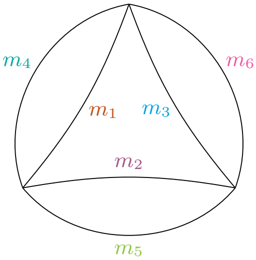
I now want to argue that hyperelliptic curves are ubiquitous in perturbative quantum field theory

$\Rightarrow$ This class of Feynman integrals deserves to be studied in much more depth!

Simple Example

Consider the following four-loop vacuum graph:



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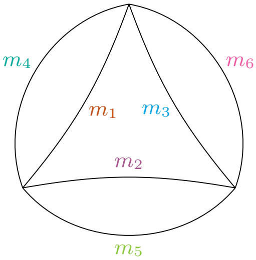
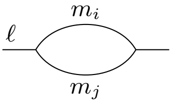


- The max cut is easy to compute in two dimensions using the max cut of the bubble

MaxCut(



)

$$= \frac{1}{\sqrt{\big(\ell^2 - \underbrace{(m_i + m_j)^2}_{\text{threshold}}\big)\big(\ell^2 - \underbrace{(m_i - m_j)^2}_{\text{pseudothreshold}}\big)}}$$

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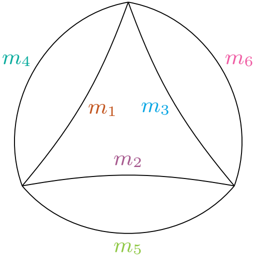

$$r_i^{\pm} = (m_i \pm m_{i+3})^2$$

$$\text{MaxCut}\Big(\quad \Big) = \int d^2\ell \, \frac{1}{\prod_{i=1}^{3} \sqrt{(\ell^2 - r_i^+)(\ell^2 - r_i^-)}}$$

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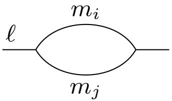

threshold

pseudothreshold

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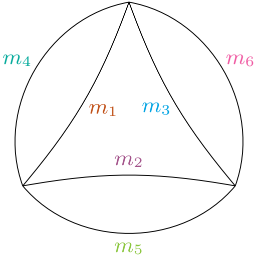

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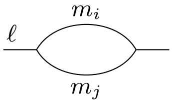

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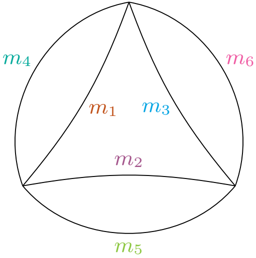

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genus 2 curve

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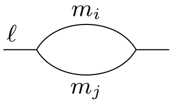
$\mathsf{MaxCut}\Bigg($  $\Bigg)$

$$= \frac{1}{\sqrt{\big(\ell^2 - (m_i + m_j)^2\big)\big(\ell^2 - (m_i - m_j)^2\big)}}$$

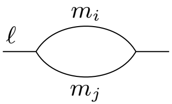
threshold

pseudothreshold

Simple Example

In fact, we can do even better—instead of cutting the bubbles, we can integrate them out

$$\boxed{\frac{\ell^2 + m_i^2 + m_j^2 + \sqrt{\left(\ell^2 - (m_i + m_j)^2\right)\left(\ell^2 - (m_i - m_j)^2\right)}}{2 m_i m_j}}$$

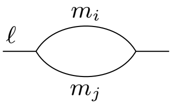


$$= \frac{\log(\quad)}{\sqrt{\left(\ell^2 - (m_i + m_j)^2\right)\left(\ell^2 - (m_i - m_j)^2\right)}}$$

Simple Example

In fact, we can do even better—instead of cutting the bubbles, we can integrate them out

$$\frac{\ell^2 + m_i^2 + m_j^2 + \sqrt{\left(\ell^2 - (m_i+m_j)^2\right)\left(\ell^2 - (m_i-m_j)^2\right)}}{2 m_i m_j}$$



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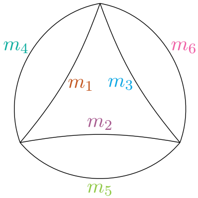
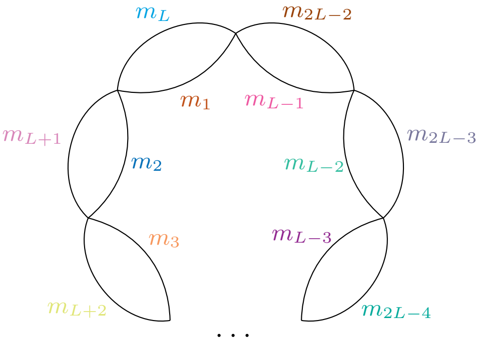


$$\sim \int d\ell^2 \frac{\prod_{i=1}^{3} \log(x_i)}{\prod_{i=1}^{3} \sqrt{(\ell^2 - r_i^+)(\ell^2 - r_i^-)}}$$

$$r_i^{\pm} = (m_i \pm m_{i+3})^2 \qquad x_i = \frac{\ell^2 + m_i^2 + m_{i+3}^2 + \sqrt{\left(\ell^2 - r_i^+\right)\left(\ell^2 - r_i^-\right)}}{2 m_i m_{i+3}}$$

All-Loop Necklace Integrals

$$r_i^{\pm} = (m_i \pm m_{i+L-1})^2 \qquad x_i = \frac{\ell^2 + m_i^2 + m_{i+L-1}^2 + \sqrt{(\ell^2 - r_i^+)(\ell^2 - r_i^-)}}{2 m_i m_{i+L-1}}$$

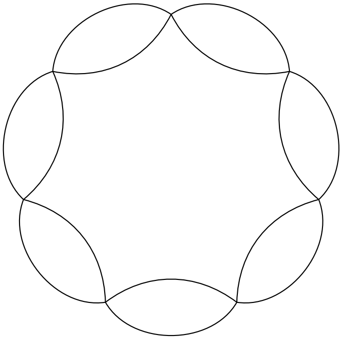


$$\sim \int d\ell^2 \frac{\prod_{i=1}^{L-1} \log(x_i)}{\prod_{i=1}^{L-1} \sqrt{(\ell^2 - r_i^+)(\ell^2 - r_i^-)}}$$

$\Rightarrow$ an integral over a **hyperelliptic curve of genus** $L-2$

Minimal Mass Configurations

However, these examples depend on a large number of masses... can we do better?



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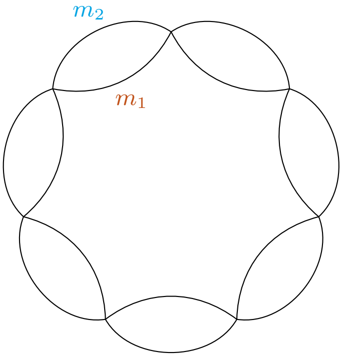

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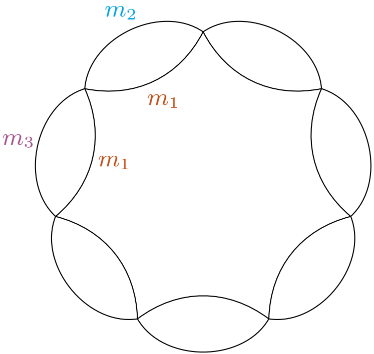

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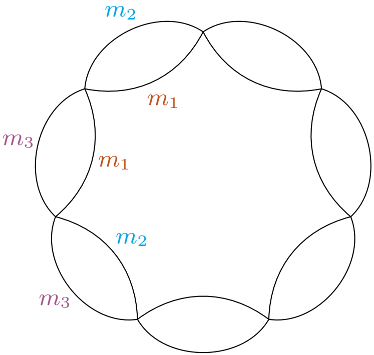

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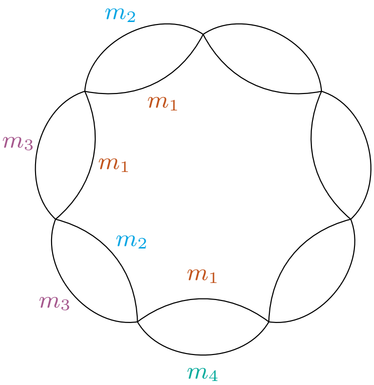

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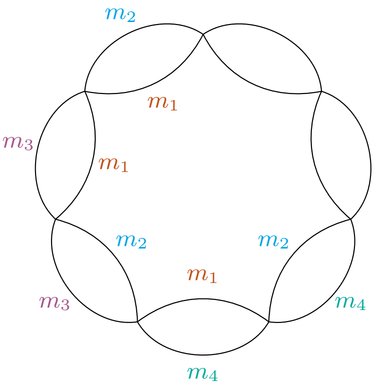

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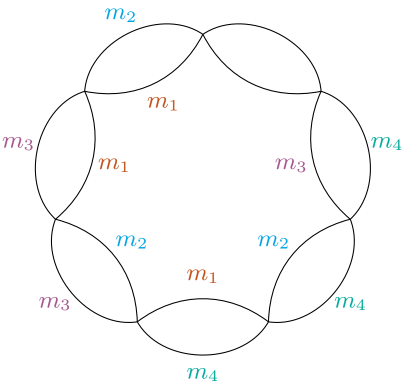

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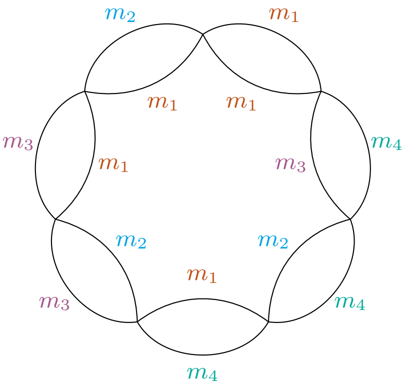

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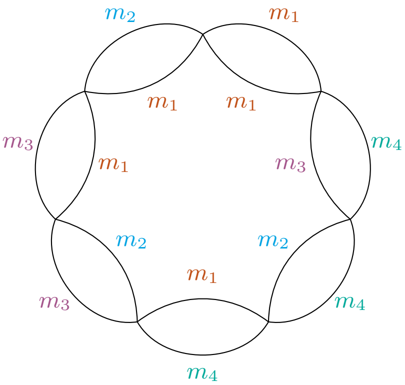

$$\Rightarrow \qquad n_L^{\mathsf{min}} = \left\lceil \frac{1+\sqrt{-15+8L}}{2} \right\rceil$$

$\Rightarrow$ the number of independent kinematic variables can be chosen to grow quite slowly (3 masses through 5 loops, 4 masses through 8 loops, 5 masses through 12 loops, . . .)

Conclusions

- Hyperelliptic curves seem to be quite common in perturbative quantum field theory
  - $\Rightarrow$ Even if you don't care about vacuum integrals, the necklaces 'lower bound' the complexity of Feynman integrals for which they appear in soft limits
- Surprising simplifications can occur in the types of functions needed to evaluate Feynman integrals beyond the elliptic case
  - $\Rightarrow$ We have identified Feynman integrals in which the periods associated with the max cut can be re-expressed as linear combinations of lower-genus curves
  - $\Rightarrow$ Can this happen beyond the hyperelliptic case? Or higher-dimensional geometries?
- Much more to exploration to be done for hyperelliptic Feynman integrals!

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Thanks!