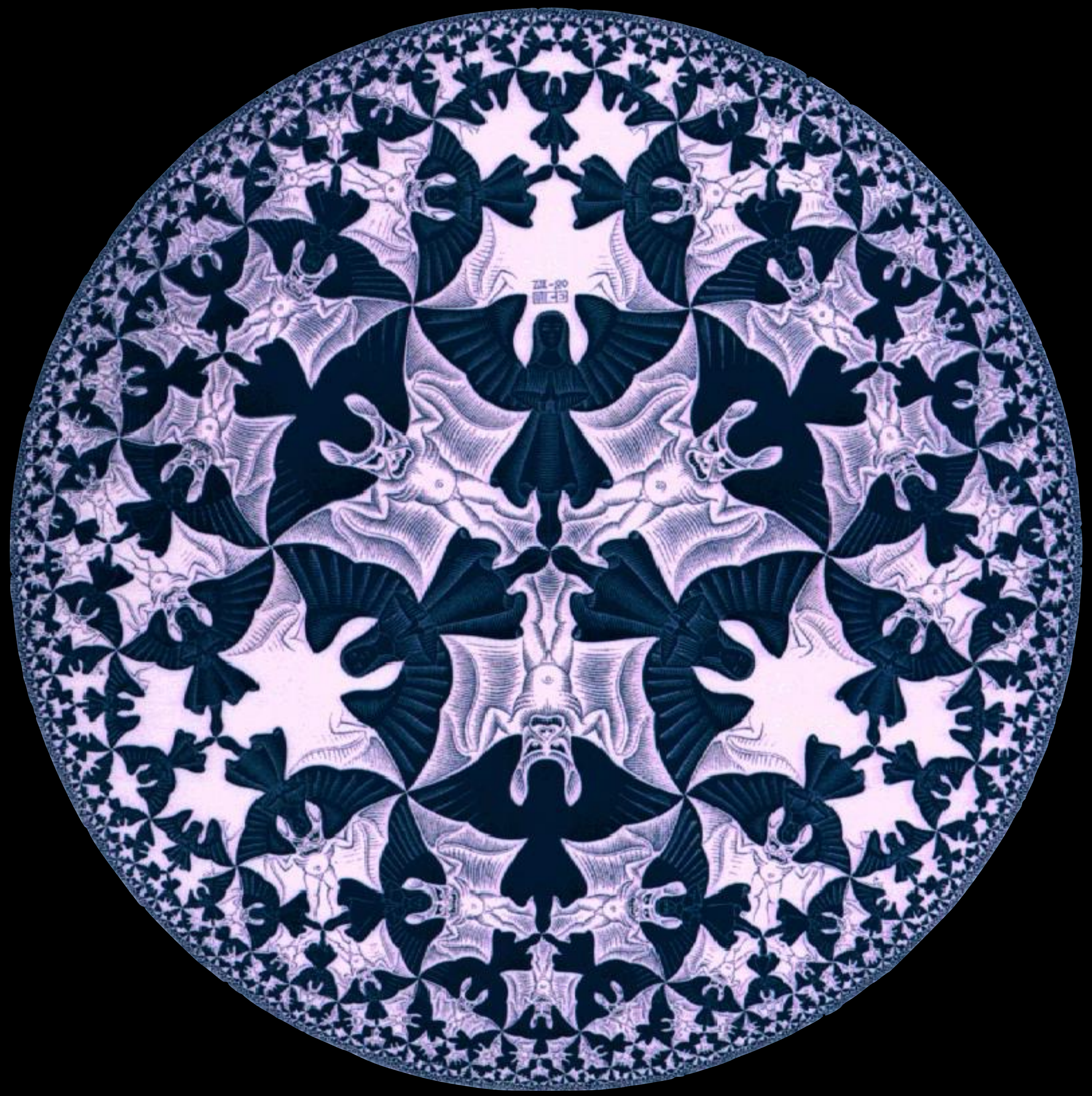
Geometries and symmetries of elliptic Feynman amplitude



Elliptics '23

Speaker: YuJiao Zhu

Section 1
A general discussion of symmetries

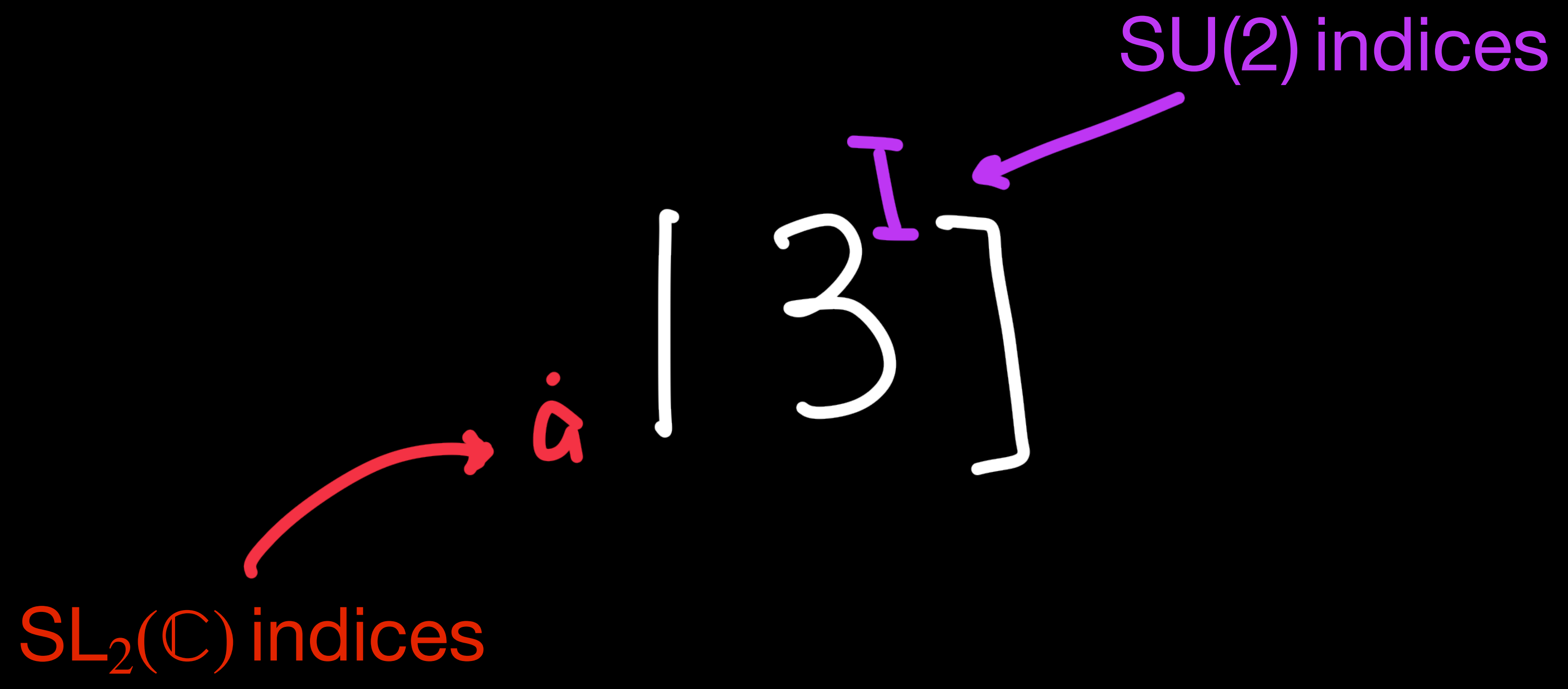
Symmetries of an amplitude

- SL(2,ℂ) Lorentz invariance (well-defined probability interpretations, regardless of any reference of frame)
- SU(3) or U(1) singlet (color-charge conservation)

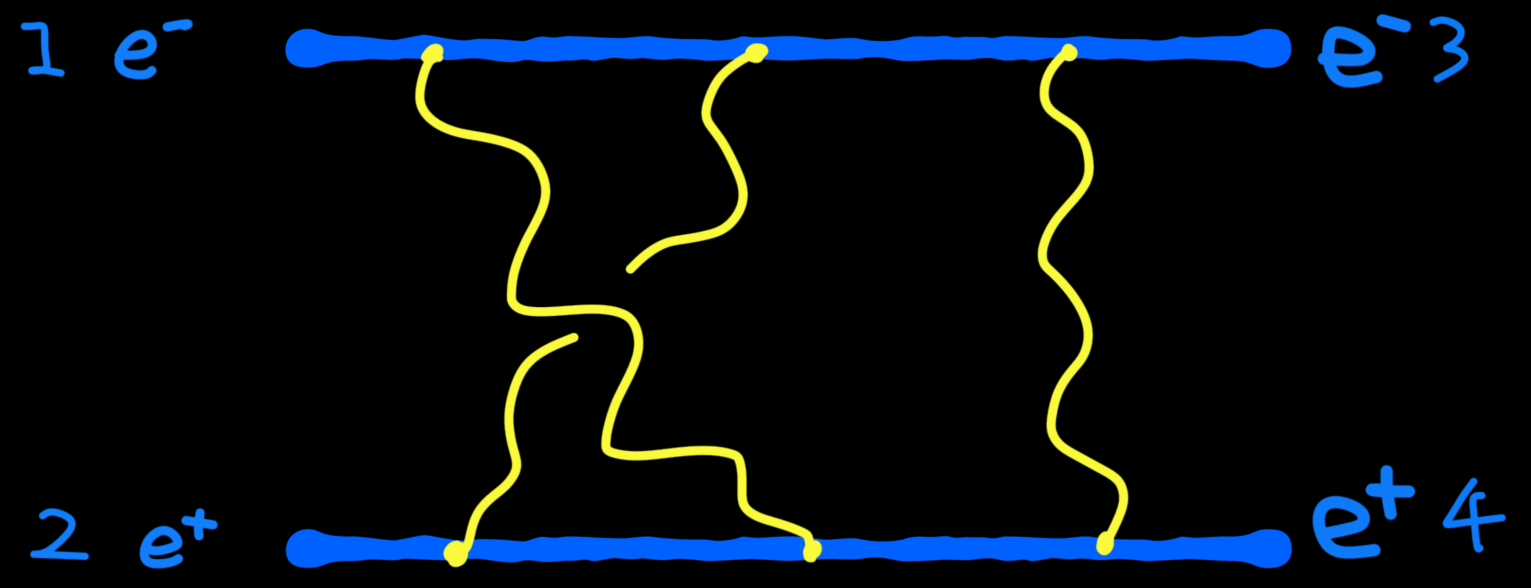
$$e^{-i\theta\cdot T}|\mathscr{A}\rangle = |\mathscr{A}\rangle \iff T^a|\mathscr{A}\rangle = 0\,, \quad T^a \equiv \sum_{i\in\{\text{in\&out}\}} T_i^a$$

- SU(2)(massive) or U(1)(massless) tensors — little group covariance

$$P_{a\dot{a}} = {}_a|P^I\rangle[P_I|_{\dot{a}} = {}_a|P^I\rangle[P_I|_{\dot{a}} \Longrightarrow {}_a|P^I\rangle = {}_a|P^J\rangle R_J^I\,, \quad R \in SU(2)$$



Bhabha scattering $S(\bar{1}^{I_1}, 2^{I_2}, \bar{3}^{I_3}, 4^{I_4}) =$

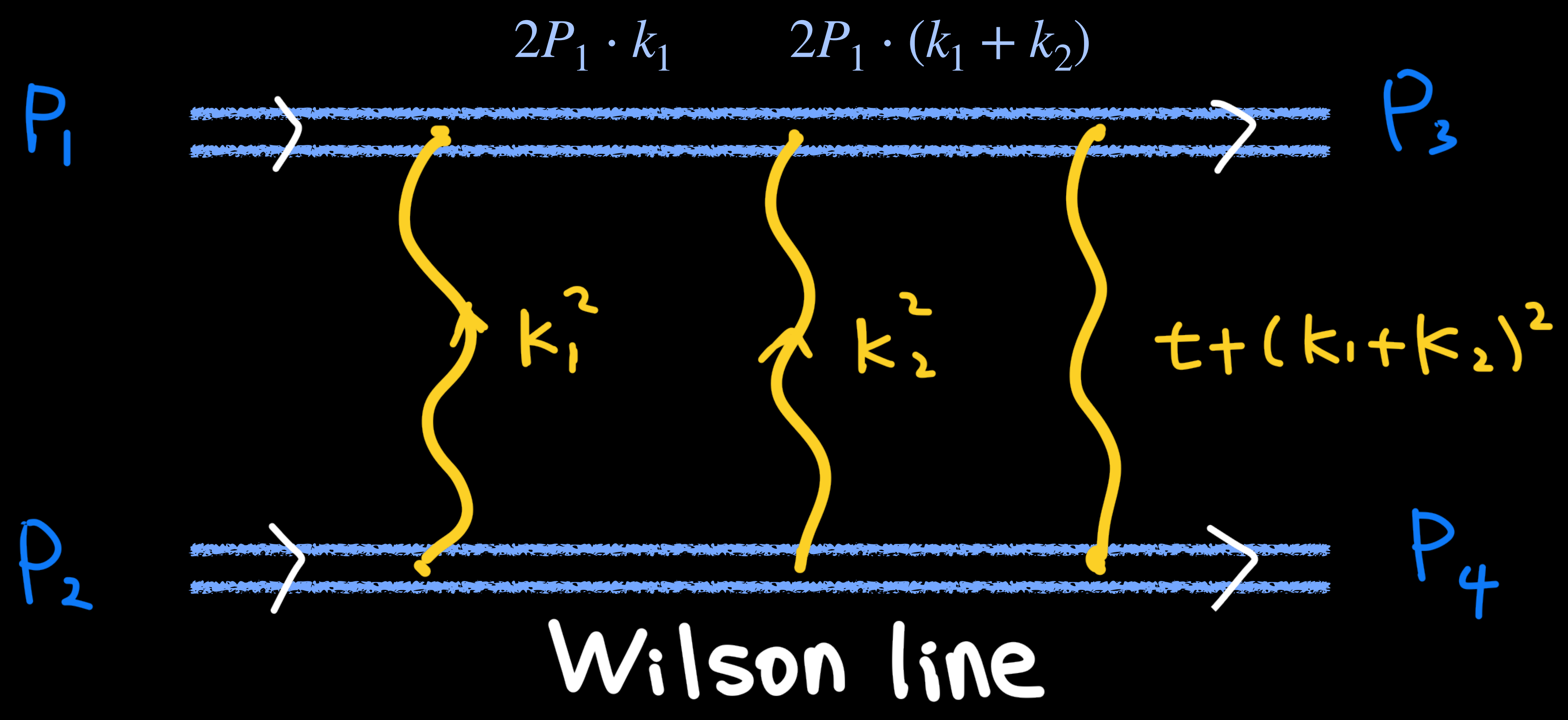


★**Symmetries: SU(2) little group covariance**

$$S(\bar{1}^{I_1}, 2^{I_2}, \bar{3}^{I_3}, 4^{I_4}) = \bar{W}^{I_1}{}_{J_1}\, W^{I_2}{}_{J_2}\, \bar{W}^{I_3}{}_{J_3}\, W^{I_4}{}_{J_4}\, S(\bar{1}^{J_1}, 2^{J_2}, \bar{3}^{J_3}, 4^{J_4})$$

$$W \in SU(2)!$$

- **Regge Limit $s, m^2 \gg t$ [Korchemsky 1996]**



$$\sim \left(\frac{t}{\mu}\right)^{-2\epsilon} \left(\frac{P_1^2\, P_2^2}{(P_1 \cdot P_2)^2}\right)^{-2\epsilon}$$

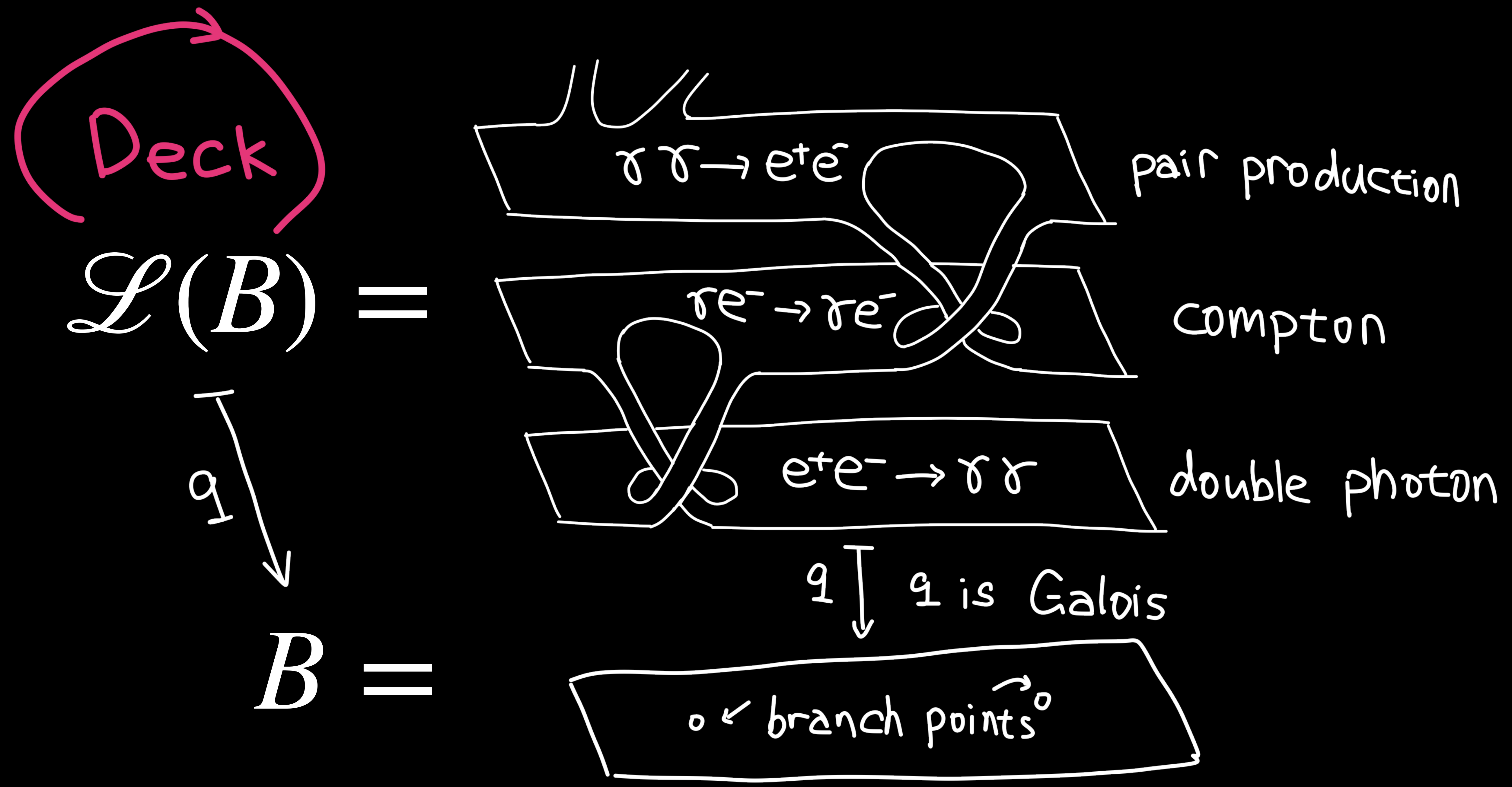
$$t \equiv (P_3 - P_1)^2$$

Naive picture: Amplitude as sheaf of germs of analytic functions $(\mathscr{L}(B), q)$ over kinematic base space

$[s : t : \ldots : m^2] \in$ Base space $= \mathbb{CP}^n \backslash \{$kinematic branch points$\}$

kinematic branch points=linear varieties

Question: The amplitude, as a geometric object, what is it's automorphism group?— the deck transformation, automorphisms of covering



Deck transformation≃Monodromy, for Galois covering

- GTM202, for a normal (Galois) covering:

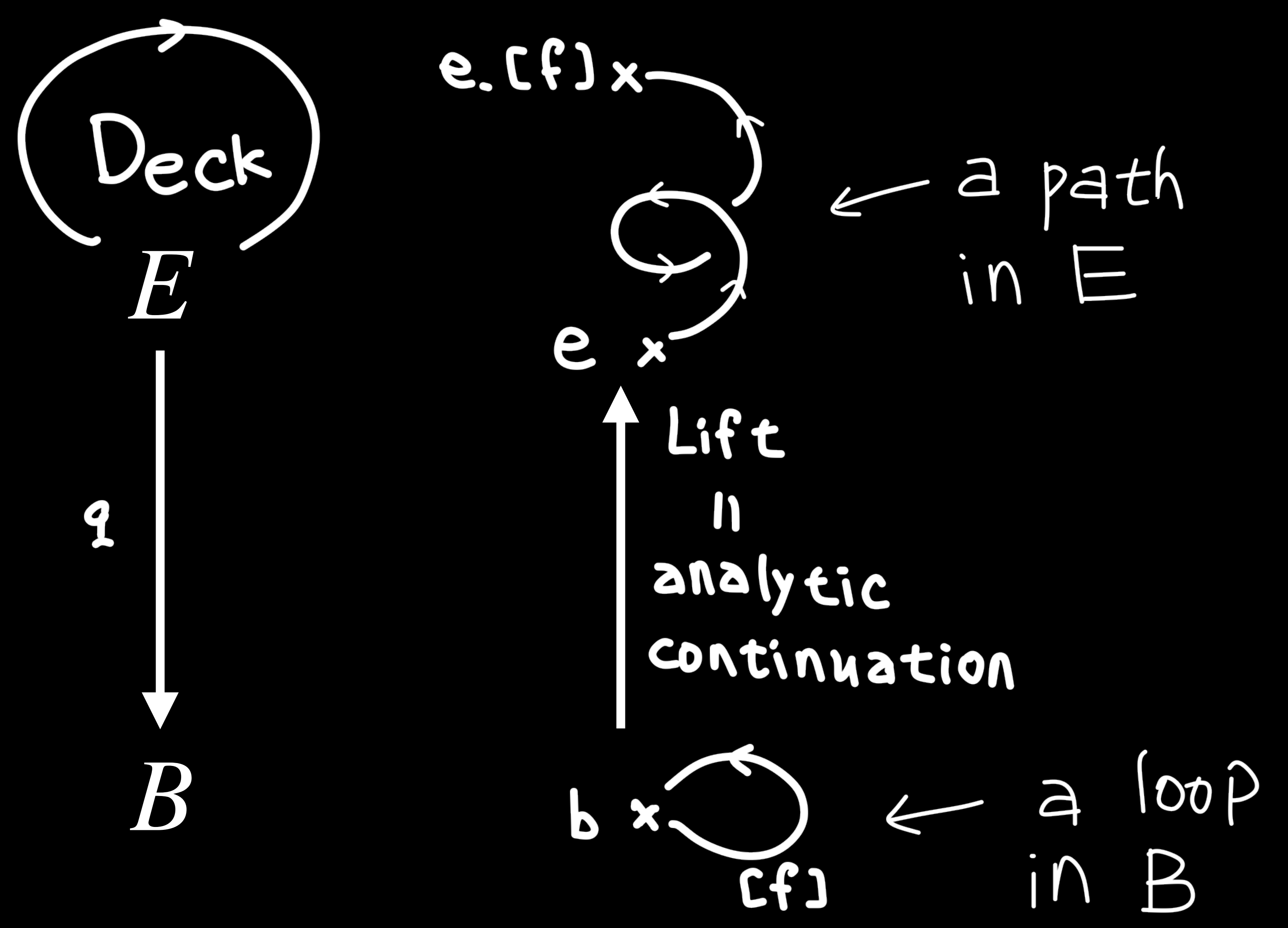
$$\mathrm{Deck}(E \overset{q}{\mapsto} B) \simeq \frac{N_{\pi_1(B,b)}(q_*\pi_1(E,e))}{q_*\pi_1(E,e)} \overset{\text{normal}}{\simeq} \pi_1(B,b)/q_*\pi_1(E,e) \overset{\text{normal}}{\simeq} \text{Monodromy}$$

E = amplitude = sheaf of germs of analytic functions!

$[s : t : \ldots : m^2] \in$ B = $\mathbb{CP}^n \backslash \{$kinematic branch points$\}$

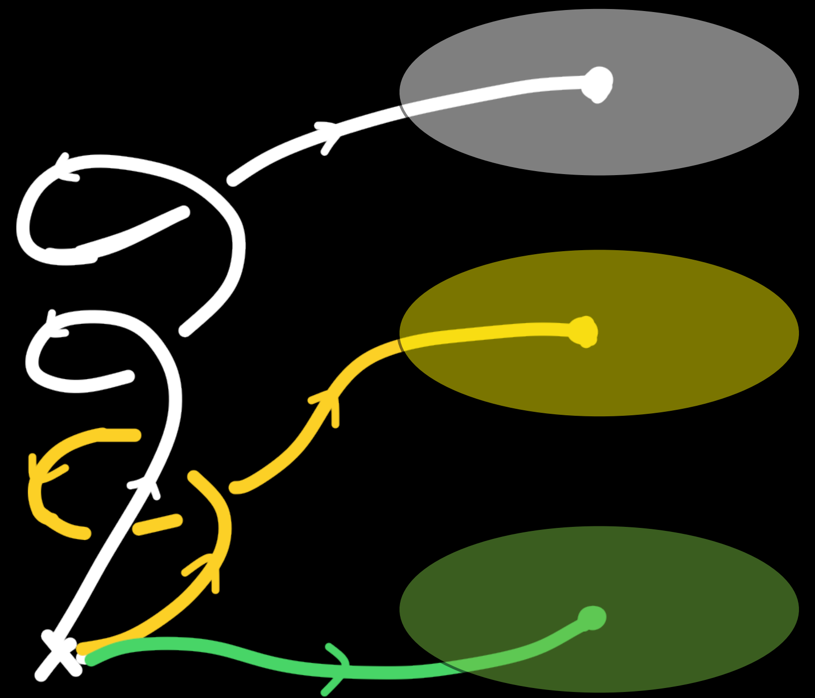
$q_*\pi_1(E,e)$: isotropy groups of the monodromy action

$N_{\pi_1(B,b)}\left(q_*\pi_1(E,e)\right)$: the normalizer



Example: uniformization of the 'amplitude' $\mathscr{A}(x) = x^{\frac{1}{3}}, x \in \mathbb{C}\backslash\{0\}$

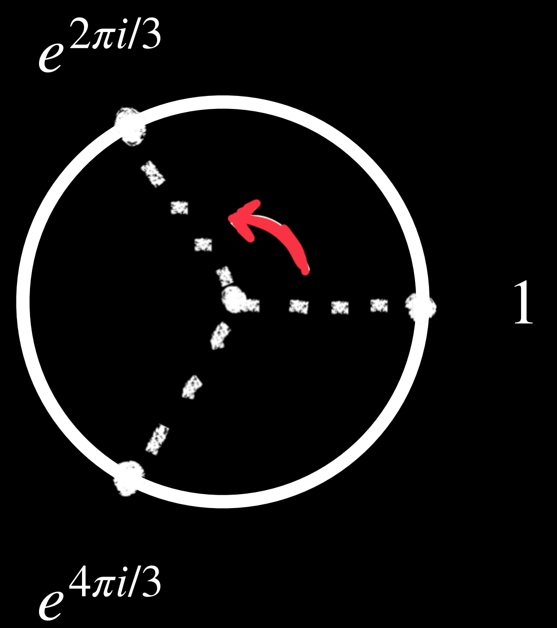
Method 1. Space of non-equivalent paths



$x^{\frac{1}{3}}e^{4\pi i/3}$

$x^{\frac{1}{3}}e^{2\pi i/3}$

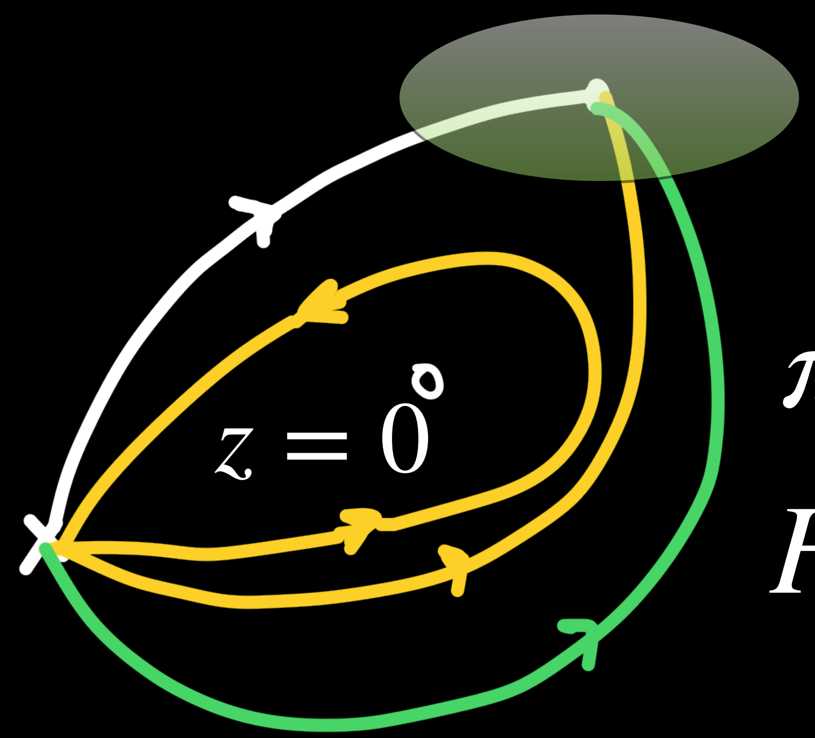
$x^{\frac{1}{3}}$



Lift ↑ ↓ q

Deck $= \{(), (123), (132)\} < S_3 \simeq \mathbb{Z}/3\mathbb{Z}$

$\simeq$ Group of 3th roots of unity



$\pi_1(\mathbb{C}\backslash\{0\}, \cdot\,) = \mathbb{Z}$

$H = 3\mathbb{Z} < \mathbb{Z}$

Method 2. Modulo from universal cover

covering map p from $\mathbb{C}$ to $\mathbb{C}\backslash\{0\}$: $x = p(z) = e^{2\pi i z}$

Deck ⟲ $z \in \mathbb{C}$

$z \in \mathbb{C} \to \mathbb{C}\backslash\hat{H}$

$x^{1/3} \overset{p^*}{\longmapsto} e^{\frac{2\pi i}{3}z}$

p: $z \in \mathbb{C} \to x \in \mathbb{C}\backslash\{0\}$; q: $\mathbb{C}\backslash\hat{H} \to x \in \mathbb{C}\backslash\{0\}$

$x \in \mathbb{C}\backslash\{0\}$

redundant symmetry $z \mapsto z + 3n,\ n \in \mathbb{Z}$

$3\mathbb{Z} = \hat{H} \subset \mathrm{Deck}_p \simeq \pi_1(B, b) = \mathbb{Z}$

The idea of the deck transformations at amplitude-level is useless, nor did we know if the covering to the kinematic base space is normal(Galois)

Section 2

Symbol letters of an amplitude

Symbol letters from canonical forms

- Amplitude through canonical bases Residues and periods

$$\mathscr{A}(s,t) = \sum_i R_i(s,t) \times \mathrm{J}_i(s,t)$$

canonical bases

- Canonical form for the differential equations [Johannes M. Henn 2014]

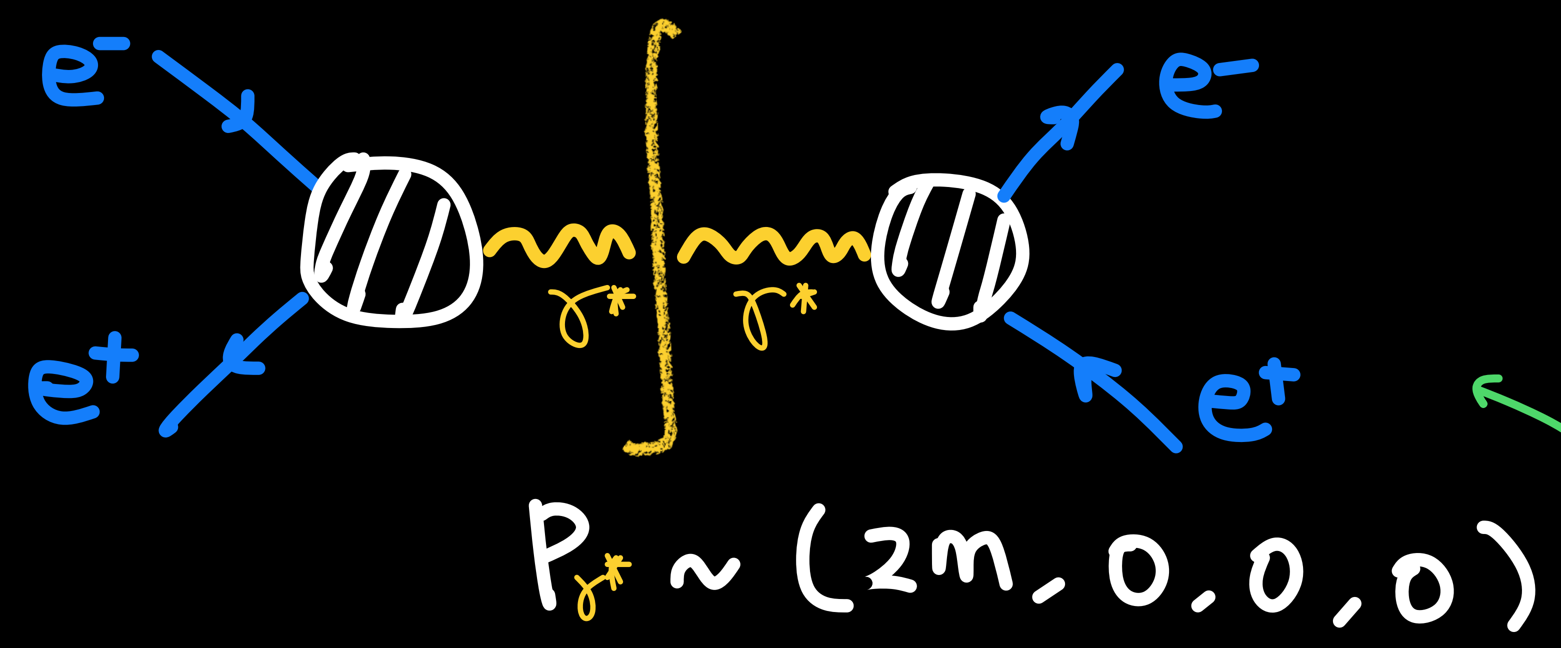
$$\mathrm{d}\vec{\mathrm{J}} = \epsilon\, M(s,t) \cdot \vec{\mathrm{J}}$$

- Kernel $M(s,t)$ as linear array over the symbol letters $\omega_i(s,t)$

$$M(s,t) = \sum_i c_i \times \omega_i(s,t)\,, \quad c_i \in \mathbb{Q}\,, \quad \mathrm{d}\omega_i(s,t) = 0$$

The role of the symbol letters

- They are closed 1-forms which encode the analytic structures of a Feynman amplitude, an example for Bhabha scattering:



$$\frac{\mathrm{d}s}{\sqrt{-s}\sqrt{4m^2-s}} \overset{\text{ramified covering}}{\underset{h:\,\mathbb{CP}^1 \mapsto \mathbb{CP}^1}{\longmapsto}} \frac{\mathrm{d}x}{x}, \quad s = h(x) := -\frac{(1-x)^2}{x}$$

threshold branch point $s \to 4m^2$

- They are in general multi-valued! After uniformization, they have at most simple poles! Integrating over simple poles generates logarithms, this is why QFT has at most logarithmic singularities!

Symbol letters for the planar Bhabha

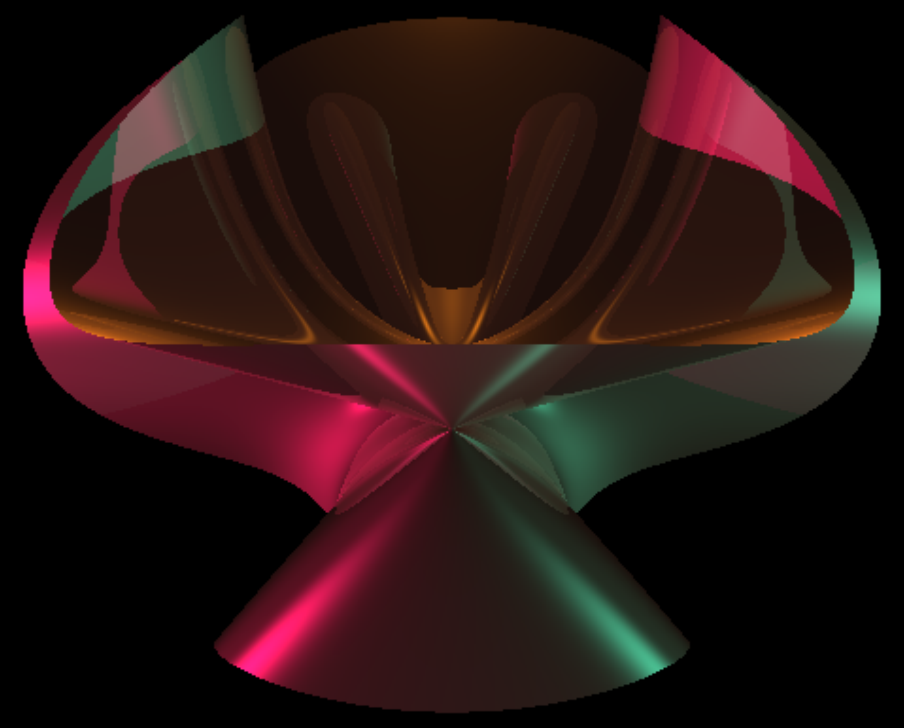
- The square roots [1307.4083, 2108.03828]

$$r_s = \sqrt{-s}\sqrt{4m^2 - s}\,, \quad r_t = \sqrt{-t}\sqrt{4m^2 - t}\,, \quad r_u = \sqrt{-s-t}\sqrt{4m^2 - s - t}$$

- Coordinates on the elliptic K3 surface

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x} \text{ and } \frac{-t}{m^2} = \frac{(1-y)^2}{y}$$

$$r_s \mapsto \frac{1}{x} - x\,, \quad r_t \mapsto \frac{1}{y} - y\,, \quad r_u \mapsto \frac{z}{xy}$$



K3 : $z^2 = (x+y)(xy+1)((x+y)(xy+1) - 4xy)$

A typical symbol letter (closed 1-form) for the non-planar Bhabha

$$
\begin{aligned}
&\mathrm{ds}\times\Bigg\{\frac{-4\sqrt{(t-4)t}(2s^2+3st-10s-2t+8)}{(s-4)s(4-t)(s+t-4)}T_2(s,t)\\
&+\frac{2s^3t^2-4s^3t+80s^3+s^2t^3+2s^2t^2+288s^2t-480s^2+4st^3+346st^2-1224st+640s+169t^3-776t^2+400t}{4(s-4)s(t-4)(s+t-4)(s+t)}T_1(s,t)\\
&+\frac{s^3t^2-2s^3t+8s^3+2s^2t^3-10s^2t^2+56s^2t-64s^2-2st^3+81st^2-260st+128s+49t^3-264t^2+272t}{(s-4)s(4-t)^2(s+t-4)}2\sqrt{(t-4)t}\Psi(s,t)\\
&+\left[\frac{6(t-4)\sqrt{(t-4)t}}{(s-4)s(4-t)^2t}T_1^2(s,t)+\frac{(s+1)(2s+t-4)}{(s-4)s(s+t-4)(s+t)}T_1(s,t)T_2(s,t)-\frac{\sqrt{(t-4)t}}{(s-4)s(4-t)t}T_2^2(s,t)\right]\frac{1}{\Psi(s,t)}\\
&+\left[\frac{2s+t-4}{4(s-4)st(s+t-4)(s+t)}T_1^3(s,t)+\frac{2s+t-4}{(s-4)st(s+t-4)(s+t)}T_1(s,t)T_2^2(s,t)\right]\frac{1}{\Psi^2(s,t)}\Bigg\}\\
&+\mathrm{dt}\times\Bigg\{2\frac{2s^2-st^2+11st-4s+7t^2-8t-16}{(4-t)^2(s+t-4)}\sqrt{(t-4)t}\Psi(s,t)+\frac{-s^2t^2+10s^2t+8s^2+12st^2+40st-32s+8t^3+39t^2-92t}{4(t-4)t(s+t-4)(s+t)}T_1(s,t)\\
&-\left[\frac{1}{4t^2(s+t-4)(s+t)}T_1^3(s,t)+\frac{1}{t^2(s+t-4)(s+t)}T_2^2(s,t)T_1(s,t)\right]\frac{1}{\Psi^2(s,t)}-\frac{s+1}{t(s+t-4)(s+t)}\frac{T_1(s,t)T_2(s,t)}{\Psi(s,t)}\\
&+\frac{4\sqrt{(t-4)t}}{(4-t)(s+t-4)}T_2(s,t)\Bigg\}
\end{aligned}
$$

The period function (mapping)

Naive definition: period functions are complete elliptic integrals of first kind, e.g.,

$$E_\lambda : Y^2 = X(X-1)(X-\lambda)\,,\lambda \in \mathbb{CP}^1\backslash\{0,1,\infty\}$$

$$\Psi(\lambda) \equiv \int_0^\lambda \frac{\mathrm{d}X}{Y} = 2\mathrm{K}(\lambda)$$

$\Psi(\lambda)$ **is multi-valued, it has branch cuts at** $\lambda = 1\,,\infty$**, e.g.,**

$$\Psi(1+ix) \xrightarrow{x\to 0^+} \frac{i\pi}{2} + 4\ln 2 + \ln x\,, \quad \Psi(1+ix) \xrightarrow{x\to 0^-} -\frac{i\pi}{2} + 4\ln 2 + \ln(-x)$$

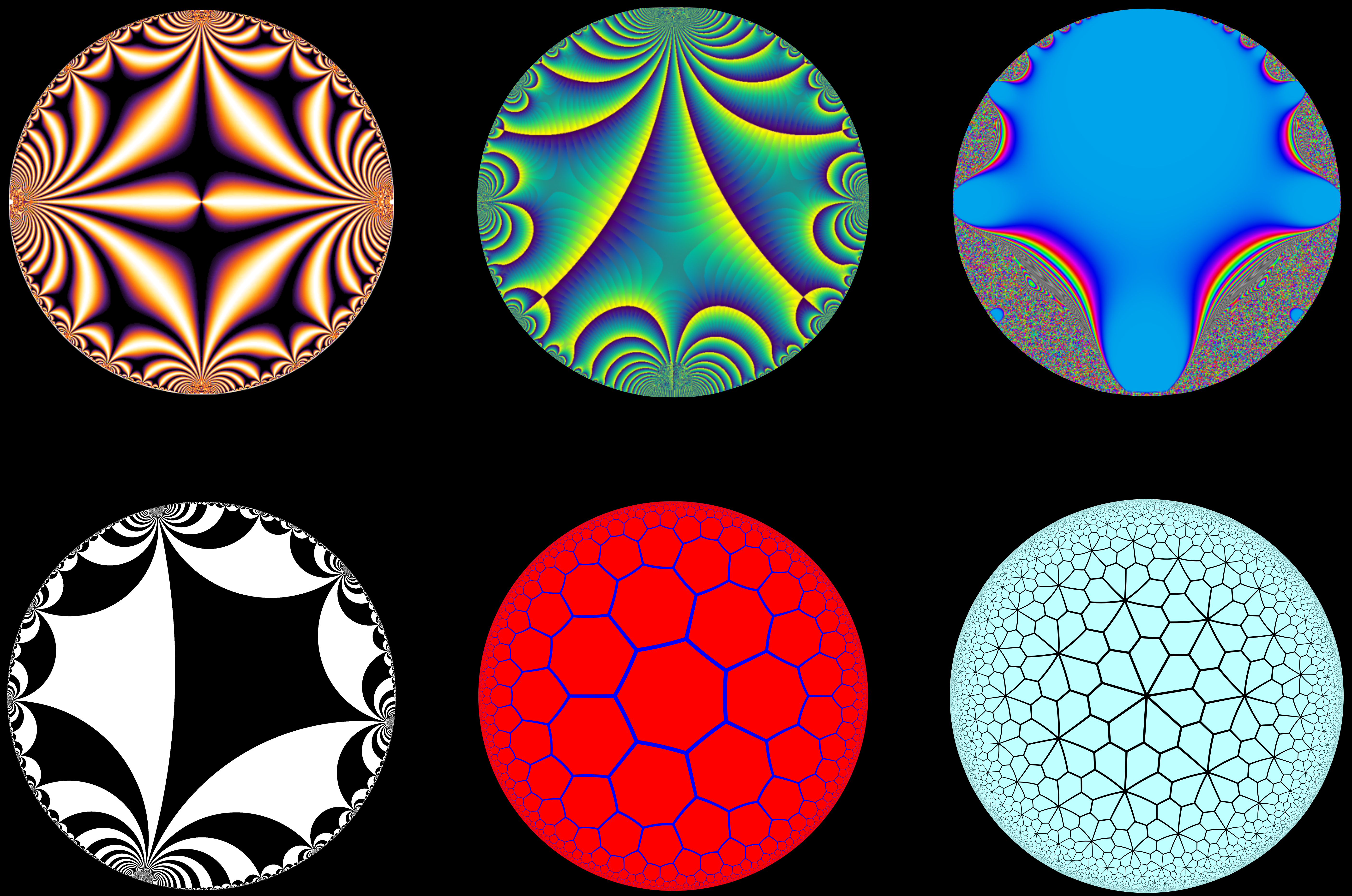
Question: How is $\Psi(\lambda)$ related to a Modular form?

Section 3

$\Gamma \subset$ **SL(2,$\mathbb{Z}$)** Modular groups and Modular forms

Modular forms & Hyperbolic tessellation

Our goal: uniformization, that is, to find the proper domain for the multi-valued period function $\Psi(\_)$ such that on that 'domain' $\Psi(\_)$ is single-valued!



How can we relate examples 1. and 2. to modular forms?

- Example 1: 1-dimensional

$$E_\lambda : Y^2 = X(X-1)(X-\lambda)\,,\, \lambda \in \mathbb{CP}^1 \backslash \{0,1,\infty\}$$

$$\Psi(\lambda) \equiv \int_0^\lambda \frac{\mathrm{d}X}{Y} = 2\mathrm{K}(\lambda)$$

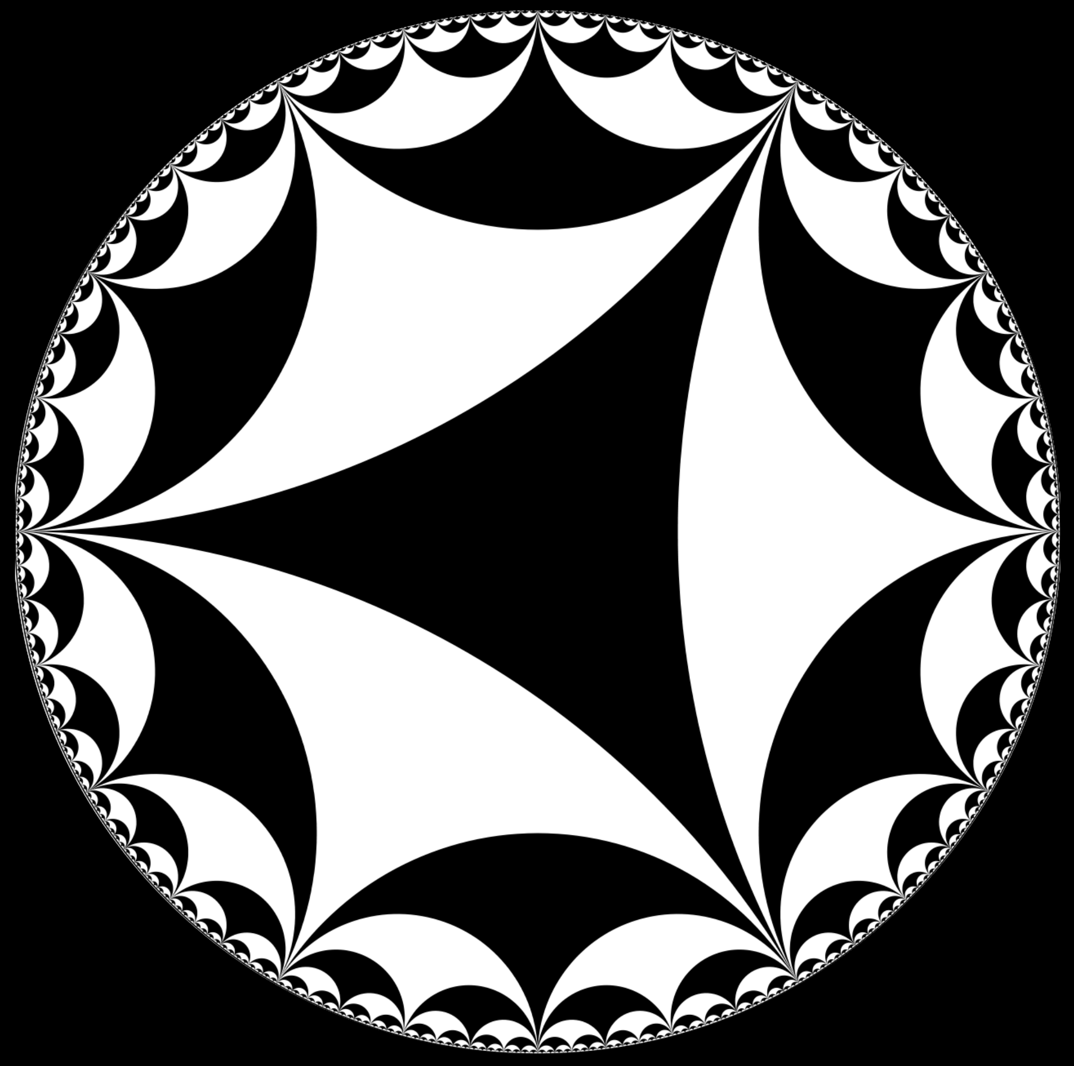
- Example 2: 2-dimensional

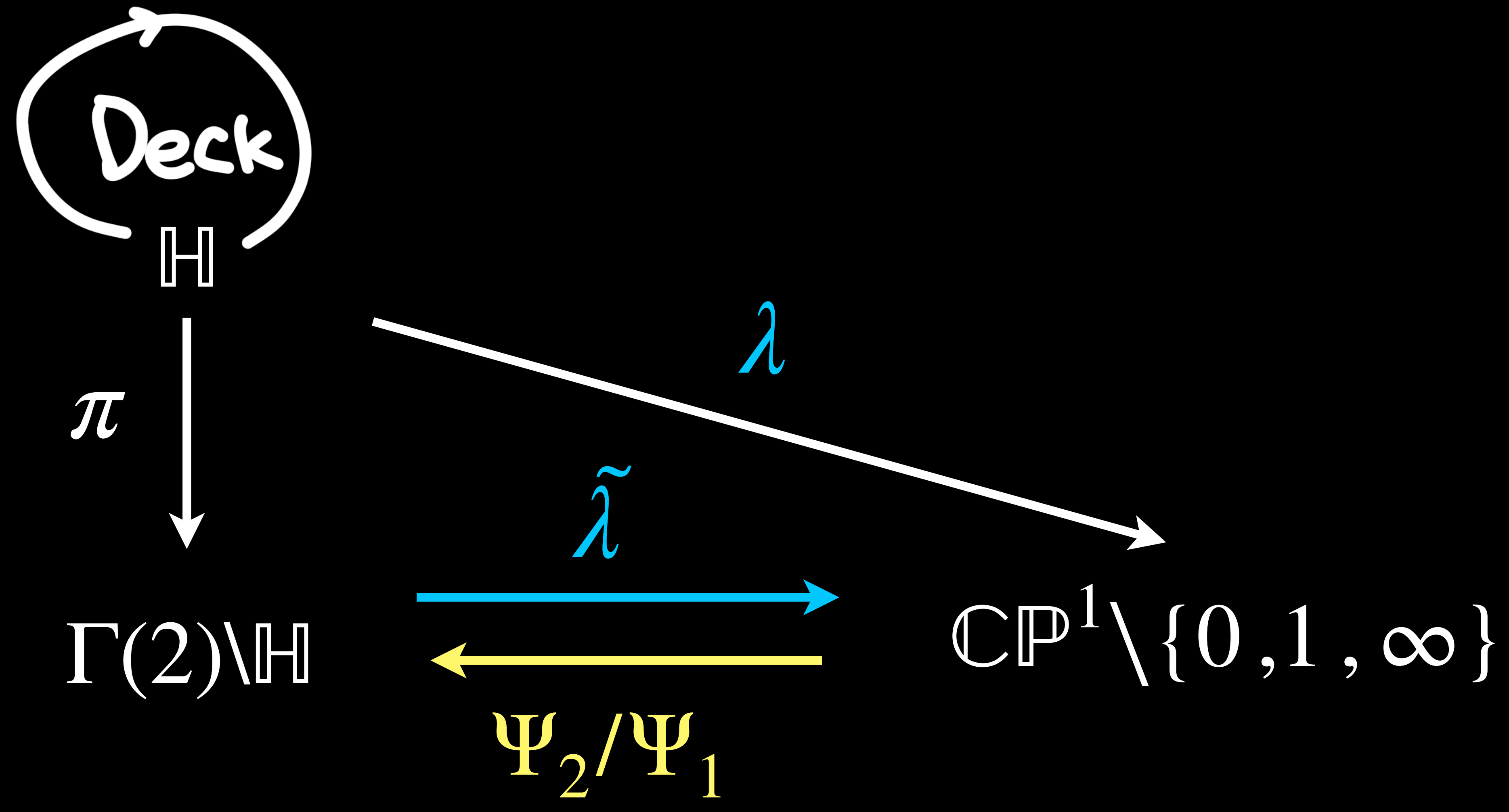
$$Y^2 = \left(X^2 - 2\frac{st}{t-4}X + \frac{(s-4)st}{t-4}\right)\left(X^2 - 2(s-2)X + s(s-4)\right)$$

$[s:t:m^2] \in$ Base space $= \mathbb{CP}^2 \backslash \{$kinematic branch points$\}$

$$\Psi_{\text{bhabha}}\left(\frac{s}{m^2}, \frac{t}{m^2}\right) \equiv 2\int_{e_2}^{e_3} \frac{\mathrm{d}X}{Y} = \frac{4\mathrm{K}\left(\frac{4m^2}{2m^2 + \sqrt{\frac{-m^2 s(s+t-4m^2)}{-t}}}\right)}{\sqrt{(e_1-e_3)(e_2-e_4)}}$$

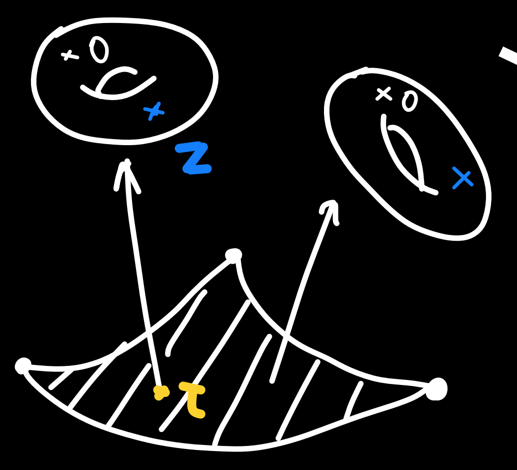
Uniformization & hyperbolic tilling of the Poincaré disk





Uniformization & universal family of curves

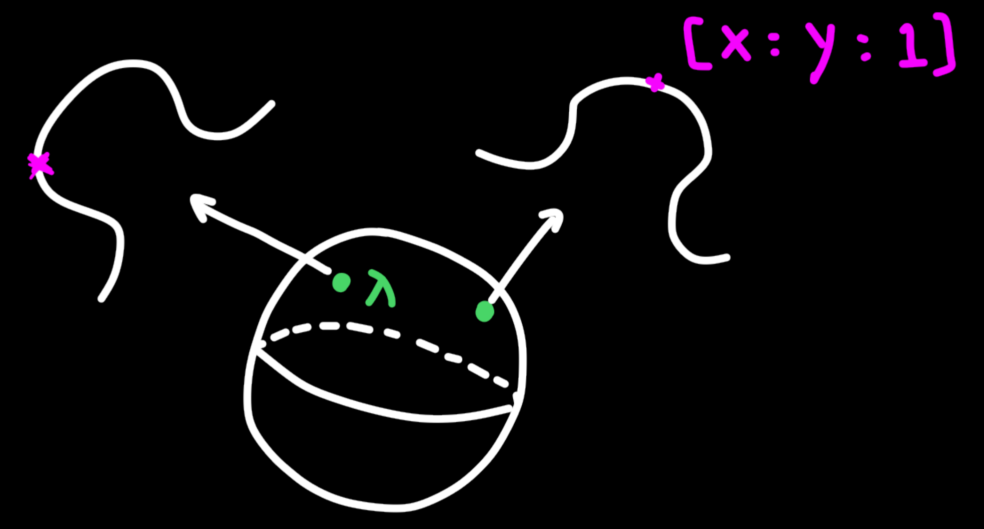
Equivalence between the two universal families $\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}}$ and $E_\tau[2]$



$$\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}} \underset{\text{Abel map}}{\overset{\tilde{f}_{[2]}}{\rightleftarrows}} E_\tau[2] \rightleftarrows E_\lambda$$

$$\begin{array}{ccccc} E_\tau[2] & & & & E_\lambda \\ \downarrow & & & & \downarrow \\ \Gamma(2)\backslash\mathbb{H} & & \underset{\Psi_2/\Psi_1}{\overset{\tilde{\lambda}}{\rightleftarrows}} & & \mathbb{CP}^1\backslash\{0,1,\infty\} \end{array}$$

$$\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}} \to \Gamma(2)\backslash\mathbb{H}$$



$$\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma(2))\backslash\mathbb{C} \times \mathbb{H}$$

$$E_\tau[2] : Y^2 = X(X-1)(X-\lambda(\tau))\,, \quad \tau \in \Gamma(2)\backslash\mathbb{H}$$

$$E_\lambda : Y^2 = X(X-1)(X-\lambda)\,, \quad \lambda \in \mathbb{C}\backslash\{0,1\}$$

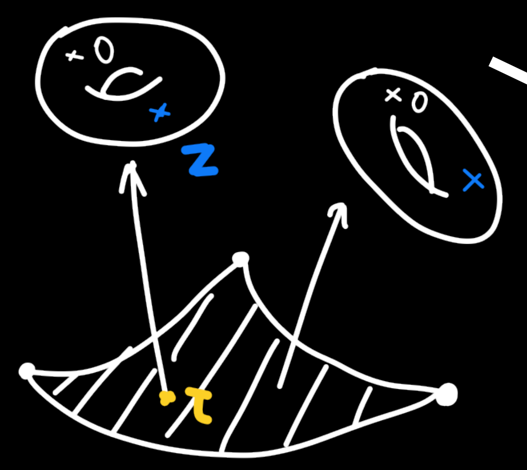
} conformal equivalence

Uniformization & universal family of curves

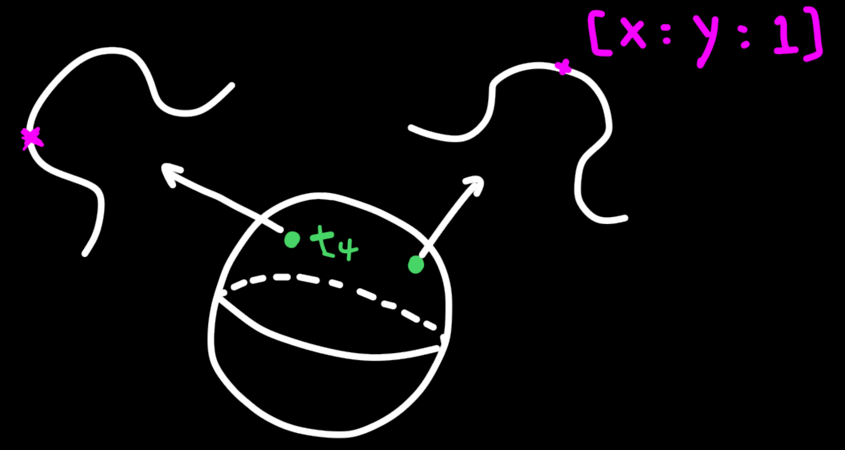
The equivalence between the two universal families $\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}}$ and E_{t4}

$$\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \underset{\text{Abel map}}{\overset{\tilde{f}_{[4]}}{\rightleftarrows}} E_\tau[4] \rightleftarrows E_{t_4}$$

$$\Gamma_1(4)\backslash\mathbb{H} \underset{\Psi_2/\Psi_1}{\overset{\tilde{t}_4}{\rightleftarrows}} \mathbb{CP}^1\backslash\{0,1,\infty\}$$



①



②

Why are ① and ② isomorphic?

Answer: because the monodromy of ② is $\Gamma_1(4)$!

$\simeq \mathbb{Z} * \mathbb{Z} = \pi_1(\mathbb{CP}^1\backslash\{0,1,\infty\},\cdot)$

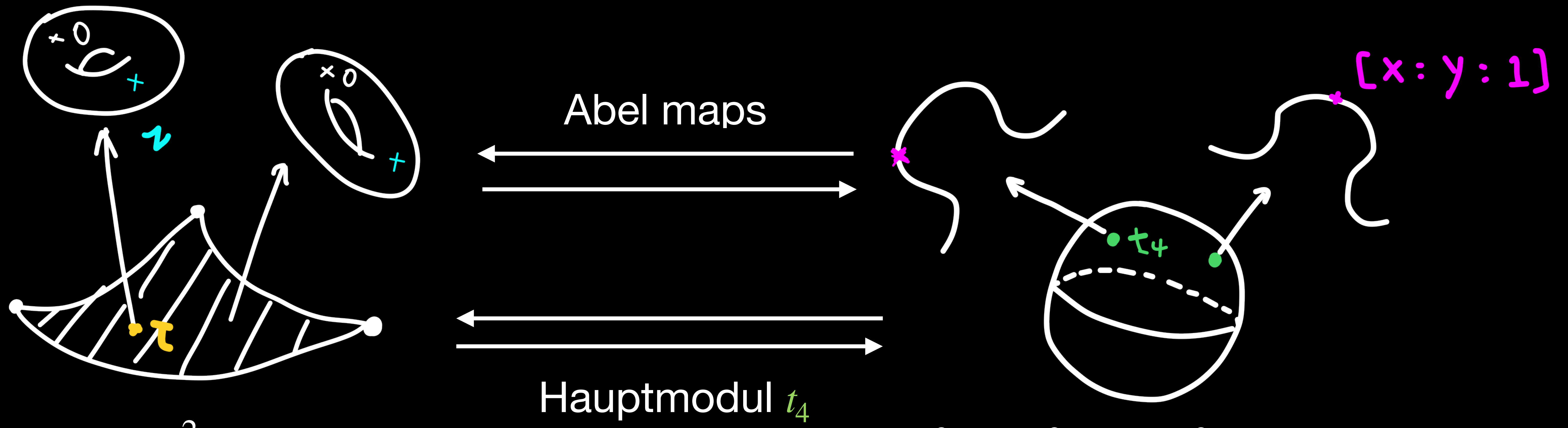
① $\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C}\times\mathbb{H}$

$E_\tau[4] : Y^2 = (X^2-1)(X^2-t_4(\tau))\,, \quad \tau \in \Gamma_1(4)\backslash\mathbb{H}$

② $E_{t_4} : Y^2 = (X^2-1)(X^2-t_4)\,, \quad t_4 \in \mathbb{C}\backslash\{0,1\}$

Algebraic realizations of Kronecker's differential forms

Given some congruence subgroup e.g. $\Gamma_1(4)$, on which family of elliptic curves such that the relevant torsion data of $\Gamma_1(4)$ is realized?



$\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C}\times\mathbb{H}$

$E_{t_4} : y^2 = (x^2-1)(x^2-t_4)\,, \quad t_4 \in \mathbb{C}\backslash\{0,1\}$

$$(z,\tau) \simeq ([x:y:1], t_4)$$

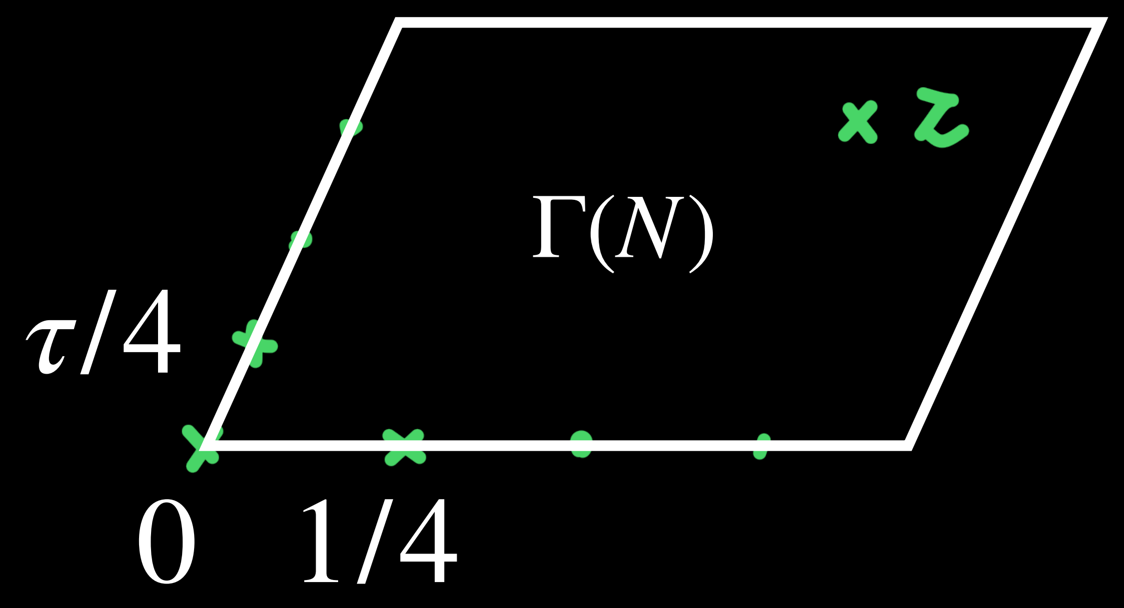
isomorphism

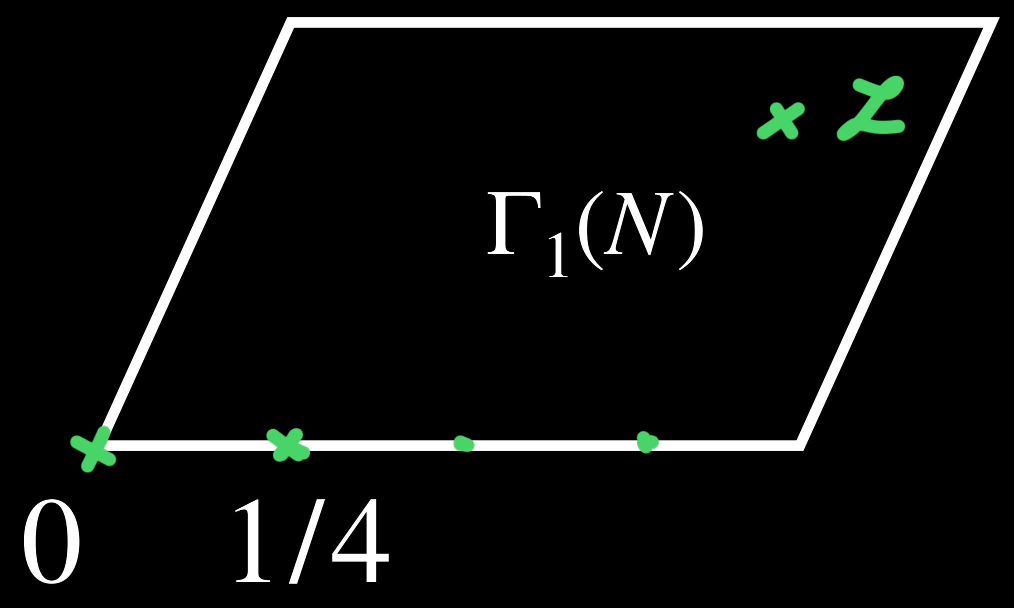
$$\begin{cases} t_4(\tau) = \left(\dfrac{\theta_3^2(q)-\theta_4^2(q)}{\theta_3^2(q)+\theta_4^2(q)}\right)^2 \\[2ex] x(z) = \dfrac{2\theta_4^2(0,q)\theta_1^2(\pi z,q)}{2\theta_3^2(0,q^2)\theta_1^2(\pi z,q)-\theta_2^2(0,q)\theta_4^2(\pi z,q)} \end{cases}$$

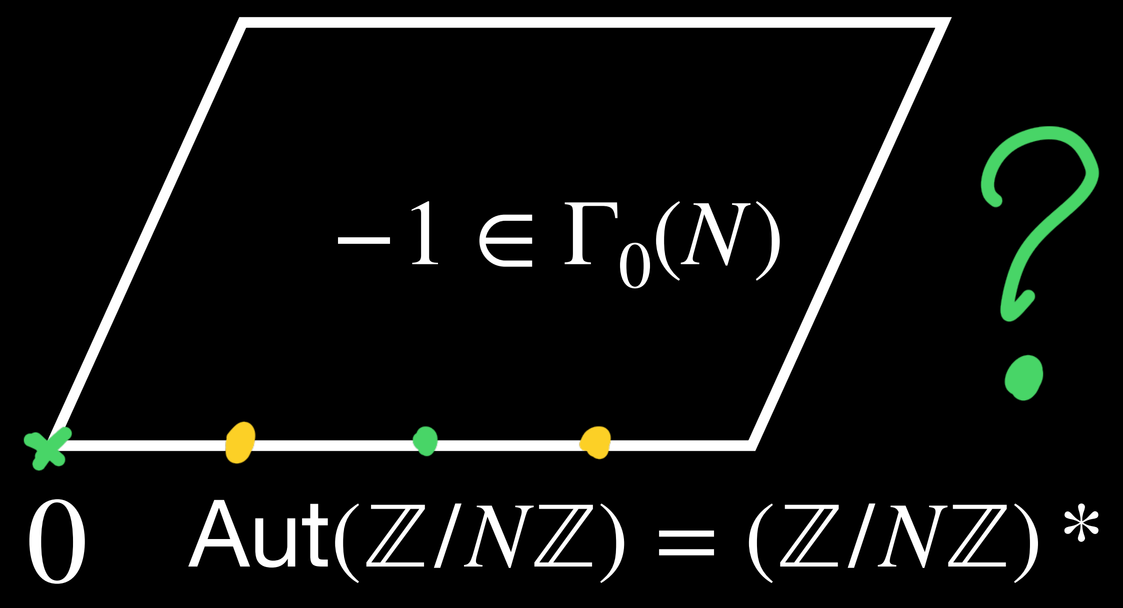
$$i\pi\,\mathrm{d}\tau \mapsto \frac{1}{8}\frac{1}{t_4(1-t_4)}\frac{\pi^2}{\mathrm{K}^2(t_4)}\mathrm{d}t_4\,, \quad 2\pi\,\mathrm{d}z \mapsto \frac{\pi}{2\mathrm{K}(t_4)}\frac{\mathrm{d}x}{y} + \mathscr{F}(x,t_4)\frac{\pi\,\mathrm{d}t_4}{2\mathrm{K}(t_4)}$$

Uniformization & universal curves on $\mathscr{M}_{1,2}[N]$

$[E_\tau, (\tau/N + \Lambda_\tau, 1/N + \Lambda_\tau)]$ | $[E_\tau, 1/N + \Lambda_\tau]$ | $[E_\tau, \langle 1/N + \Lambda_\tau \rangle]$



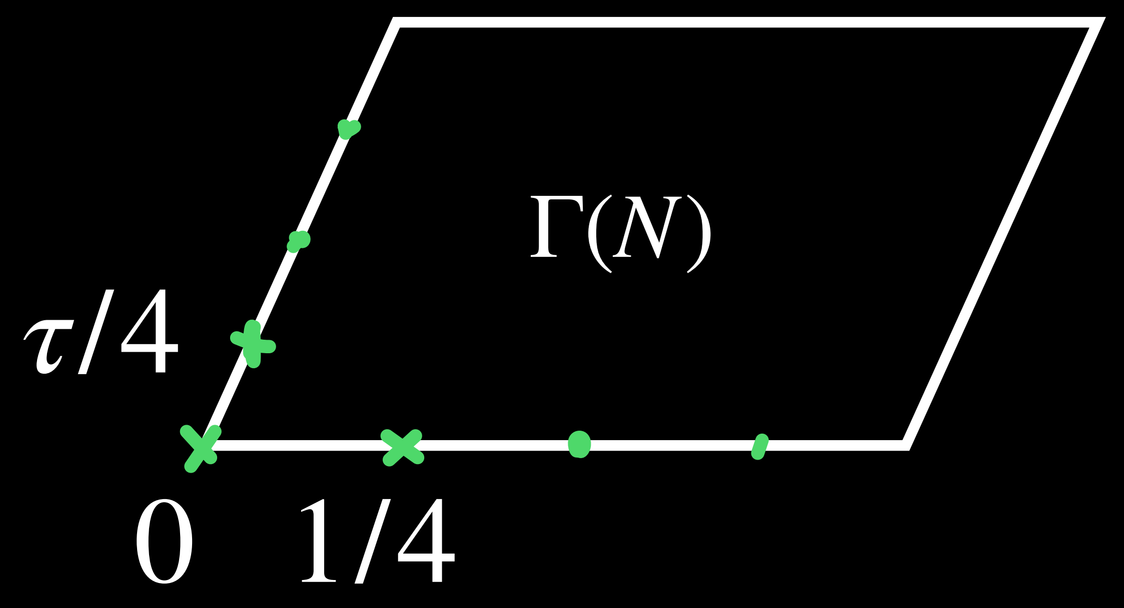


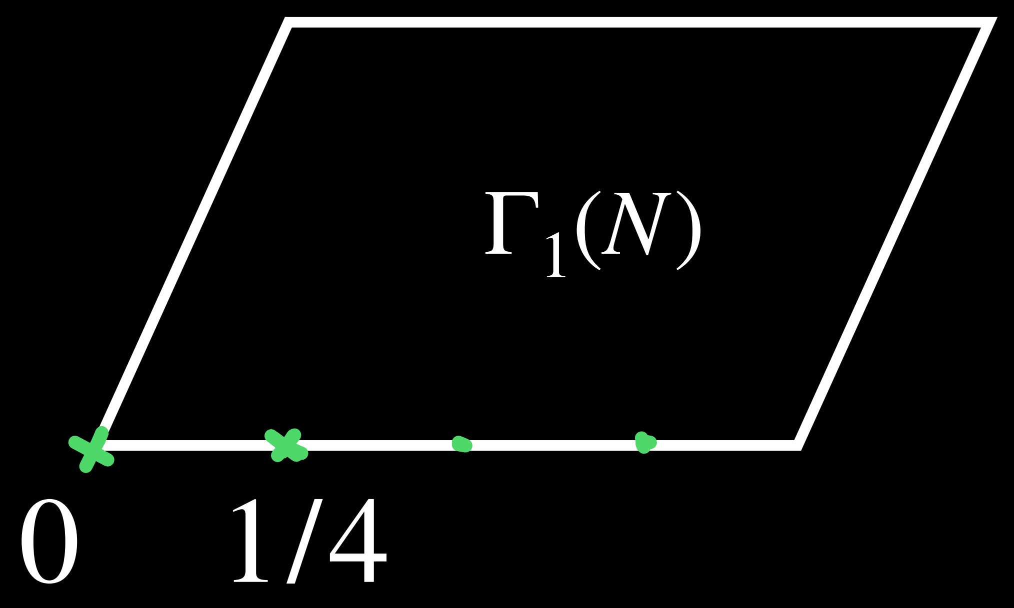


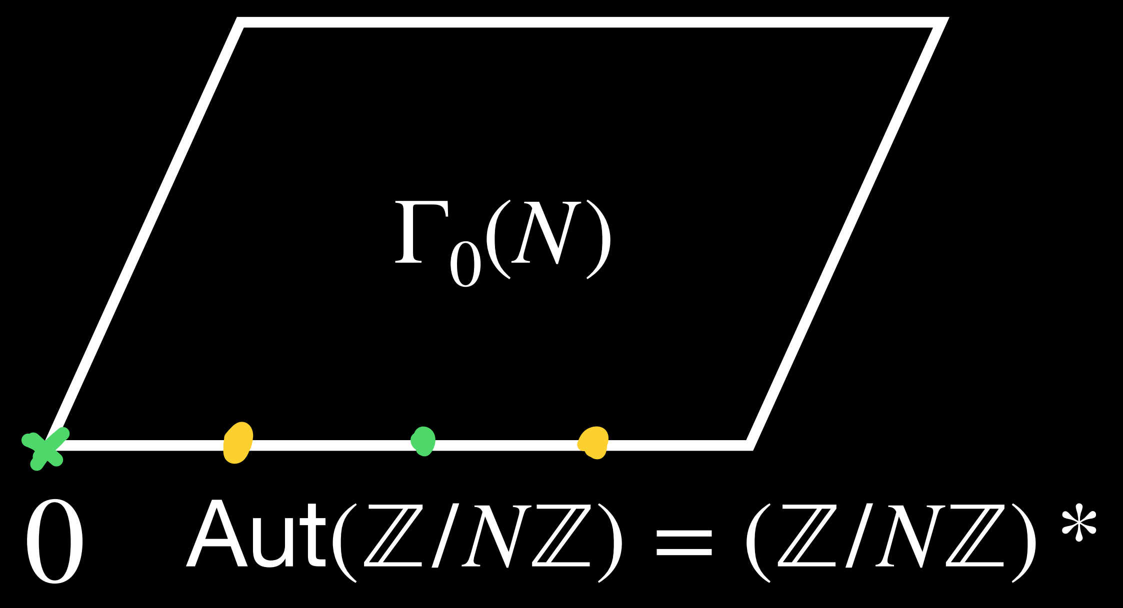
Section 4

Uniformization & universal curves on $\mathscr{M}_{1,1}[N]$

$[E_\tau, (\tau/N + \Lambda_\tau, 1/N + \Lambda_\tau)]$ | $[E_\tau, 1/N + \Lambda_\tau]$ | $[E_\tau, \langle 1/N + \Lambda_\tau \rangle]$







The universal family of complex tori

$$\mathscr{E}_{\Gamma\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma)\backslash\mathbb{C} \times \mathbb{H}$$

- $\mathbb{Z}^2 \rtimes \Gamma$ is isomorphic to the following action

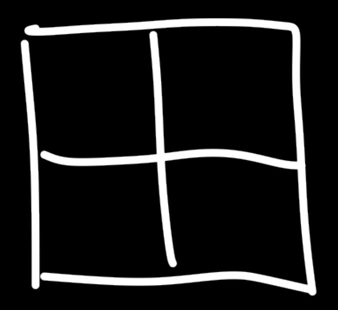
$$\begin{pmatrix} z \\ \tau \\ 1 \end{pmatrix} \mapsto \frac{1}{c\tau + d} \begin{pmatrix} 1 & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} = \begin{pmatrix} \frac{z + m\tau + n}{c\tau + d} \\ \gamma \cdot \tau \\ 1 \end{pmatrix}, \quad (m, n) \in \mathbb{Z}^2, \quad \gamma \in \Gamma$$

- Is each fiber a complex torus?

$-1 \notin \Gamma$ and that the action of Γ is free

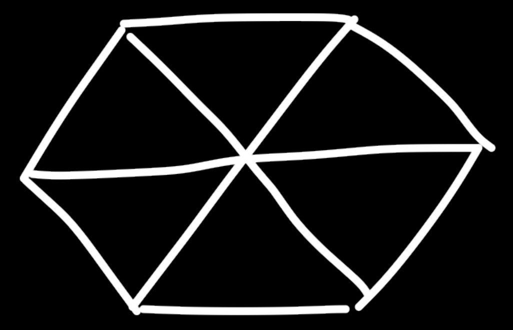
potential candidates: $\Gamma_1(N)$, with $N > 3$

$e^{i\frac{\pi}{2}}$



$\mathbb{Z}_4$

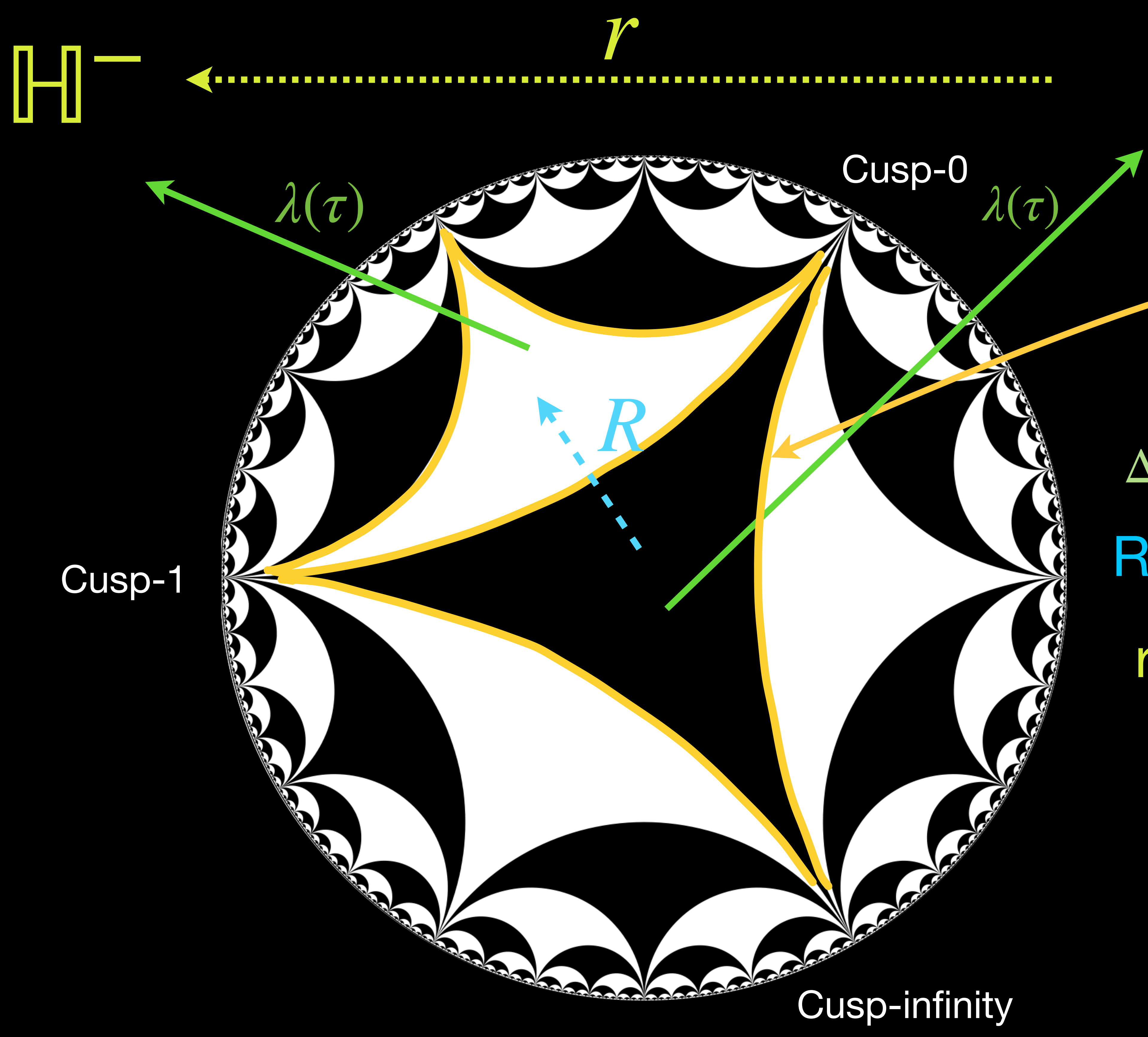
$e^{i\frac{\pi}{3}}$



$\mathbb{Z}_6$

Poincaré polygon theorem

Geometric construction by the Fuchsian Triangle Group $\Gamma_{\infty\infty\infty}$

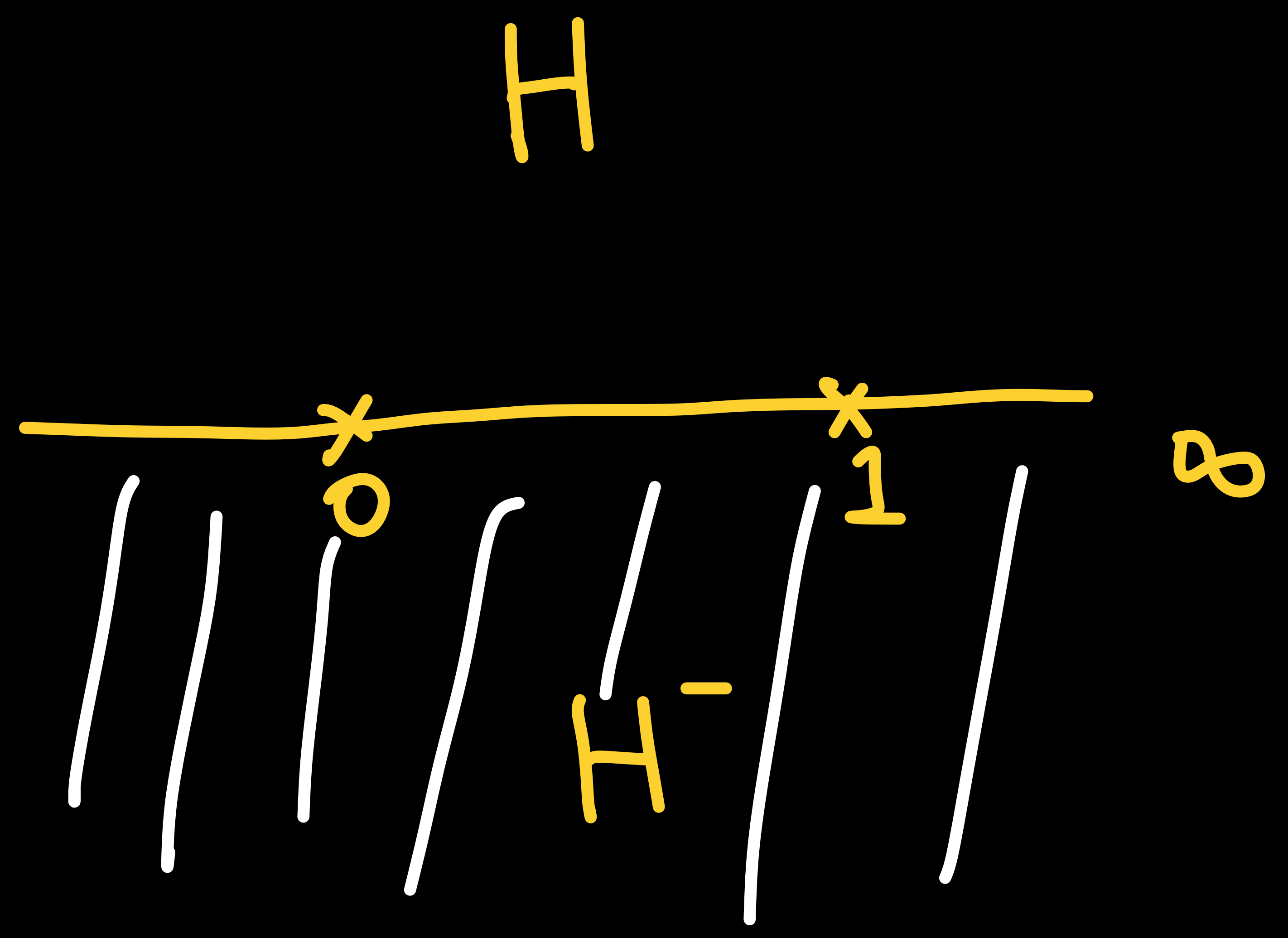


Δ = Fundamental triangle

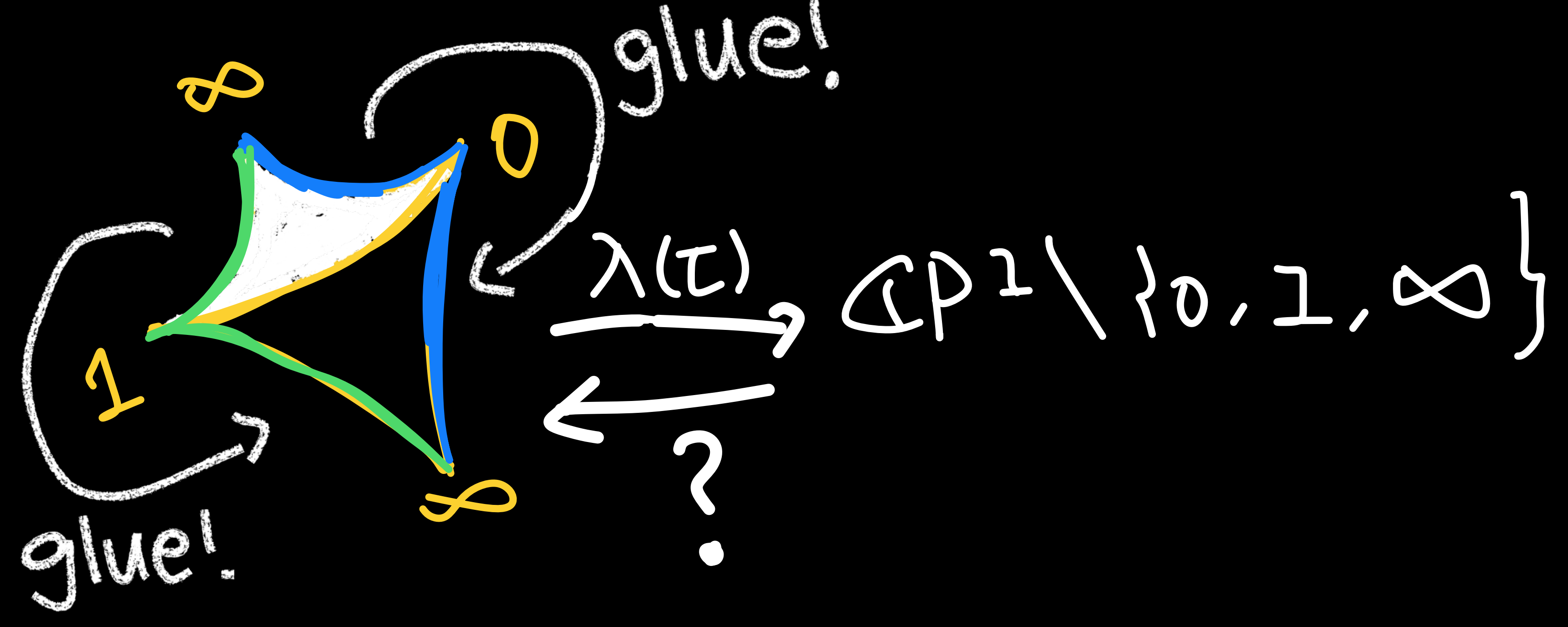
$\Delta \overset{\lambda(\tau)}{\longleftrightarrow} \mathbb{H}$: Riemann Mapping Theorem

R: Schwarz Reflection

r: Schwarz Reflection Principle

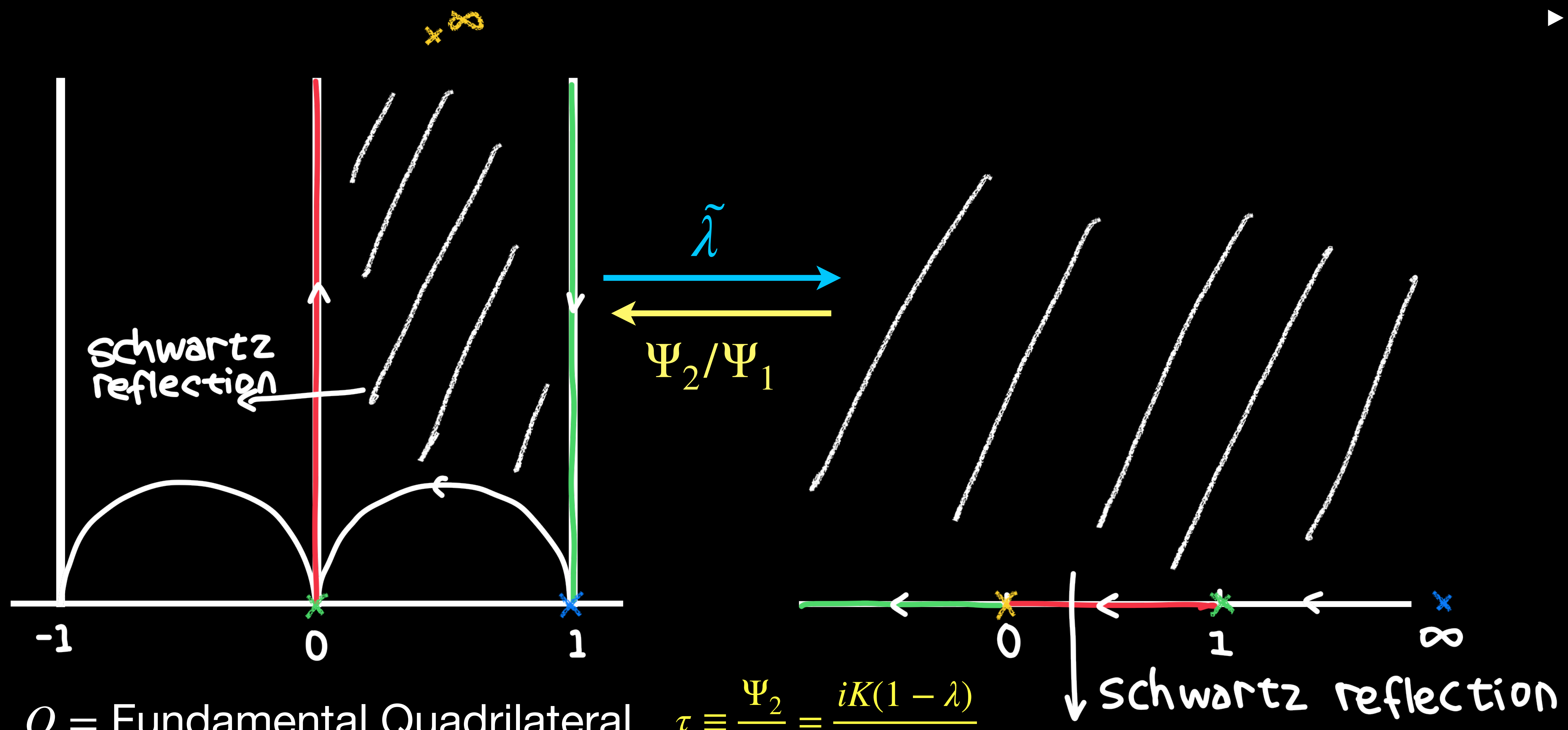


$$\Delta \cup \Delta^- \cup \partial(\Delta \cup \Delta^-) \xrightarrow{\lambda(\tau)} \mathbb{H} \cup \mathbb{H}^- \cup \overline{\mathbb{R}} \backslash \{0, 1, \infty\}$$



Tiling by $\Gamma_{\infty\infty\infty} \simeq \Gamma(2) \simeq \mathbb{Z} * \mathbb{Z}$

Elliptic curves with extra torsion data



Q = Fundamental Quadrilateral $\quad \tau \equiv \frac{\Psi_2}{\Psi_1} = \frac{iK(1-\lambda)}{K(\lambda)}$

- Period mappings for a family of elliptic curves

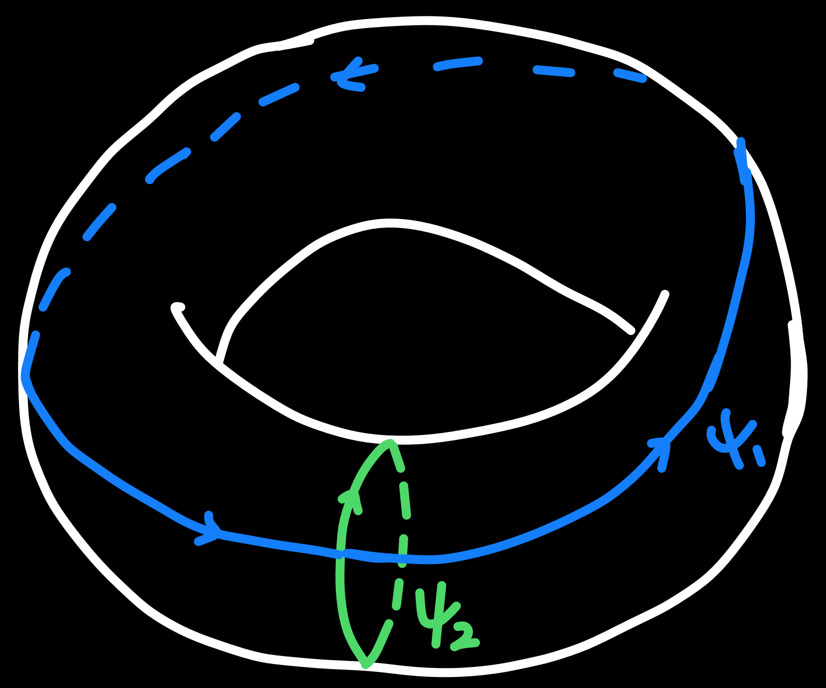
$$E_\lambda : Y^2 = X(X-1)(X-\lambda)\,,\lambda \in \mathbb{CP}^1 \backslash \{0,1,\infty\}$$

$$j(\lambda) = 256\frac{(1-\lambda(1-\lambda))^3}{\lambda^2(1-\lambda)^2}$$

which is ramified at $\lambda = 0,1$ and ∞, each with ramification index 2 so that $deg(j) = 6 = [\mathbb{P}\mathrm{SL}(2,\mathbb{Z}) : \Gamma(2)]$

$$\int \frac{dX}{Y} : H_1(E_\lambda) \to \mathbb{C}$$

$$\Psi_1(\lambda) \equiv \int_0^\lambda \frac{\mathrm{d}X}{Y} = 2K(\lambda)\,, \quad \Psi_2(\lambda) \equiv \int_1^\lambda \frac{\mathrm{d}X}{Y} = 2iK(1-\lambda)$$



Monodromy=Analytic continuation, the next steps is to show the effect for the analytic continuation of $\tau(\lambda)$ is equivalent to

$$\tau \to \tau_{\circlearrowleft} \equiv \rho \cdot \tau\,, \rho \in \Gamma(2)\,, \quad \text{so that } \mathbb{H} = \cup\,\{\rho \cdot Q \,|\, \rho \in \Gamma(2)\}$$

Torsion data from monodromy group

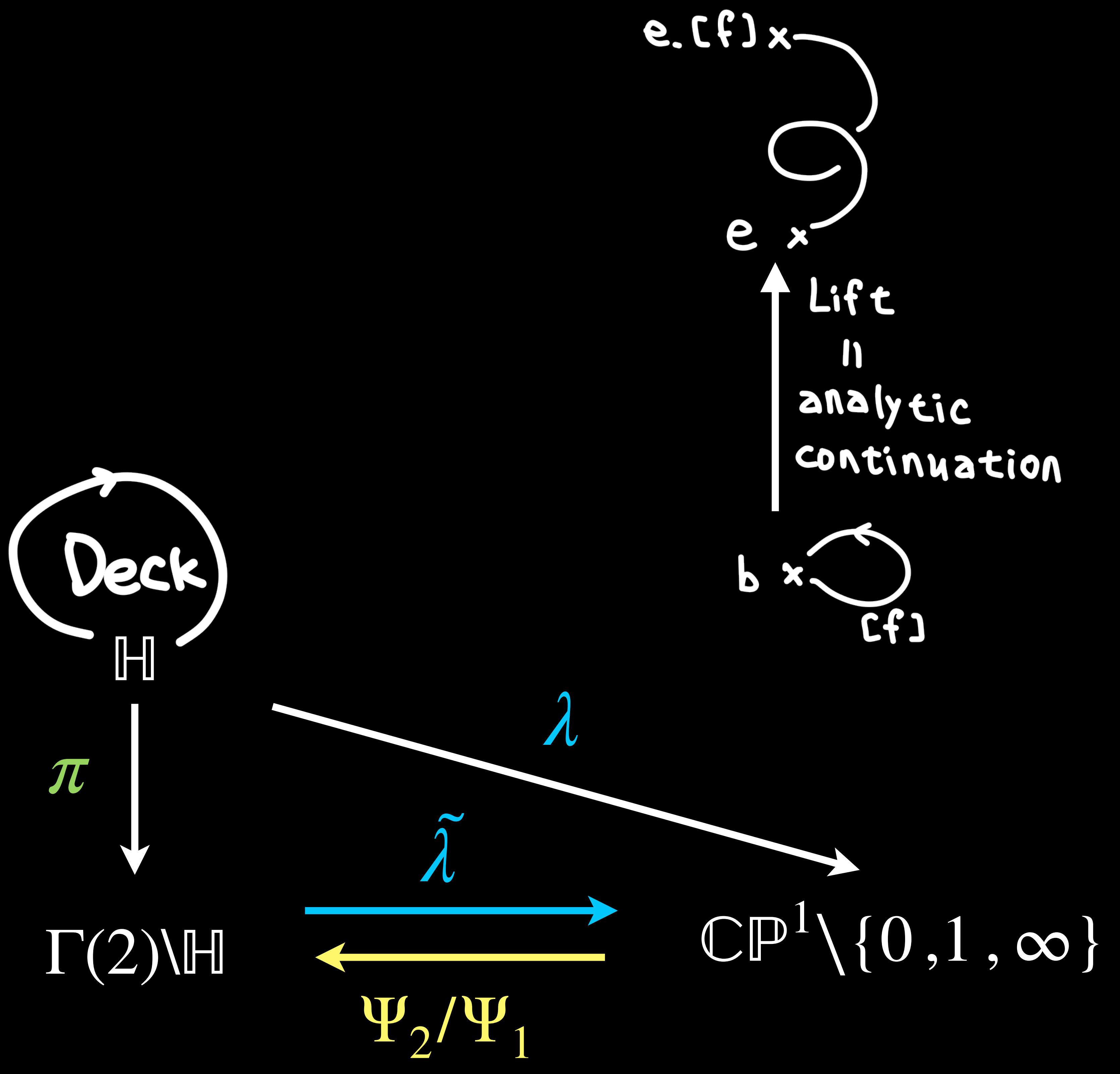
- Picard-Fuchs differential equation

$$\left[4\lambda(1-\lambda)\frac{d^2}{d\lambda^2}+4(1-2\lambda)\frac{d}{d\lambda}-1\right]\Psi_i=0\,,\quad i=1\,,2$$

- Monodromy representation $\rho_{[\gamma_1][\gamma_2]}=\rho_{[\gamma_1]}\cdot\rho_{[\gamma_2]}$

$$\rho:\pi_1(X,\cdot\,)\to \mathrm{GL}_2(\mathbb{C})$$
$$[\gamma]\mapsto\rho_{[\gamma]}\,.$$

$$\tau\to\tau_{\circlearrowleft}\equiv\rho_{[\circlearrowleft]}\cdot\tau$$



- Images of the generators in $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$

$$\rho_{[\circlearrowleft_0]}=\begin{pmatrix}1&2\\0&1\end{pmatrix},\quad\rho_{[\circlearrowleft_1]}=\begin{pmatrix}1&0\\-2&1\end{pmatrix}$$

- Covering automorphism group structure theorem

$$\rho(\pi_1(X,\cdot\,))=\langle\rho_{[\circlearrowleft_0]},\rho_{[\circlearrowleft_1]}\rangle=\mathrm{Deck}_\pi(\mathbb{H})\simeq\Gamma(2)\simeq\mathbb{Z}*\mathbb{Z}\simeq\mathrm{Deck}_\lambda(\mathbb{H})$$

π

- Covering space quotient theorem

Pullback of the period function

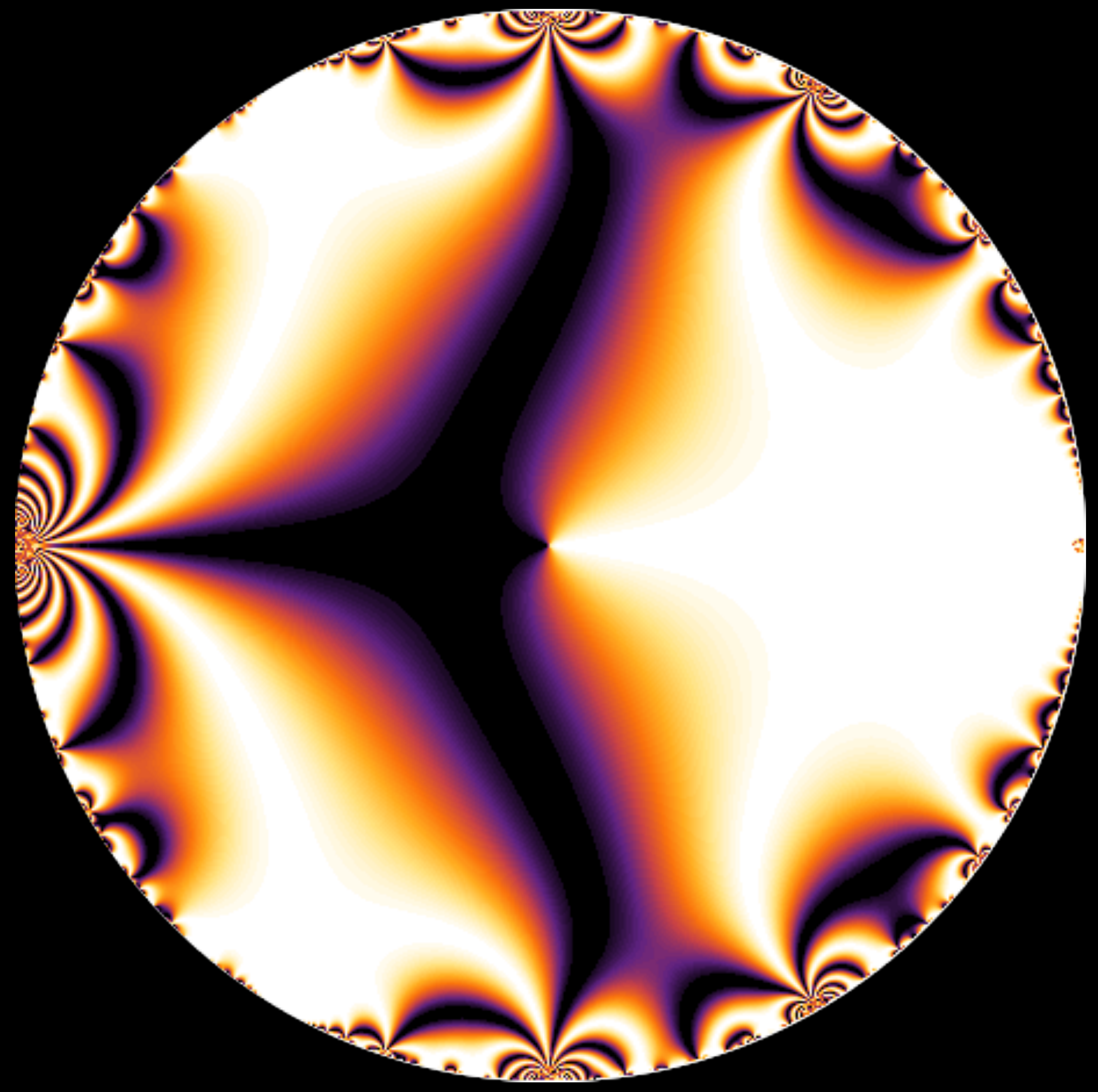
Deck $= \Gamma(2)$

$\mathbb{H} \simeq \mathbb{D}$

$\pi \downarrow$; $\lambda : \mathbb{H} \to \mathbb{CP}^1 \backslash \{0, 1, \infty\}$

$Y(2) = \Gamma(2) \backslash \mathbb{H} \underset{\tilde{\lambda}^{-1}}{\overset{\tilde{\lambda}}{\rightleftarrows}} \mathbb{CP}^1 \backslash \{0, 1, \infty\}$

$\tilde{\lambda}^{-1} = \Psi_2 / \Psi_1$



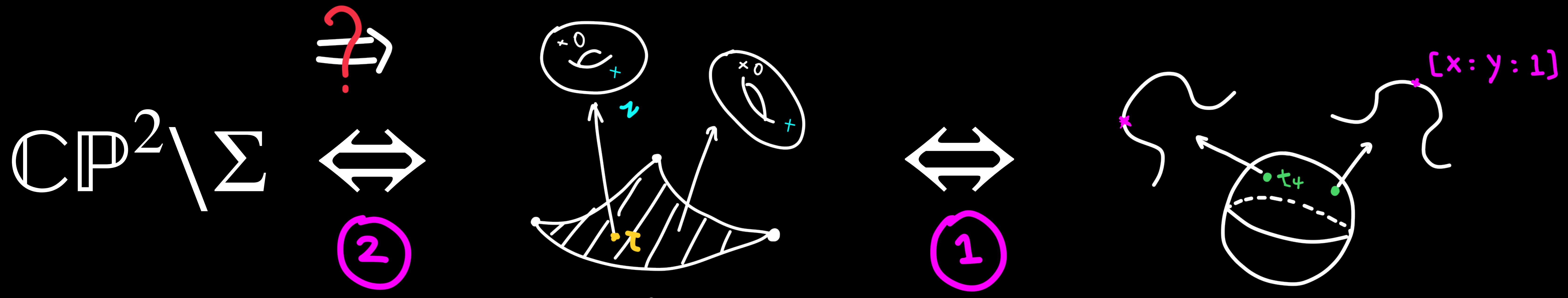
$\lambda(\tau)$: Modular function for $\Gamma(2)$

$$\Psi_1(\lambda) \equiv \int_0^\lambda \frac{\mathrm{d}X}{Y} = 2K(\lambda)$$

Pull back $(\lambda^* \Psi_1)(\tau) \equiv \Psi_1(\lambda(\tau)) = \pi \theta_3^2(0, \tau)$

modular form for $\Gamma(2)$!

$$\mathbb{CP}^2\backslash\Sigma \Leftrightarrow (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C}\times\mathbb{H} \Leftrightarrow y^2 = (x^2-1)(x^2-t_4), \quad t_4 \in \mathbb{C}\backslash\{0,1\}$$



$\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C}\times\mathbb{H}$

$E_{t_4} : y^2 = (x^2-1)(x^2-t_4), \quad t_4 \in \mathbb{C}\backslash\{0,1\}$

$$(z, \tau) \simeq ([x : y : 1], t_4)$$

kinematic base space:

$$[s : t : m^2] \in \mathbb{CP}^2\backslash\Sigma$$

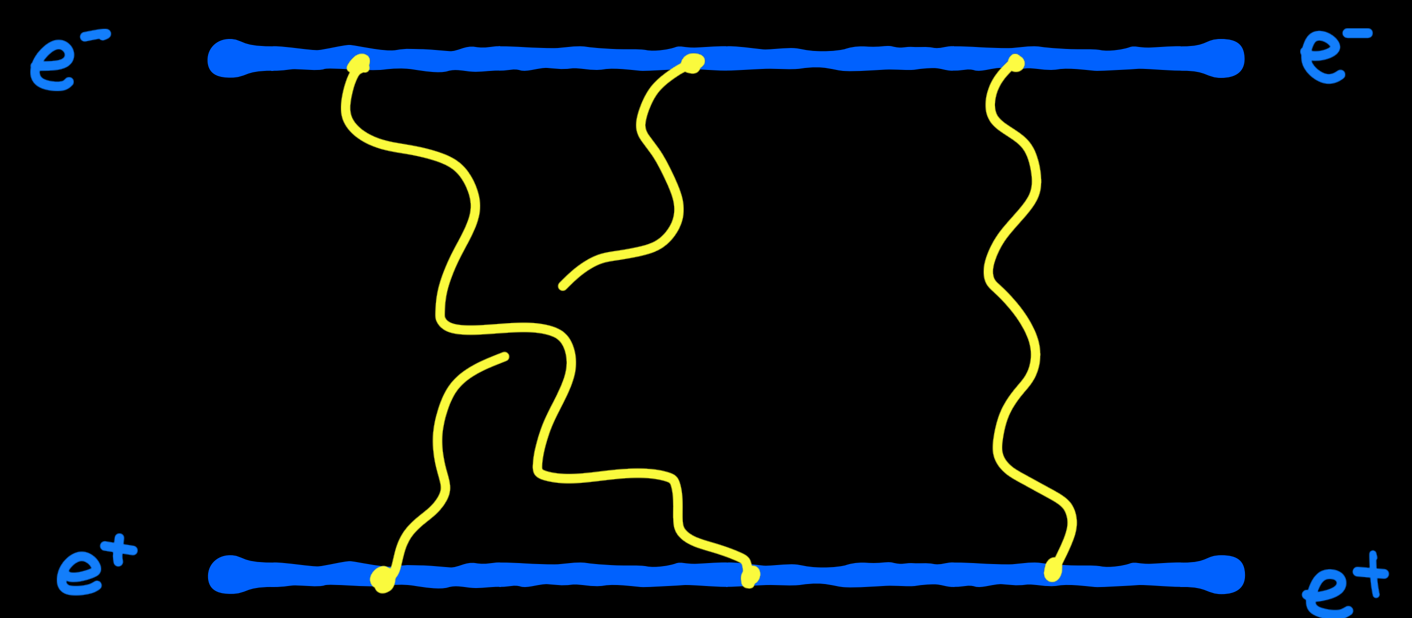
Σ is the union of linear varieties, given by the zero locus of linear equations, e.g.

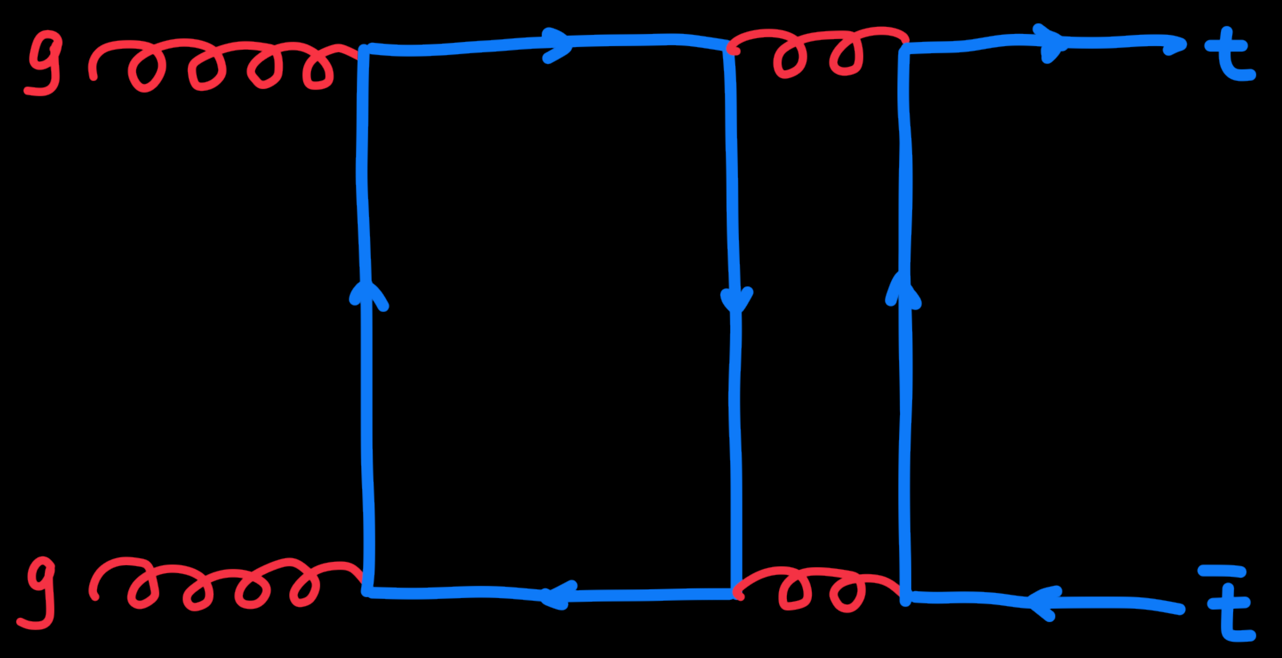
$$\Sigma = \langle s+t=0\rangle \cup \langle s+t-4=0\rangle \cup \dots$$

① done!

② will show you till the end

Unified description of Bhabha and top quark production (for several sectors) through canonical coordinates on Moduli space $\mathscr{M}_{1,2}[4]$





$$Y^2 = \left(X^2 - 2\frac{st}{t-4}X + \frac{(s-4)st}{t-4}\right)\left(X^2 - 2(s-2)X + s(s-4)\right) \simeq E_{\mathscr{M}_{1,2}[4]} \simeq Y^2 = \left(X^2 - 2(t-2)X + t(t-4)\right)\left(X^2 + 2X + 1 - 4t - 4\frac{(1-t)^2}{s}\right)$$

$$\Psi_{\text{bhabha}}(s,t) = \frac{4\mathrm{K}\left(\frac{4}{2+\sqrt{\frac{-s(s+t-4)}{-t}}}\right)}{\sqrt{\ldots.}} \overset{?}{\Longleftrightarrow} \Psi_{\text{tquark}}(s,t) = \frac{4\mathrm{K}\left(\frac{16\sqrt{\frac{1+t(s+t-2)}{s}}}{3 - t(t-6) + \frac{4(1-t)^2}{s} + 8\sqrt{\frac{1+t(s+t-2)}{s}}}\right)}{\sqrt{\ldots.}}$$

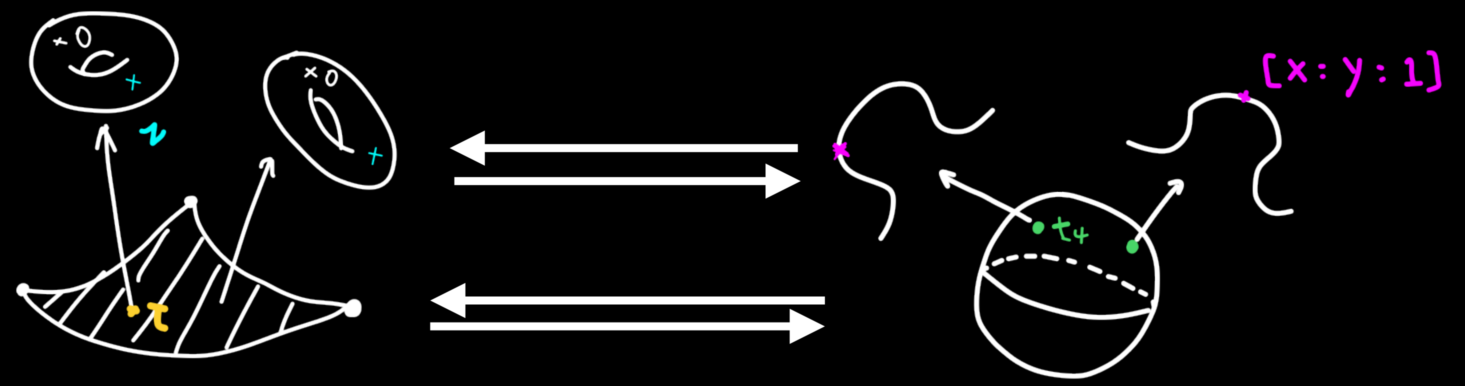
$$\Psi_{\text{bhabha}}(z,\tau) = \Psi_{\text{tquark}}(z,\tau) = \frac{\pi\theta_2^2(0,q)}{2}\frac{\theta_3(\pi z,q)\theta_4(\pi z,q)}{\theta_1(\pi z,q)\theta_2(\pi z,q)} \Rightarrow$$

$$\Psi_{\text{bhabha}}\left(\frac{z+m\tau+n}{c\tau+d}, \gamma\cdot\tau\right) = \frac{1}{c\tau+d}\Psi_{\text{bhabha}}(z,\tau), \forall\gamma\in\Gamma_1(4)$$

The two processes are partially described by the same set of function space!!

Algebraic realizations of the moduli space $\mathscr{M}_{1,2}[4]$

$$\mathbb{CP}^2\backslash\Sigma \;\Leftrightarrow$$



$\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C}\times\mathbb{H}$

$E_{t_4} : y^2 = (x^2-1)(x^2-t_4)\,, \quad t_4 \in \mathbb{C}\backslash\{0,1\}$

$$(z,\tau) \simeq ([x:y:1], t_4)$$

- $\mathbb{CP}^2\backslash\Sigma$ $\Leftrightarrow$ universal family of complex tori

$$\text{s} = -\frac{4(-1+R)\times(-2+\lambda)}{-2+\lambda+R\times\lambda}\,, \quad \text{t} = \frac{4(-1+R)\times R\times\lambda^2}{(-2+R\times\lambda)(-2+\lambda+R\times\lambda)}$$

$$\Psi_{\text{tquark}}(z,\tau) = \Psi_{\text{bhabha}}(z,\tau) = \frac{\pi\theta_2^2(0,q)}{2}\frac{\theta_3(\pi z,q)\theta_4(\pi z,q)}{\theta_1(\pi z,q)\theta_2(\pi z,q)}$$

$$R = \frac{\theta_2^2(0,q)}{\theta_3^2(0,q)}\frac{\theta_1^2(\pi z,q)}{\theta_4^2(\pi z,q)}\,, \quad \lambda = \frac{\theta_2^4(0,q)}{\theta_3^4(0,q)}$$

- $\mathbb{CP}^2\backslash\Sigma \Leftrightarrow$ universal family of elliptic curves

$$s = 2\frac{(1+t_4)(1-x)}{t_4-x}\,, \quad t = 4\frac{t_4(x^2-1)}{(t_4-x)(t_4+x)}$$

$$\Psi_{\text{tquark}}(z,\tau) = \Psi_{\text{bhabha}}(x,t_4) = 2\frac{\sqrt{t_4-x^2}}{\sqrt{1-x^2}}K(t_4)\,, \quad 4K(t_4) = 2\pi\theta_3^2(q^2) \in \mathscr{M}_1(\Gamma_1(4))$$

A translation table for $\Gamma_1(4)$ […, 2023 ⇔ J.Broedel, C.Duhr, F.Dulat, L.Tancredi 2017,…,D.Zagier 1991]

- Meromorphic differentials on the base space modular curve $\mathscr{M}_{1,1}[4]$
  - Weight-2 $\quad \Theta_{\mathbb{D}^4}(q^2)\frac{\mathrm{d}q}{q} = \left(\theta_3^4(q^2)+\theta_2^4(q^2)\right)\frac{\mathrm{d}q}{q} \mapsto \left(\frac{1}{2t_4}+\frac{1}{1-t_4}\right)\mathrm{d}t_4\,, \quad \Theta_{\mathbb{Z}^4}(e^{\pi i}q^2)\frac{\mathrm{d}q}{q} = \theta_3^4(e^{\pi i}q^2)\frac{\mathrm{d}q}{q} \mapsto \frac{\mathrm{d}t_4}{2t_4}$
  - Weight-4 $\quad \Theta_{\mathbb{E}^8}(q)\frac{\mathrm{d}q}{q} = \frac{1}{2}\left(\theta_2^8(q)+\theta_3^8(q)+\theta_4^8(q)\right)\frac{\mathrm{d}q}{q} \mapsto 8\left(\frac{1}{t_4}+\frac{16}{1-t_4}-1\right)\frac{\mathrm{K}^2(t_4)}{4\pi^2}\mathrm{d}t_4$
- Meromorphic differentials on moduli space $\mathscr{M}_{1,2}[4]$ $\quad \mathscr{F}(x,t_4) = K(t_4)\times\partial_{t_4}\left[\frac{1}{K(t_4)}\int_{-1}^{x}\frac{dX}{\sqrt{(X^2-1)(X^2-t_4)}}\right]$
  - Weight-0 & Weight-1 $\quad 2\pi\mathrm{d}z \overset{f^*}{\longmapsto} \frac{\pi}{2K(t_4)}\frac{\mathrm{d}x}{Y}+\mathscr{F}(x,t_4)\frac{\pi\,\mathrm{d}t_4}{2K(t_4)}\,, \quad i\pi\mathrm{d}\tau \overset{f^*}{\longmapsto} \frac{1}{8}\frac{1}{t_4(1-t_4)}\frac{\pi^2}{K^2(t_4)}\mathrm{d}t_4$
  - Weight-2 $\quad \omega_2^{\mathrm{Kro}}(z,\tau) \mapsto \mathrm{d}x\left(\frac{t_4(1-t_4)}{Y}\mathscr{F}(x,t_4)-\frac{t_4+x}{2(x^2-t_4)(x+1)}\right)+\mathrm{d}t_4\left(\frac{1}{2}t_4(1-t_4)\mathscr{F}^2(x,t_4)+\frac{1}{8(x^2-t_4)}+\frac{t_4-2}{24(t_4-1)t_4}\right)$
  - Weight-3 $\quad 2\pi i\omega_3^{\mathrm{Kro}}(z,\tau) \mapsto \mathrm{d}x\mathrm{K}(t_4)\left[\frac{2(1-t_4)^2t_4^2}{Y}\mathscr{F}^2(x,t_4)+2(t_4-1)t_4\left(\frac{x}{x^2-t_4}-\frac{1}{1+x}\right)\mathscr{F}(x,t_4)+\frac{1}{6Y}\left(\frac{3(t_4-1)t_4}{x^2-t_4}+t_4-2\right)\right]$

    $+\mathrm{d}t_4\mathrm{K}(t_4)\left[\frac{2}{3}(1-t_4)^2t_4^2\mathscr{F}^3(x,t_4)+\left(\frac{t_4-t_4^2}{2(x^2-t_4)}+\frac{2-t_4}{6}\right)\mathscr{F}(x,t_4)+\frac{1}{Y}\left(\frac{x^2}{2(t_4-1)}+\frac{(1-t_4)x}{6(x^2-t_4)}+\frac{1}{2(1-t_4)}-\frac{x}{6}\right)\right]$
  - Weight-4 $\quad (2\pi i)^2\omega_4^{\mathrm{Kro}}(z,\tau) \mapsto \mathrm{d}x\mathrm{K}^2(t_4)\left[\frac{8}{3}\frac{(1-t_4)^3t_4^3}{Y}\mathscr{F}^3(x,t_4)-4(1-t_4)^2t_4^2\left(\frac{x}{x^2-t_4}-\frac{1}{1+x}\right)\mathscr{F}^2(x,t_4)+\frac{2}{3}(t_4-1)t_4\left(\frac{3t_4(1-t_4)}{x^2-t_4}-t_4+2\right)\frac{\mathscr{F}(x,t_4)}{Y}\right.$

    $\left.+\frac{t_4}{3}\left(\frac{3}{x^2-t_4}+(1-t_4)\frac{x}{(x^2-t_4)^2}\right)\right]+\mathrm{d}t_4\mathrm{K}^2(t_4)\left[\frac{2}{3}(1-t_4)^3t_4^3\mathscr{F}^4(x,t_4)+\frac{t_4(1-t_4)}{3}\left(\frac{3t_4(1-t_4)}{x^2-t_4}-t_4+2\right)\mathscr{F}^2(x,t_4)\right.$

    $\left.+2t_4\left(\frac{(1-t_4)^2x}{3(x^2-t_4)}+\frac{1}{3}(t_4-1)x-x^2+1\right)\frac{1}{Y}\mathscr{F}(x,t_4)+\frac{1}{120}\left(\frac{15(t_4^2-t_4)}{(x^2-t_4)^2}+\frac{30(t_4-4x)}{x^2-t_4}+\frac{7}{t_1-1}+\frac{8}{t_4}+7\right)\right]$

$$\mathscr{F}(x,t_4) = \frac{1}{4t_4}\frac{Z_4(x,t_4)}{\sqrt{t_4}-1}-\frac{xY}{2t_4(t_4-1)(x^2-t_4)}$$

$$4K(t_4) = 2\pi\theta_3^2(q^2) \in \mathscr{M}_1(\Gamma_1(4))$$

Section 5

Pull back of the symbol letters to $\mathscr{M}_{1,2}[4]$

Pullback of the closed 1-forms for Bhabha

$$\begin{array}{ccc} f^*\omega & & \mathbb{C}\times\mathbb{H} \\ f^*\big\uparrow & & \big\downarrow f_{[4]} \\ \omega & & \mathbb{CP}^2\backslash\Sigma \end{array}$$

- Fundamental differentials

$$\omega_z = \mathrm{d}t\,\frac{-1}{4t^2(s+t-4m^2)(s+t)}\,\frac{\mathrm{T}_1(s,t)}{\Psi_1^2(s,t)} + \mathrm{d}s\left(\frac{2s+t-4m^2}{4s(s-4m^2)t(s+t)(s+t-4m^2)}\,\frac{\mathrm{T}_1(s,t)}{\Psi_1^2(s,t)} + \frac{2\sqrt{-t}\sqrt{4m^2-t}}{s(s-4m^2)t(t-4m^2)}\,\frac{1}{\Psi_1(s,t)}\right),$$

$$\omega_\tau = \frac{\mathrm{d}t\,(s-4m^2)s - \mathrm{d}s\,t(2s+t-4m^2)}{2st^2(s-4m^2)(s+t-4m^2)(s+t)\Psi_1^2(s,t)}$$

$$\omega_\tau \xrightarrow{f^*} i\pi\mathrm{d}\tau \quad \text{and} \quad \omega_z \xrightarrow{f^*} 2\pi\mathrm{d}z$$

$$\mathrm{T}_1(s,t) = \int \mathrm{d}s\left[\frac{-t}{s}\,\frac{4s^2+4s(t-4m^2)+t(t-4m^2)}{\sqrt{-t}\sqrt{4m^2-t}}\Psi_1 - 8t\,\frac{(s+t-4m^2)(s+t)}{\sqrt{-t}\sqrt{4m^2-t}\,(t+2s-4m^2)}\partial_s\Psi_1\right] + \mathrm{d}t\left[\frac{-t}{4m^2-t}\,\frac{-48m^4+4m^2s+2s^2+12m^2t+st}{\sqrt{-t}\sqrt{4m^2-t}\,(t+s-4m^2)}\Psi_1\right]$$

Pullback of the closed 1-forms for Bhabha

- 4-dimensional cubic lattice $\mathbb{Z}^4$

$$\omega_{11} = \mathrm{d}t \frac{\sqrt{(s-4m^2)s}}{t\sqrt{(s+t-4m^2)(s+t)}} - \mathrm{d}s \frac{2s+t-4m^2}{\sqrt{(s-4m^2)s}\sqrt{(s+t-4m^2)(s+t)}}$$

$$\omega_{11} \xrightarrow{f^*} 2\,\Theta_{\mathbb{Z}^4}(e^{\pi i}q^2)\frac{\mathrm{d}q}{q} = 2\theta_3^4(e^{\pi i}q^2)\frac{\mathrm{d}q}{q} \in \mathscr{M}_2(\Gamma_1(4))$$

- Jacobi's four square theorem $\Omega = \mathbb{Z}^4$

$$\Theta_\Omega(\tau) = \sum_{x\in\Omega} e^{2i\pi\tau||x||^2} = \sum_{n=0}^{\infty} r(n,k)(e^{2\pi i\tau})^n, \quad r(n,k) = \#\left\{v \in \mathbb{Z}^k \ : \ n = v_1^2 + \ldots + v_k^2\right\}, \quad \mathrm{Im}\tau > 0$$

$$\Theta_{\mathbb{Z}^4}(\tau) \equiv \theta_3^4(\tau) \Longrightarrow r(n,4) = 8 \sum_{0<d|n,4\nmid d} d, \quad n \geq 1$$

Pullback of the closed 1-forms for Bhabha

$$\omega_{41} = \mathrm{d}t\left[\frac{1}{2t^2(s+t-4m^2)(s+t)}\frac{\mathrm{T}_1^2(s,t)}{\Psi_1^2(s,t)} + \frac{2(s-4m^2)}{(t-4m^2)(s+t-4m^2)}\right]$$

$$+\,\mathrm{d}s\left[\frac{2s+t-4m^2}{2(s-4m^2)st(s+t-4m^2)(s+t)}\frac{\mathrm{T}_1^2(s,t)}{\Psi_1^2(s,t)} + \frac{\sqrt{t(t-4m^2)}}{(s-4m^2)s(4m^2-t)t}\frac{\mathrm{T}_1(s,t)}{\Psi_1(s,t)} - \frac{2t(2s^2+st+4m^2s+12m^2t-48m^4)}{(s-4m^2)s(t-4m^2)(s+t-4m^2)}\right]$$

- D_4 root lattice $D_4 = \frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k})\mathbb{Z} \oplus \mathbf{i}\,\mathbb{Z} \oplus \mathbf{j}\,\mathbb{Z} \oplus \mathbf{k}\,\mathbb{Z} = \frac{1}{2}\mathbb{Z} \oplus \frac{1}{2}\mathbf{i}\,\mathbb{Z} \oplus \frac{1}{2}\mathbf{j}\,\mathbb{Z} \oplus \frac{1}{2}\mathbf{k}\,\mathbb{Z}$

$$\omega_{41} \xmapsto{f^*} 8\omega_2^{\mathsf{Kro}}(2z\,,q) - 8\omega_2^{\mathsf{Kro}}(2z\,,q^2) + \frac{4}{3}\frac{\mathrm{d}q}{q}\Theta_{D_4}(q^2)$$

$$\Theta_{D_4}(q^2) = \theta_3^4(q^2) + \theta_2^4(q^2) \in \mathscr{M}_2(\Gamma_0(2)) \subset \mathscr{M}_2(\Gamma_1(4))$$

Function space of symbol letters for Bhabha

- square roots from lower sectors

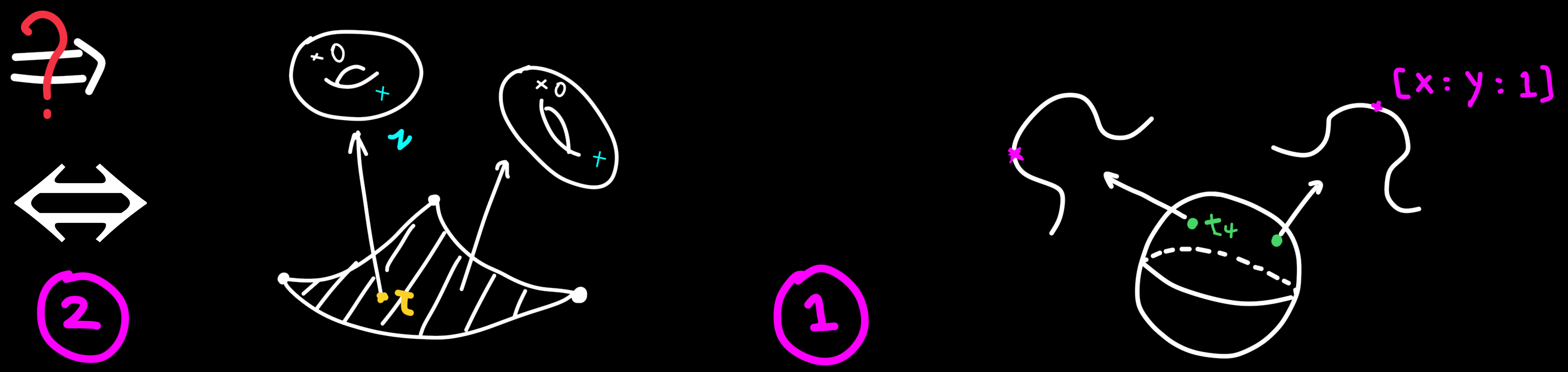
$$\left\{ \sqrt{1-x^2},\ \ \sqrt{t_4-x^2},\ \ \sqrt{t_4},\ \ \sqrt{1-t_4},\ \ \sqrt{1+t_4} \right\}$$

multi-valued!

- Transcendental objects
  - $K(t_4)$
  - $\mathscr{F}(x, t_4)$
  - $f(t_4)$

$$\frac{\partial f}{\partial t_4} = 2\frac{1-t_4}{\sqrt{t_4}(1+t_4)^{3/2}}K(t_4)$$

$\mathbb{CP}^2\backslash\Sigma \Leftrightarrow$



$\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C} \times \mathbb{H}$

$E_{t_4} : y^2 = (x^2 - 1)(x^2 - t_4)\,, \quad t_4 \in \mathbb{C}\backslash\{0,1\}$

$(z, \tau) \simeq ([x : y : 1], t_4)$

Section 6

Uniformization of punctured $\mathbb{CP}^2$

A family of curves over punctured $\mathbb{CP}^2$

- The family of elliptic curves for Bhabha scattering, with coordinates $[s : t : m^2]$

$$E_4 : Y^2 = (X - e_1)(X - e_2)(X - e_3)(X - e_4)$$

$$e_1 = \frac{s}{m^2} - 4\,, \quad e_2 = -\frac{st + 2\sqrt{m^2 s\, t(s + t - 4m^2)}}{m^2(4m^2 - t)}\,, \quad e_3 = -\frac{st - 2\sqrt{m^2 s\, t(s + t - 4m^2)}}{m^2(4m^2 - t)}\,, \quad e_4 = \frac{s}{m^2}$$

- What is the base space ? Answer: equating the roots in all possible ways. But Why? Answer: cusps correspond to elliptic curves with nodes or monomial singularities

Union of the following linear varieties is deleted :

$\mathbb{CP}^2 \backslash \Sigma$ $\quad \Sigma = \langle s\,, s - 4\,, s + t\,, s + t - 4\,, t\,, t - 4 \rangle \cup \{[1 : 0 : 0]\}$

The Mordell-Weil group for a family of elliptic curves

- Theorem of Mordell-Weil

For elliptic curves over $\mathbb{Q}$ (or its finite extensions), the group of rational points is finitely generated

- Sections of rational points $\{[n]p_0 \,|\, p_0 \in A(E_3) \simeq T \oplus r\mathbb{Z}\,, n \in \mathbb{Z}\} \simeq (\mathbb{Z}\,, +)$

$$E_3 : Y^2 = \prod_{i=1}^{3} (e_i - e_4)\left(X + \frac{e_i}{e_4(e_i - e_4)} \right)$$

$$p_0 = \left[\frac{s-4}{s(s+t)} : \frac{(s-4)(s+t-4)}{s(s+t)} : 1 \right]$$

$$[2]p_0 = \left[\frac{16 + t(8 - 3t + s(s+t-4))}{4s(t-4)(s+t)} : \ldots : 1 \right]$$

$$\vdots$$

Generators of the Mordell-Weil group as marked points

$$\begin{array}{ccc} E_{\text{bhabha}} & \overset{?}{\longleftrightarrow} & E_{\text{universal}} \\ \downarrow & & \downarrow \\ \mathbb{CP}^2\backslash\Sigma & \overset{?}{\longleftrightarrow} & \mathcal{M}_{1,2}[N] \end{array}$$

- The generator of Mordell-Weil group $A(E_{[s:t:m^2]}) \simeq T \oplus r\mathbb{Z}$

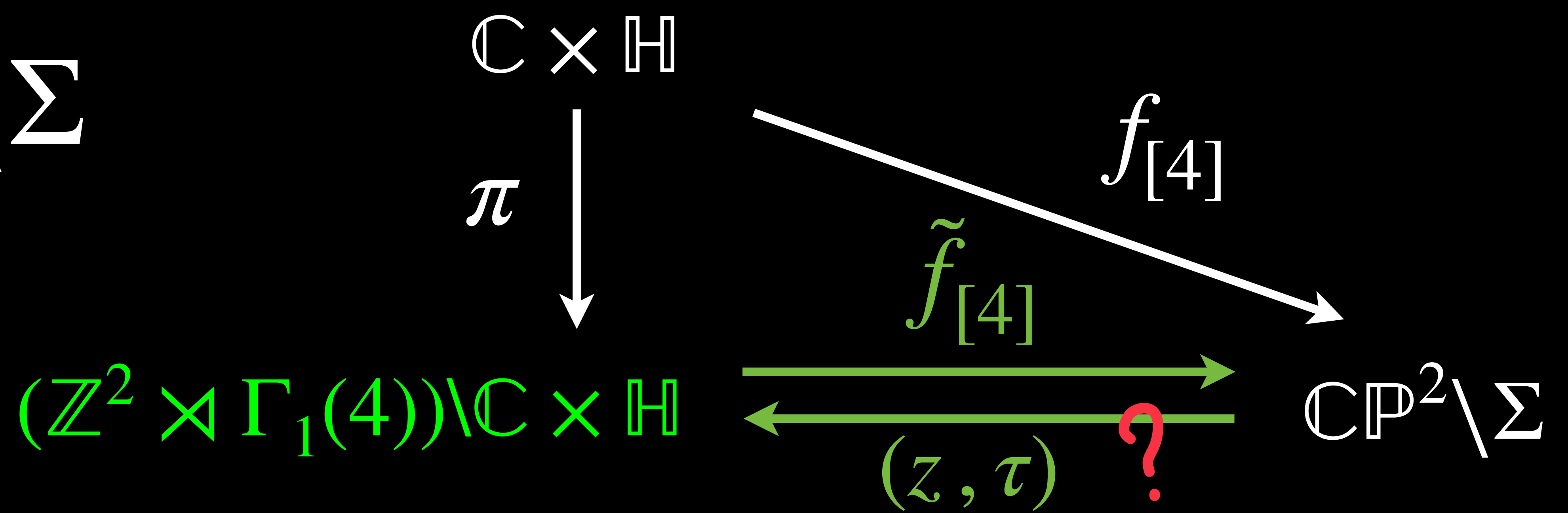
$$p_0 = [X : Y : 1] = \left[\frac{(s-4m^2)s}{-4m^2+2s+t} : \frac{(s-4m^2)s/m^2(s+t-4m^2)}{(2s+t-4m^2)^2/(s+t)} : m^2\right]$$

- Mapping to a universal family of complex tori

Abel map: $\dfrac{(e_2-e_4)(e_1-X)}{(e_1-e_4)(e_2-X)} = \dfrac{\theta_2^2(0,q)}{\theta_3^2(0,q)}\dfrac{\theta_1^2(\pi z,q)}{\theta_4^2(\pi z,q)}$, Modular lambda: $\dfrac{4m^2}{2m^2+\sqrt{\dfrac{(-m^2-i0)s(s+t-4m^2)}{-t}}} = \dfrac{\theta_2^4(0,q)}{\theta_3^4(0,q)}$

Uniformization of punctured $\mathbb{CP}^2$

$\Sigma = \{$kinematic branch points$\}$



- Definition of the map $f_{[4]} : \mathbb{C}\times\mathbb{H} \mapsto \mathbb{CP}^2\backslash\Sigma$

$$\mathrm{s} = -\frac{4(-1+R)\times(-2+\lambda)}{-2+\lambda+R\times\lambda}, \quad \mathrm{t} = \frac{4(-1+R)\times R\times\lambda^2}{(-2+R\times\lambda)(-2+\lambda+R\times\lambda)}$$

- $f_{[4]}$ is invariant under $\mathbb{Z}^2 \rtimes \Gamma_1(4) \Longrightarrow \tilde{f}_{[4]}$ is well-defined

$$R = \frac{\theta_2^2(0,q)}{\theta_3^2(0,q)}\frac{\theta_1^2(\pi z,q)}{\theta_4^2(\pi z,q)}, \quad \lambda = \frac{\theta_2^4(0,q)}{\theta_3^4(0,q)}$$

$$f_{[4]}[z,\tau] = f_{[4]}[((m,n),\gamma)\cdot(z,\tau)], \forall (m,n)\in\mathbb{Z}^2, \gamma\in\Gamma_1(4), \quad ((m,n),\gamma)\cdot(z,\tau) = \left(\frac{z+m\tau+n}{c\tau+d}, \gamma\cdot\tau\right)$$

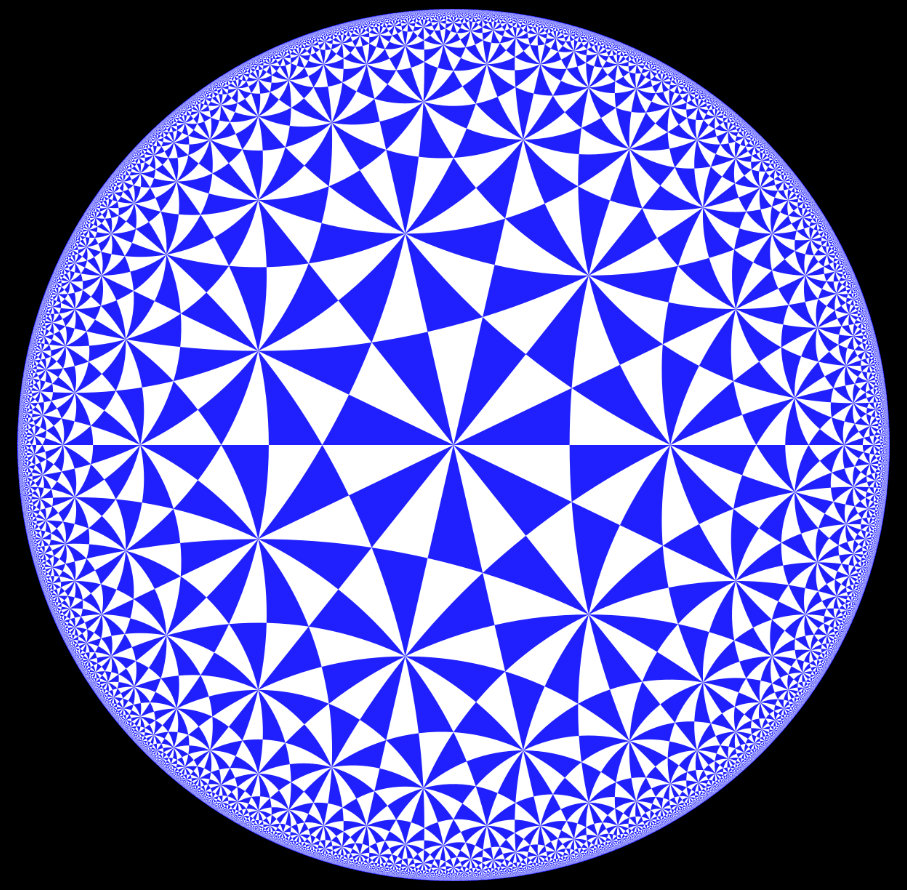
- **The period is a modular form of weight 1 under the action of $\mathbb{Z}^2 \rtimes \Gamma_1(4)$**

$$\Psi_{\text{bhabha}}(s,t) = 4\mathrm{K}\left(\frac{4}{2+\sqrt{\frac{-s(s+t-4)}{-t}}}\right)/\sqrt{\ldots}$$

$$\Psi_{\text{bhabha}}(z,\tau) = \frac{\pi\theta_2^2(0,q)}{2}\frac{\theta_3(\pi z,q)\theta_4(\pi z,q)}{\theta_1(\pi z,q)\theta_2(\pi z,q)}, \quad \Psi_1\left(\frac{z+m\tau+n}{c\tau+d}, \gamma\cdot\tau\right) = \frac{1}{c\tau+d}\Psi_1(z,\tau), \forall\gamma\in\Gamma_1(4)$$

Summary and Outlook

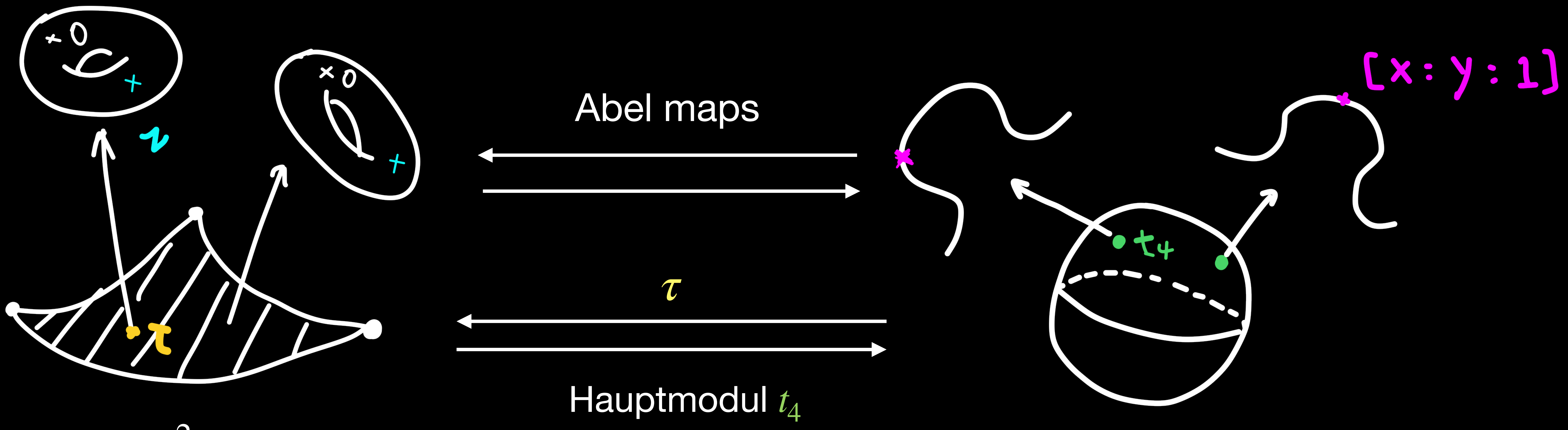
- **Summary**
  - Bhabha scattering—the first amplitude beyond genus 0 in QED
  - Underlying connections to arithmetic geometries of elliptic curves, e.g. the hyperbolic tesselation and Mordell-Weil group of rational points
  - Unified description to several sectors of Bhabha scattering and top quark production through moduli space $\mathscr{M}_{1,2}[4]$
  - Correspondence between Kronecker's differential forms and letters of eMPLs
- **Future applications**
  - Go beyond genus 1, Hurwitz automorphisms
  - Elliptic integrals and modular forms in gravitational wave physics



Hurwitz (2,3,7)

Appendix

Isomorphism between universal family of complex tori and universal family of elliptic curves



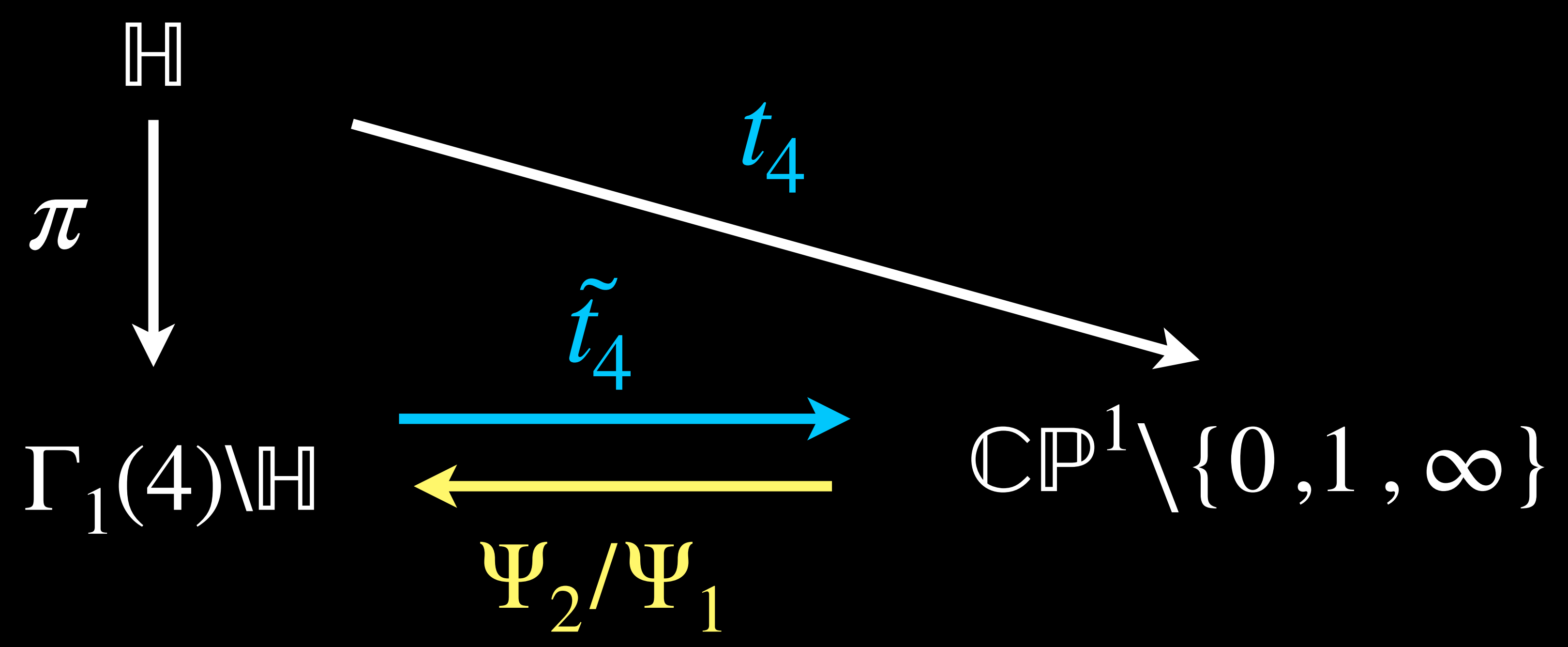
$\mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C} \times \mathbb{H}$

$E_{t_4} : y^2 = (x^2 - 1)(x^2 - t_4)\,, \quad t_4 \in \mathbb{C}\backslash\{0,1\}$

$$(z, \tau) \simeq ([x : y : 1], t_4)$$

Reversed problem: how can one find a family of elliptic curves with given monodromy e.g., the Fuchsian Triangle Group $\Gamma_1(4)$? Answer: from 'the book'

- Covering by Hauptmodul t_4, $\{0, 1, \infty\} \overset{t_4}{\longmapsto} \{\infty, 1/2, 0\} \in \partial\overline{\mathbb{H}}$



$$t_4(\tau) = \left(\frac{\theta_3^2(q) - \theta_4^2(q)}{\theta_3^2(q) + \theta_4^2(q)}\right)^2$$

[S. MAIER, 2006]

- A family of elliptic curves 'from the book'

$$E_{t_4} : Y^2 = (X^2 - 1)(X - t_4)\,, \quad t_4 \in \mathbb{CP}^1 \backslash \{0,1,\infty\}\,, \quad j(t_4) = 16\frac{(t_4(t_4 + 14) + 1)^3}{t_4(1 - t_4)^4}$$

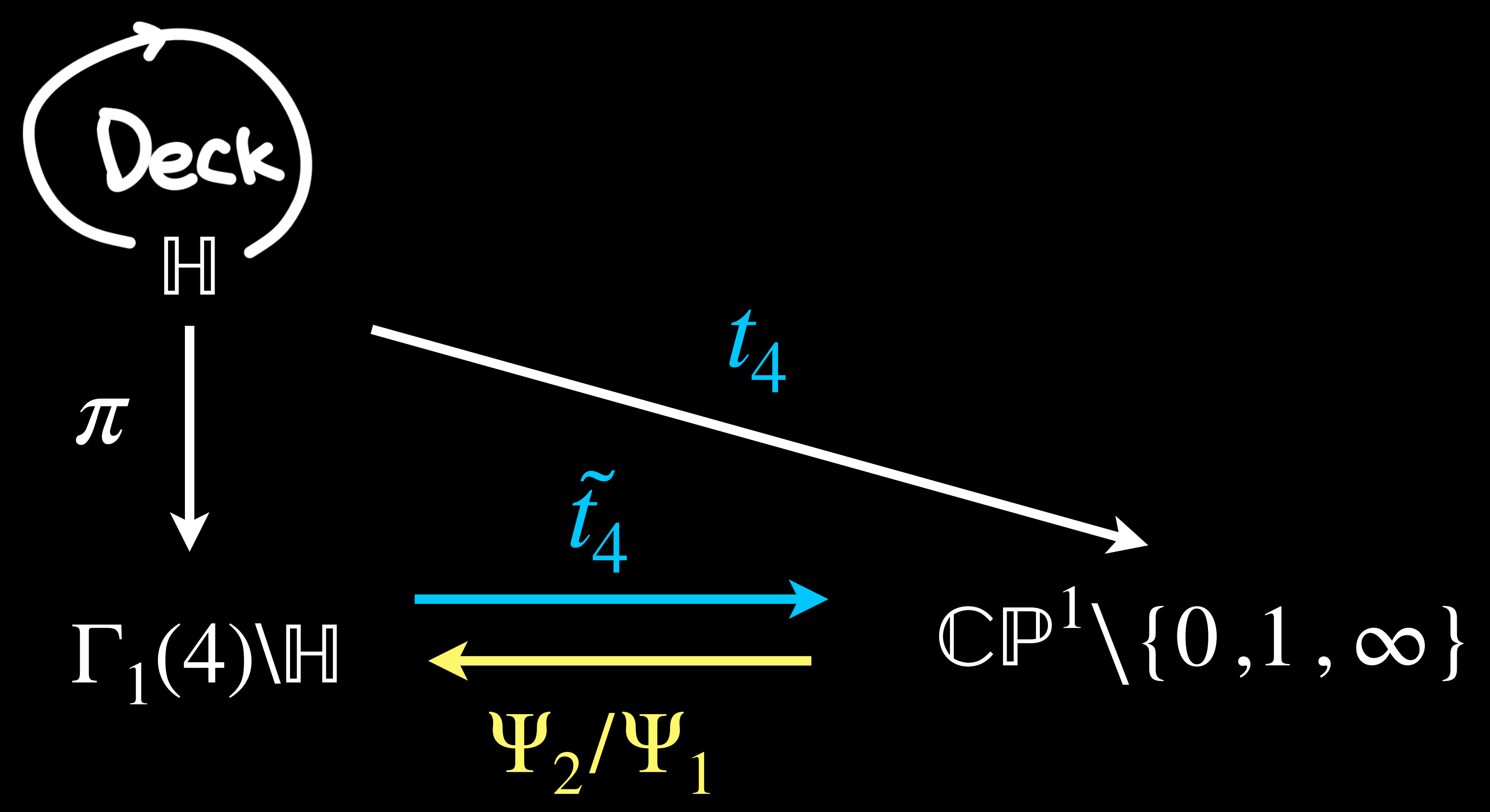
which is ramified at $t_4 = 0, 1$ and ∞, each with ramification index { 1 ,4 ,1} so that $deg(j) = 6 = [\mathbb{P}\mathrm{SL}(2,\mathbb{Z}) : \Gamma_1(4)]$

$$\int \frac{dX}{Y} : H_1(E_\lambda) \to \mathbb{C}\,, \qquad \tau \equiv \frac{\Psi_2}{\Psi_1} = \frac{iK\left(1 - \frac{4\sqrt{t_4}}{(1+\sqrt{t_4})^2}\right)}{K\left(\frac{4\sqrt{t_4}}{(1+\sqrt{t_4})^2}\right)}$$

Monodromy representations

- Picard-Fuchs differential equation

$$\left[\frac{d^2}{dt_4^2} + \left(\frac{1}{t_4} - \frac{1}{1-t_4}\right)\frac{d}{dt_4} + \frac{1}{4(t_4-1)t_4}\right]\Psi_i = 0\,, \quad i = 1\,,2$$



- The monodromy matrices generators in $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$

$$\rho_{[\circlearrowleft_0]} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \quad \rho_{[\circlearrowleft_1]} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$$

- Covering automorphism group structure theorem

$$\rho(\pi_1(X,\cdot\,)) = \langle \rho_{[\circlearrowleft_0]}\,, \rho_{[\circlearrowleft_1]} \rangle = \underbrace{\mathrm{Deck}_\pi(\mathbb{H}) \simeq \Gamma_1}_{\pi}(4) \simeq \mathbb{Z} * \mathbb{Z} \simeq \mathrm{Deck}_{t_4}(\mathbb{H})$$

- Covering space quotient theorem

- The pullback of the period function

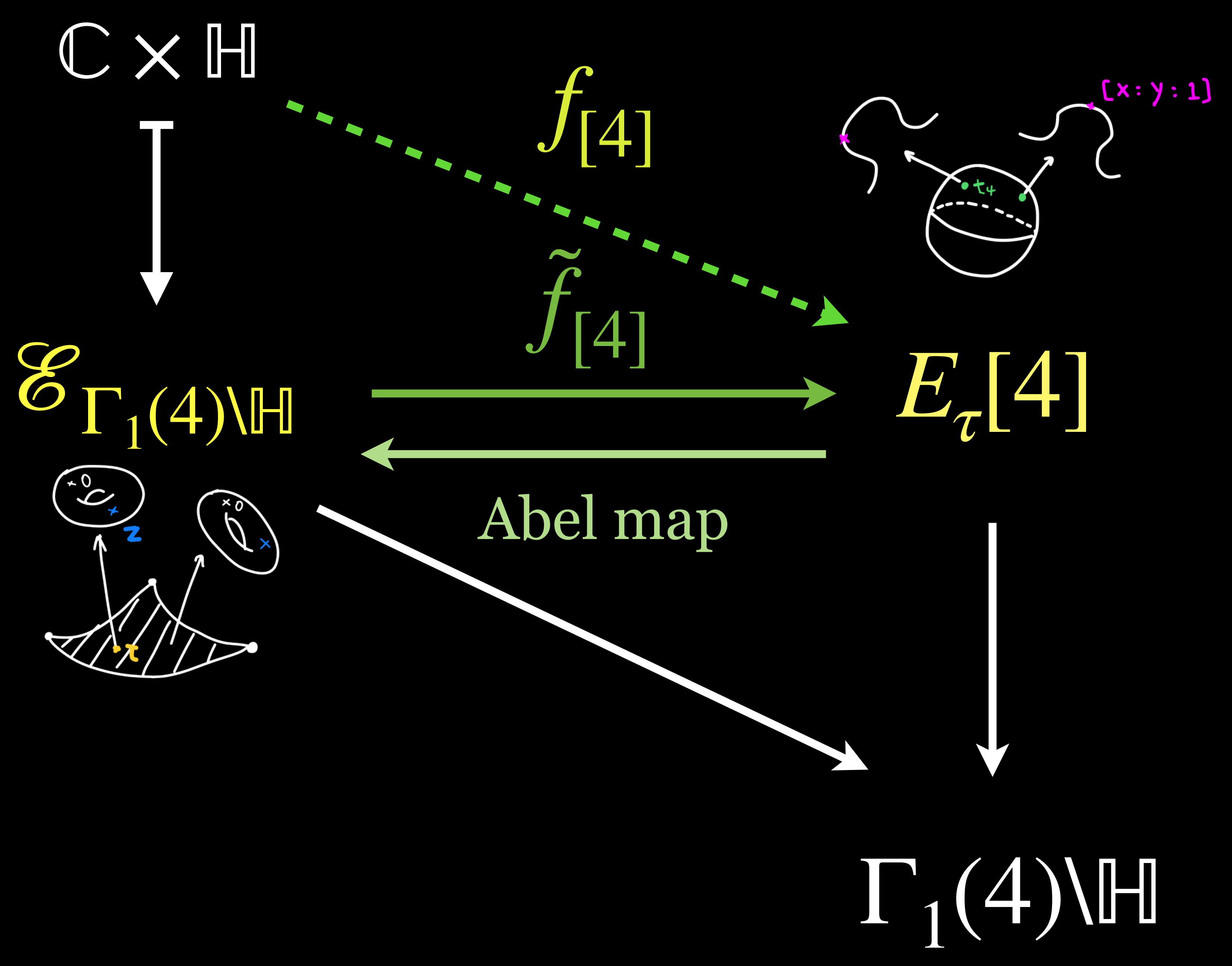
$$\Psi_1(\tau) = 4K(t_4) = \pi(\theta_3^2(q) + \theta_4^2(q)) = 2\pi\theta_3^2(q^2) = 2\pi\frac{\eta^{10}(2\tau)}{\eta(\tau)\eta^4(\tau)} \in \mathscr{M}_1(\Gamma_1(4))\,, \quad \dim(\mathscr{M}_1(\Gamma_1(4))) = 1$$

The isomorphism between $E_\tau[4] \simeq \mathscr{E}_{\Gamma_1(4)\backslash\mathbb{H}}$

- the isomorphism map

$$(z,\tau) \in \mathbb{C}\times\mathbb{H} \overset{f_{[4]}}{\longmapsto} \left[X : \frac{1}{\Psi_1(\tau)}\partial X/\partial z : 1\right] \in E_\tau[4]$$

$$\Psi_1(\tau) = 2\pi\theta_3^2(q^2)\,, \quad X(z) = \frac{2\theta_4^2(0,q)\theta_1^2(\pi z, q)}{2\theta_3^2(0,q^2)\theta_1^2(\pi z, q) - \theta_2^2(0,q)\theta_4^2(\pi z, q)}$$



- invariance under $\mathbb{Z}^2 \rtimes \Gamma_1(4) \Longrightarrow \tilde{f}_{[4]}$ is well-defined

$$f_{[4]}[z\,,\tau] = f_{[4]}[((m,n),\gamma)\cdot(z\,,\tau)]\,, \forall (m\,,n) \in \mathbb{Z}^2\,, \gamma \in \Gamma_1(4)\,, \quad ((m,n),\gamma)\cdot(z\,,\tau) = \left(\frac{z+m\tau+n}{c\tau+d}\,, \gamma\cdot\tau\right)$$