ϵ-factorised Differential Equations Beyond Polylogarithms

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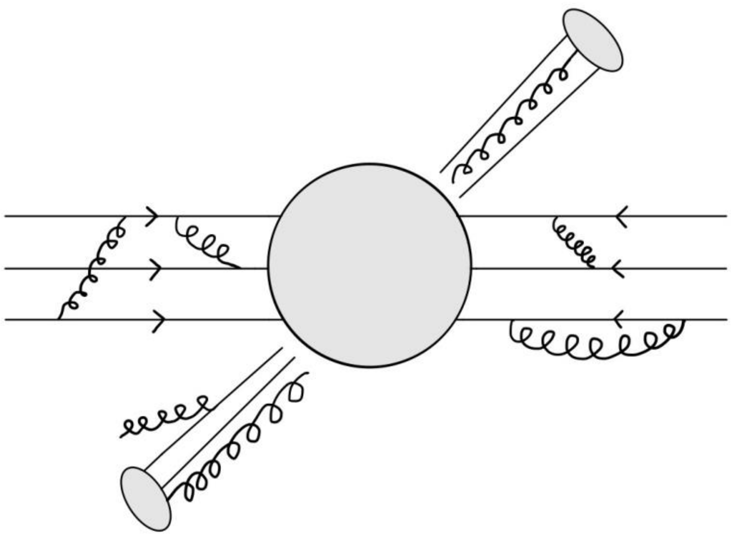
Joint work with L. Görges, C. Nega, L. Tancredi [arXiv:2305.14090]
with C. Duhr, S. Maggio, C. Nega, L. Tancredi [ongoing work]

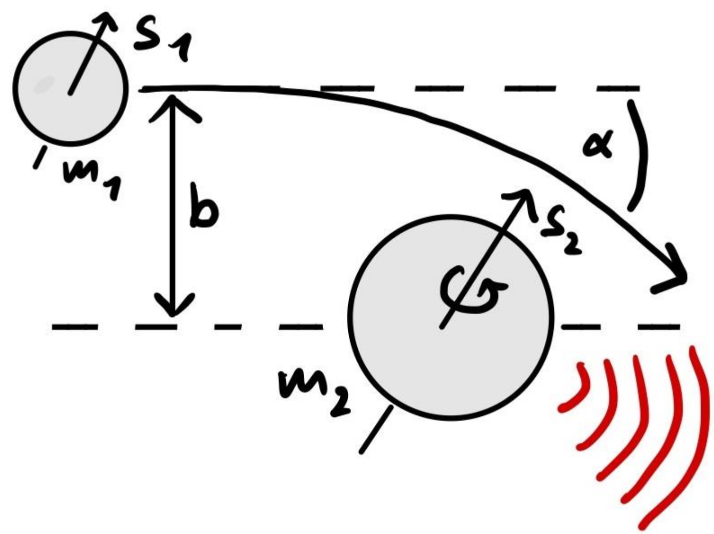
Technical University of Munich

Why study Feynman integrals?

- Traditionally: ubiquitous in scattering amplitude calculations for **collider observables** in perturbative Quantum Field Theory framework



- More recently: **gravitational-wave observables** calculated in Post-Minkowskian expansion (black hole / neutron star scattering)



⇨ **Key for precise theoretical predictions**

Differential Equations Method

dimensionally regularised scalar
Feynman integral families

$$\int \left(\prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \right) \frac{N_1^{b_1} \dots N_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}}$$

$$d = d_0 - 2\epsilon \, , \, d_0 \in \mathbb{N}$$

Integration-by-parts

identities

[Chetyrkin Tkachov `81]

vector space structure with
basis of ***master integrals***

$$\vec{I} = \big(I_1(\vec{z}, \epsilon), \dots, I_N(\vec{z}, \epsilon) \big)$$

Choice is a
priori free

kinematic
variables

The master integrals satisfy a system of **partial differential equations** (DEs) w.r.t. $\vec{z}$:

$$d\vec{I} = GM(\vec{z}, \epsilon) \, \vec{I}$$

[Kotikov `93; Remiddi `97;
Gehrmann Remiddi `99; ...]

Gauss-Manin connection
matrix of differential 1-froms
(**fuchsian** & entirely **rational**!)

ϵ-factorised form of differential equations

DEs are **hard** to solve for **arbitrary** choice of basis, solution becomes straight-forward in **ϵ-factorised form**.

$$d\vec{I} = GM(\vec{z},\epsilon)\,\vec{I} \quad \xrightarrow[\vec{J} = T(\vec{z},\epsilon)\,\vec{I}]{\text{Change of basis}} \quad d\vec{J} = \epsilon\, GM_\epsilon(\vec{z})\,\vec{J}$$

[Kotikov `12; Henn `13; Lee `13]

does not depend on ϵ!

Solution: $$\vec{J}(\vec{z},\epsilon) = \mathbb{P}\, exp\left(\epsilon \int_\gamma GM_\epsilon(\vec{z}')\right)\vec{J}(\vec{z}_0,\epsilon)$$

path ordering

path from $\vec{z}_0$ to $\vec{z}$

boundary condition

At every order in ϵ, find **Chen iterated integrals:** [Chen `77]

$$\vec{J}(\vec{z},\epsilon) = \sum_{k=0}^{\infty} \epsilon^k\, \vec{J}^{(k)}(\vec{z}) \quad \Longrightarrow \quad \vec{J}^{(k)}(\vec{z}) \sim \text{``} \sum_{j=0}^{k} \int_\gamma \underbrace{GM_\epsilon \cdot \cdots \cdot GM_\epsilon}_{j}\, \vec{J}^{(k-j)}(\vec{z}_0) \text{``}$$

assume: normalised with a power of ϵ such that its ϵ-expansion starts at $\mathcal{O}(\epsilon^0)$

j-fold iterated integral

Canonical differential equations

Conjecturally, a basis satisfying differential equations in ϵ-factorised form **always exists**. Even up to constant rotations, it is in fact **not unique**! Some bases are better than others.

<u>Simple example</u>: massless Box family (2 master integrals, $z = s/t$)

$$d\vec{I} = \begin{pmatrix} 0 & 0 \\ \frac{2(2\epsilon-1)}{z\,(1+z)} & -\frac{1+z+\epsilon}{z\,(1+z)} \end{pmatrix} dz\,\vec{I} \xrightarrow{T_1 = \begin{pmatrix} 1 & 0 \\ 2\ln(1+z) & z/2 \end{pmatrix}} d\vec{J}_1 = \epsilon \begin{pmatrix} 0 & 0 \\ \frac{2z+\ln(1+z)}{z\,(1+z)} & \frac{-1}{z\,(1+z)} \end{pmatrix} dz\,\vec{J}_1$$

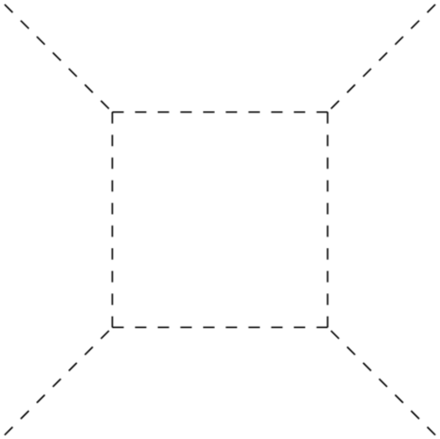
$$T_2 = \begin{pmatrix} 2\epsilon-1 & 0 \\ 0 & \varepsilon\, z \end{pmatrix}$$

$$d\vec{J}_2 = \epsilon \underbrace{\begin{pmatrix} 0 & 0 \\ \frac{2}{1+z} & \frac{-1}{z\,(1+z)} \end{pmatrix} dz}_{= \begin{pmatrix} 0 & 0 \\ 2\,d\ln(1+z) & d\ln(1+z) - d\ln(z) \end{pmatrix}} \vec{J}_2$$

Canonical Basis [Henn `13]
$d$log-forms with rational (algebraic) arguments

⇩

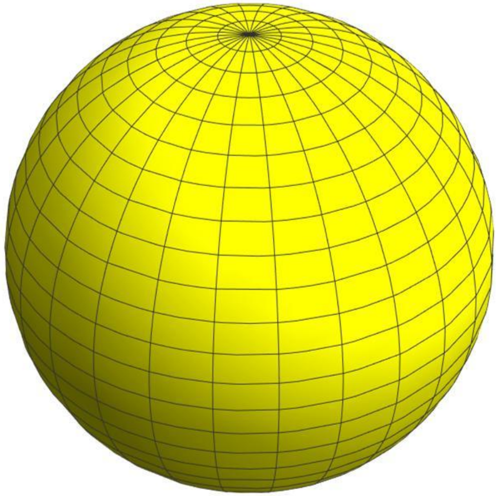
Resulting iterated integrals evaluate to *Multiple Polylogarithms (MPLs)*



Multiple Polylogarithms (MPLs)

[…, Remiddi Vermaseren `99, Goncharov `00, …]

In a nutshell, they can be defined as iterated integrals of rational functions with **simple poles** on the Riemann sphere



$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt_1}{t_1 - a_1} G(a_2, \ldots, a_n; t_1) =$$

$$= \int_0^x \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_2}{t_2 - a_2} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n - a_n}$$

length / transcendental weight

They have at most logarithmic singularities and satisfy a simple inhomogeneous, **unipotent** differential equation:

$$\frac{d}{dx} G(a_1, a_2, \ldots, a_n; x) = \frac{1}{x - a_1} G(a_2, \ldots, a_n; x)$$

transcendental weight / length decreased by one

⇨ such functions are called ***pure*** functions

Canonical differential equations and pure functions

Canonical Form: $$d\vec{J} = \epsilon \sum_n \underbrace{m_n}_{\text{constant, rational matrices}} \mathrm{dlog}(\underbrace{\alpha_n}_{\textbf{\textit{letters}}\text{ (rational functions)}})\vec{J}$$

Solution: $$\vec{J}(\vec{z},\epsilon) = \mathbb{P}\, exp\left(\epsilon \int_\gamma \sum_n m_n\, \mathrm{dlog}(\alpha_n(\vec{z}\,'))\right)\vec{J}(\vec{z}_0,\epsilon)$$

$\vec{J}(\vec{z},\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \vec{J}^{(k)}(\vec{z})$ satisfies a canonical differential equation	$\Rightarrow$ $(\Leftarrow)$	$\vec{J}^{(k)}(\vec{z})$ can be written in terms of a **pure** linear combination of MPLs of transcendental weight k

"$\vec{J}$ is **pure** and of **uniform transcendental weight (UT)**"

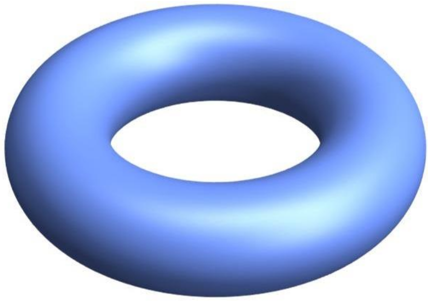
Smallest step up in complexity: the elliptic case

MPLS: iterated integrals of **rational functions** with **simple poles** on the **Riemann sphere**

⇨ Generalization on **torus**: ***elliptic multiple polylogarithms (eMPLs)***

[Brown Levin `11; Brödel Mafra Matthes Schlotterer `14; Brödel Dulat Duhr Penante Tancredi `17,`18]

BUT: cohomology of torus can't be spanned by differential forms with simples poles only!



⇩

Way to avoid higher poles: add infinite tower of **transcendental** kernels

Important property: Resulting functions satisfy generalised unipotent differential equations. We have a notion of purity!

⇨ Generalisation of the idea of a canonical basis possible?

The information encoded in the Wronskian

Knowledge of the solution of the differential equations at $\boldsymbol{\epsilon = 0}$ is **crucial** to achieve the factorisation of ϵ

$$d\vec{I} = [GM_0(\vec{z}) + \mathcal{O}(\epsilon)]\vec{I}$$

$$dW(\vec{z}) = GM_0(\vec{z})\, W(\vec{z})$$

$$\vec{J} = W^{-1}(\vec{z})\, \vec{I}$$

$$d\vec{J} = [\mathcal{O}(\epsilon)]\, \vec{J}$$

$W(\vec{z})$: fundamental matrix of solutions, also called ***Wronskian*** or ***period matrix***

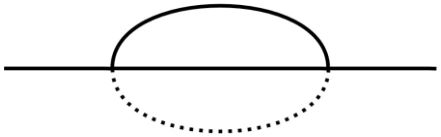
The Wronskian informs us on the **function space** required to decouple the differential equations at $\epsilon = 0$.

- polylogarithmic case: W consists of rational (algebraic) functions and (poly)logarithms
- elliptic case: W contains **complete elliptic integrals**

Important Observation: this information can be extracted by studying the **maximal cuts** of the Feynman integrals. [Primo Tancredi `16, Frellesvig Papadopoulos ´17, Bosma Sogaard Zhang `17]

From unipotent to canonical

<u>Examples:</u> **sunrise** graph with two / three equal non-vanishing internal masses on **maximal cut**



<u>MPL case (two masses)</u>

$$\frac{\partial}{\partial m^2}\vec{I} = \left[\begin{pmatrix} 0 & 1 \\ \frac{2}{m^2\,(s-4m^2)} & \frac{10m^2-s}{m^2\,(s-4m^2)} \end{pmatrix} + \mathcal{O}(\epsilon)\right]\vec{I} \qquad W = \begin{pmatrix} \frac{1}{r(s,m^2)} & \frac{\ln\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right)}{r(s,m^2)} \\ \frac{\partial}{\partial m^2}\left(\frac{1}{r(s,m^2)}\right) & \frac{\partial}{\partial m^2}\left(\frac{\ln\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right)}{r(s,m^2)}\right) \end{pmatrix}$$

$$r(s,m^2) = \sqrt{s(s-4m^2)}$$

not unipotent

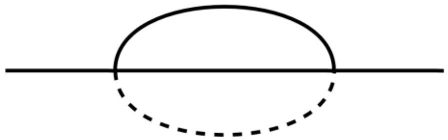
mixed transcendental weights!

Split W into a **unipotent** and **semi-simple** part: $W = W_{ss} \cdot W_u$

$$W_{ss} = \begin{pmatrix} \frac{1}{r(s,m^2)} & 0 \\ \frac{2s}{r(s,m^2)^3} & \frac{1}{m^2\,(s-4m^2)} \end{pmatrix}, \quad W_u = \begin{pmatrix} 1 & \ln\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right) \\ 0 & 1 \end{pmatrix}, \quad \frac{\partial}{\partial m^2} W_u = \begin{pmatrix} 0 & \frac{s}{m^2\, r(s,m^2)} \\ 0 & 0 \end{pmatrix} W_u$$

and rotate away the **semi-simple** part: $\vec{I} = W_{ss} \cdot \vec{I}'$

From unipotent to canonical



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rescale first integral by ϵ!

Rotate away the **semi-simple** part: $\vec{I} = W_{ss} \cdot \vec{I}'$

$$\frac{\partial}{\partial m^2}\vec{I}' = \left[\begin{pmatrix} 0 & \frac{s}{m^2\,r(s,m^2)} \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\epsilon)\right]\vec{I}' \longrightarrow d\vec{J} = \epsilon\, GM_c(s,m^2)\,\vec{J}$$

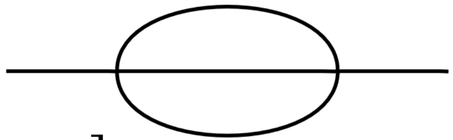
canonical (contains only dlog-forms)

$$\vec{J} = \begin{pmatrix} 1 & 0 \\ -\frac{2(s+2m^2)}{r(s,m^2)} & 1 \end{pmatrix} \cdot \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \vec{I}'$$

Integrates out a total derivative coming from $\mathcal{O}(\epsilon)$-terms

From unipotent to canonical

eMPL case (three masses)



$$\frac{\partial}{\partial m^2}\vec{I} = \left[\begin{pmatrix} 0 & 1 \\ \frac{3(s-m^2)}{m^2\,(s-m^2)(s-9m^2)} & -\frac{s^2-20s\,m^2+27\,m^4}{m^2\,(s-m^2)(s-9m^2)} \end{pmatrix} + \mathcal{O}(\epsilon)\right]\vec{I}$$

not unipotent

periods

$$W = \begin{pmatrix} \omega_0(s,m^2) & \omega_1(s,m^2) \\ \partial_{m^2}\,\omega_0(s,m^2) & \partial_{m^2}\,\omega_1(s,m^2) \end{pmatrix}$$

algebraic functions

$$\omega_0(s,m^2) \sim a_1(s,m^2)\,K(a_2(s,m^2))$$
$$\omega_1(s,m^2) \sim a_1(s,m^2)\,K(1-a_2(s,m^2))$$

Complete elliptic integral of the first kind

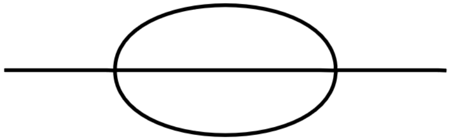
around the ***MUM point*** $m^2/s = 0$:

$\omega_0(s,m^2) = \text{power series in } m^2/s$ ⟵ holomorphic

$\omega_1(s,m^2) = \omega_0(s,m^2)\,\ln(m^2/s) + \text{power series in } m^2/s$ ⟵ single-logarithmic

Split W into a **unipotent** and **semi-simple** part: $W = W_{ss} \cdot W_u$

From unipotent to canonical



eMPL case (three masses)

Split W into a **unipotent** and **semi-simple** part: $W = W_{ss} \cdot W_u$

$$W_u = \begin{pmatrix} 1 & \frac{\omega_1}{\omega_0} \\ 0 & 1 \end{pmatrix}, \qquad \frac{\partial}{\partial \tau} W_u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} W_u \,, \qquad \tau = \frac{\omega_1}{\omega_0}$$

$$W_{ss} = \begin{pmatrix} \omega_0 & 0 \\ \partial_{m^2}\, \omega_0 & \dfrac{1}{m^2\,(s-m^2)(s-9m^2)\,\omega_0} \end{pmatrix}$$

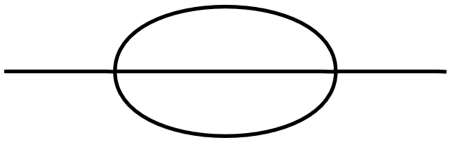
Legendre relation: $\omega_0(\partial_{m^2}\,\omega_1) - \omega_1(\partial_{m^2}\,\omega_0) = [m^2\,(s-m^2)(s-9m^2)]^{-1}$

As in the polylogarithmic two-mass case: rotate with $\boldsymbol{W_{ss}^{-1}}$, **rescale** the first integral with $\boldsymbol{\epsilon}$ and integrate out a **total derivative**

$$\vec{J} = \begin{pmatrix} 1 & 0 \\ \frac{s^2 - 30s\,m^2 + 45m^4}{2}\,{\omega_0}^2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \cdot W_{ss}^{-1}\,\vec{I} \qquad \Rightarrow \qquad d\vec{J} = \epsilon\, GM_\epsilon(s, m^2)\,\vec{J}$$

From unipotent to canonical

<u>eMPL case (three masses)</u>



Arrived at a known ϵ-factorised form [Adams Weinzierl `18]

$$GM_{\epsilon}^{m^2}(s,m^2) = \begin{pmatrix} \frac{-s^2+30s\,m^2-45m^4}{2m^2(s-m^2)(s-9m^2)} & \frac{\omega_0^2}{m^2(s-m^2)(s-9m^2)} \\ \frac{(3m^2+s)^4}{4m^2(s-m^2)(s-9m^2)\omega_0^2} & \frac{-s^2+30s\,m^2-45m^4}{2m^2(s-m^2)(s-9m^2)} \end{pmatrix}$$

only ω_0 appears, $\partial_{m^2}\,\omega_0$ does not!

Further: this basis can indeed be expressed in terms of pure eMPLs!

[Broedel Duhr Dulat Penante Tancredi `18]

⇩

Strong support that this might indeed be the elliptic generalisation of the idea of a canonical differential equation

Integrand / Leading Singularity Analysis

Parametric representation of the considered integral family (Feynman, Baikov, …):

$$I \sim \int \prod_{i=1}^{n} dx_i \ \mathcal{F}(x_i, \vec{z}) \, [\mathcal{G}(x_i, \vec{z})]^{\epsilon}$$

neglected (expanding in ϵ only adds logarithms)

Try to write this as a sum over ***d*log-forms** (in polylogarithmic case):

$$\prod_{i=1}^{n} dx_i \ \mathcal{F}(x_i, \vec{z}) = \sum_i c_i \, d\log(f_{1,i}) \wedge d\log\left(f_{2,i}\right) \wedge \cdots \wedge d\log(f_{n,i})$$

leading singularities (multi-variate / iterative residues)

Conjecturally, master integrals whose integrands admit such a ***d*log-form** with **constant leading singularities** (numbers!) evaluate to pure functions.

[Arkani Hamed et al `10; Henn `13; Henn Mistlberger Smirnov Wasser `20]

⇨ in many cases sufficient to find a canonical basis

Integrand / Leading Singularity Analysis

In the elliptic case, examples indicate that the conjecture can be generalised for integrands that look as follows:

$$\prod_{i=1}^{n} dx_i \;\; \mathcal{F}(x_i, \vec{z}) = \sum_i c_i \, d\log(f_{1,i}) \wedge \cdots \wedge d\mathcal{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix}, x_j; \vec{a}\right) \wedge \cdots \wedge d\log(f_{n,i})$$

kernel of eMPLs for constant $\vec{a}$

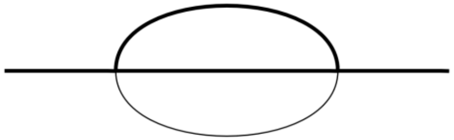
differential of 1st kind

$$d\mathcal{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix}, x_j; \vec{a}\right) = \frac{dx_j}{\sqrt{P_4(x_j)}}\,, \qquad P_4(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4), \qquad \vec{a} = (a_1, a_2, a_3, a_4)$$

Can't continue to insist on single poles: to find second candidate **double poles** are required.

⇨ **one good initial integral** and can work with its **derivative basis**

Another step up in complexity



One internal mass **different** from the other two (all non-zero): **extra master integral**!

$$\frac{\partial}{\partial m_2^2}\vec{I} = \left[\begin{pmatrix} 0 & 1 & 0 \\ f_1(s,m_1^2,m_2^2) & f_2(s,m_1^2,m_2^2) & 0 \\ -2 & 0 & 0 \end{pmatrix} + \mathcal{O}(\epsilon)\right]\vec{I}$$

Internal mass appearing twice

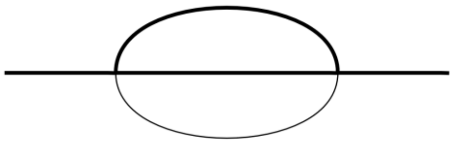
Solution at $\epsilon = 0$, requires **new function**: integral over the solution for the first integral!

$$W = \begin{pmatrix} \omega_0 & \omega_1 & 0 \\ \partial_{m_2^2}\,\omega_0 & \partial_{m_2^2}\,\omega_1 & 0 \\ G_0 & G_1 & 1 \end{pmatrix}, \qquad G_i \equiv -2\int dm_2^2\,\omega_i$$

Integrand analysis: new master integral has **extra residue** (differential of the **3rd kind**)

⇨ can indeed be written in terms of a complete **elliptic integral of the third kind**!

Another step up in complexity



Perform the splitting of W **block-by-block** (minimal irreducible complexity)

$$W_u = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \frac{\partial}{\partial \tau} W_u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} W_u$$

$$\vec{J} = T_{td} \cdot \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \cdot W_{ss}^{-1} \, \vec{I} \qquad \Rightarrow \qquad d\vec{J} = \epsilon \, GM(s, m_1^2, m_2^2) \, \vec{J}$$

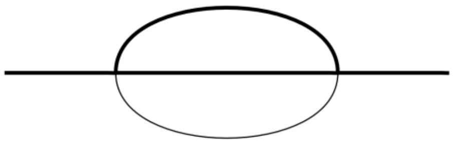
Not only total derivatives of rational functions and ω_0

introduces G_0 into the problem

contains ω_0 and $\partial_{m^2} \, \omega_0$

only ω_0 and G_0 appear, $\partial_{m^2} \, \omega_0$ does not!

Alternative: generalised splitting



Another approach leads to the **exact same result**: define a **generalised splitting** for W :

$$W_u^b = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & G_1 - \tau\, G_0 & 1 \end{pmatrix}, \qquad \frac{\partial}{\partial \tau} W_u^b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -G_0 & 0 \end{pmatrix} W_u^b$$

$$\vec{J} = T_{td} \cdot \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \cdot W_{ss}^{-1}\, \vec{I} \qquad \Rightarrow \qquad d\vec{J} = \epsilon\, GM(s, m_1^2, m_2^2)\, \vec{J}$$

removes total derivatives including also G_0

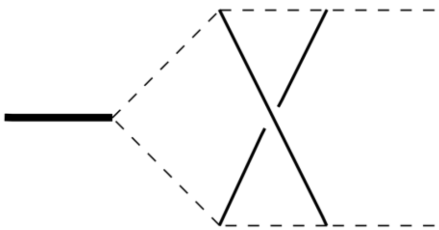
G_i and ω_i have **same weight**!

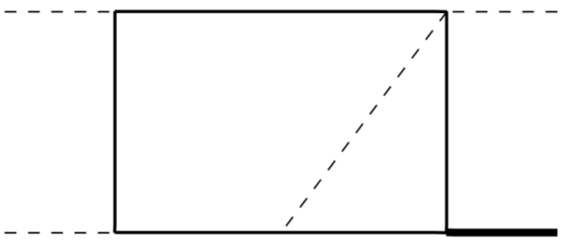
contains ω_0 , $\boldsymbol{G_0}$ and $\partial_{m^2}\, \omega_0$

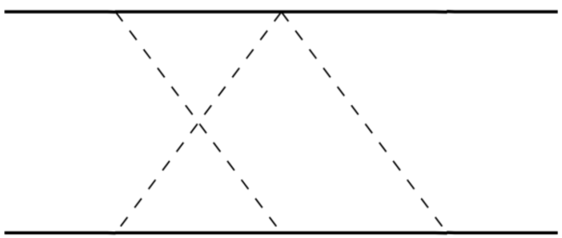
same result as with block-by-block approach

More examples from this category

Gauss-Manin connections correspond to the homogeneous system at $\epsilon =$ (derivative with respect to the internal mass squared).







$$GM = \begin{pmatrix} 0 & 1 & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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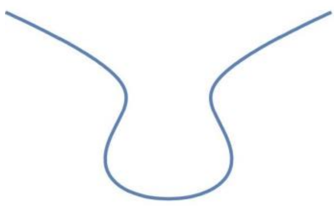
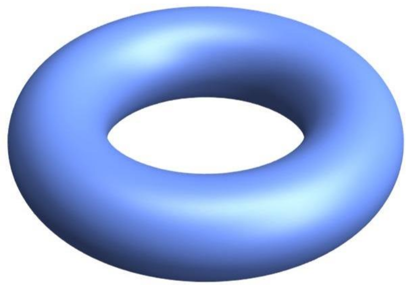
$*$ denotes rational (algebraic) functions

Going beyond (e)MPLs

Not all Feynman integral families evaluate to (e)MPLs. With increasing complexity, special functions defined on even more complicated geometries are required.



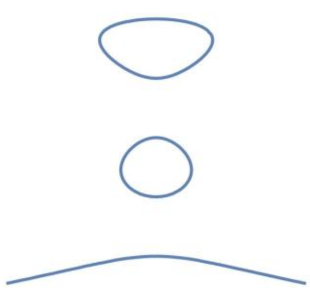
Elliptic Curve / Torus
(genus 1, complex 1D)



higher genus

generalisations

higher dimensional varieties



Hyperelliptic curves, …

Calabi-Yau manifolds

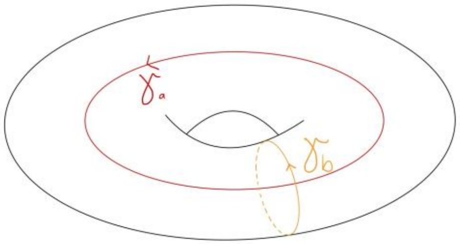
How does our previous idea of a canonical differential equation generalise further?

[Pögel Wang Weinzierl `22]

Calabi-Yau Manifolds: Collection of relevant facts

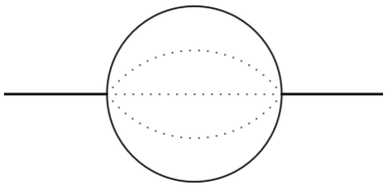
- A ***Calabi-Yau n-fold*** of complex dimension n is a compact Kähler manifold with a unique holomorphic $(n, 0)$-form that vanishes nowhere.
- can be defined through **polynomial constraints**.
- elliptic curves are Calabi-Yau 1-folds
- shape and properties described by ***period integrals***:

$$\begin{aligned} \omega: \quad H_n \times H_{dR}^n \quad &\to \quad \mathbb{C} \\ (\gamma, \alpha) \quad &\mapsto \quad \omega_{\gamma,\alpha} = \int_\gamma \alpha \end{aligned}$$



- ***period matrix*** $\Pi_{ij} = \omega_{\gamma_i,\alpha_j}$ is constructed from the independent cycles and differential forms. For one-parameter families of Calabi-Yaus parametrised by a variable z, the period matrix $\Pi(z)$
  - satisfies **linear differential equations**: $d\Pi(z) = GM(z)\,\Pi(z)$
  - takes on a **hierarchical logarithmic structure** at a point of *maximal unipotent monodromy (MUM point)*
  - obeys **quadratic relations** (*Griffiths transversality conditions*)

Banana Graphs: the Calabi-Yau case



Perfect playground: banana graph integral families with equal, non-zero internal masses m^2.
An $(n+1)$-loop banana graph corresponds to a one-parameter family of n-dim. Calabi-Yaus.

$$\frac{\partial}{\partial m^2}\vec{I}^{(n+1)}_{ban} = \left[\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ * & * & * & * & * \end{pmatrix} + \mathcal{O}(\epsilon)\right]\vec{I}^{(n+1)}_{ban}$$

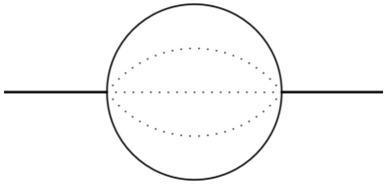
derivative basis
(tadpole neglected)

Calabi-Yau operator

$$W = \begin{pmatrix} \omega_0 & \omega_1 & \cdots & \omega_n \\ \partial_{m^2}\,\omega_0 & \partial_{m^2}\,\omega_1 & \cdots & \partial_{m^2}\,\omega_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{(n)}_{m^2}\,\omega_0 & \partial^{(n)}_{m^2}\,\omega_1 & \cdots & \partial^{(n)}_{m^2}\,\omega_n \end{pmatrix}$$

Wronskian constructed from **Calabi-Yau period integrals**

Banana Graphs: the Calabi-Yau case



Calabi-Yau period integrals satisfy a known **unipotent differential equation**

lower triangular

upper triangular with unit diagonal

$$W = W_{ss} \cdot W_u \,, \qquad \tau = \frac{\omega_1}{\omega_0} \,,$$

single-logarithmic

holomorphic

$$\frac{\partial}{\partial \tau} W_u = GM_{CY} \cdot W_u$$

$$GM_{CY} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & Y_1 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & Y_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & Y_2 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & Y_1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Y-invariants

Banana Graphs: the Calabi-Yau case

Fabian Wagner (TUM) | ϵ-factorised differential equations beyond polylogarithms | ETH Zürich, September 5th, 2023

The semi-simple part W_{ss} can be simplified using ***Griffiths transversality conditions***

$$Z = W \cdot \Sigma \cdot W^T$$

algebraic functions

intersection form

In derivative basis:

$$\Sigma = \begin{pmatrix} \iddots & 0 & 0 & 1 \\ \iddots & 0 & -1 & 0 \\ \iddots & 1 & 0 & 0 \\ \iddots & \iddots & \iddots & \iddots \end{pmatrix}$$

Construction of the basis transformation to an ϵ-factorised form in principle as before:

$$\vec{J}_{ban}^{(n+1)} = T_{nf} \cdot T_{td} \cdot \begin{pmatrix} \epsilon^n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \epsilon & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \cdot W_{ss}^{-1}\, \vec{I}_{ban}^{(n+1)} \quad \Rightarrow \quad d\vec{J}_{ban}^{(n+1)} = \epsilon\, GM_{ban}^{(n+1)}(s, m^2)\, \vec{J}_{ban}^{(n+1)}$$

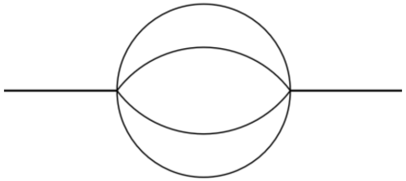
[Pögel Wang Weinzierl `22]

ϵ-rescalings reflect hierarchical logarithmic structure of periods (at *MUM point*)

<u>New step</u>: require **new functions** which are **(iterated) integrals** over the **periods**, the $\boldsymbol{Y}$-**invariants** and **algebraic** functions. Originate from the $\boldsymbol{\mathcal{O}(\epsilon)}$**-terms**!

Example: 3-loop banana

$z = m^2/s$



$$GM_{ban}^{(3)}(z) = \begin{pmatrix} \ast + \ast\,\dfrac{G_2}{\omega_0} & \ast\,\dfrac{1}{\omega_0} & 0 \\ \ast\,\omega_0 + \ast\,\dfrac{G_2^2}{\omega_0} & \ast + \ast\,\dfrac{G_2}{\omega_0} & \ast\,\dfrac{1}{\omega_0} \\ \ast\;\omega_0^2 + \ast\,G_2\,\omega_0 + \ast\,\dfrac{G_2^3}{\omega_0} & \ast\,\omega_0 + \ast\,\dfrac{G_2^2}{\omega_0} & \ast + \ast\,\dfrac{G_2}{\omega_0} \end{pmatrix}$$

$$G_2(z) = \int dz\, \frac{G_1(z)}{\sqrt{(1-4z)(1-16z)}\,\omega_0(z)}$$

$$G_1(z) = \int dz\, \frac{2(1-8z)(1+8z)^3\omega_0^2(z)}{z^2\,(1-4z)^2\,(1-16z)^2}$$

$\ast$ denotes algebraic functions

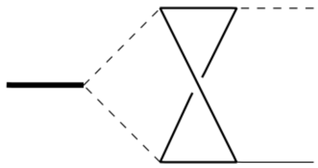
Overview of the procedure: the general recipe

Work **bottom-up**, **sector-by-sector** and employ for **every sector** the following strategy, which leads to a sequence of rotations on the initial basis:

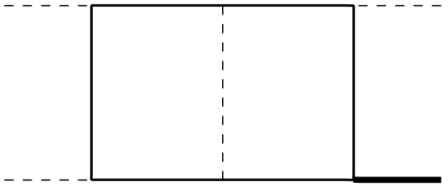
1. Choose a starting basis **free of double poles** in the UV and IR and perform **rational (algebraic)** basis transformations such that the differential equations are for $\epsilon = 0$ manifestly put into **minimal irreducible blocks.** For each block, choose the first integral such that its integrand corresponds to the **holomorphic differential of the first kind** on the underlying geometry. Fill up the basis with its **derivatives**. [This uses the information from integrand analysis and the study of the maximal cuts].
2. Construct the **Wronskian**, split it into a **unipotent** and **semi-simple** part and rotate away the latter.
3. Perform **ϵ-rescalings** to match transcendental weights (afterwards, non-ϵ-factorised terms appear frequently only below the diagonal of the Gauss-Manin connection)
4. Remove any remaining undesired terms (including **subsector contributions**)
   a. by integrating out **total derivatives** of functions already present
   b. by introducing **new functions** ((iterated) integrals of functions already present)

<u>Conjecture:</u> following these steps, it is possible to find a (generalization of the) **canonical** form for **any** Feynman integral family!

More examples (last one beyond one Calabi-Yau)



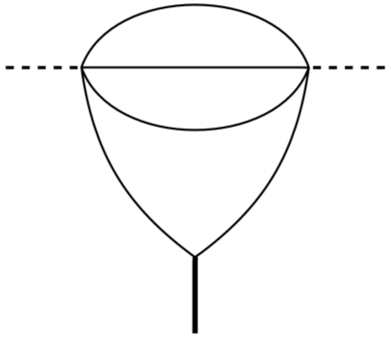
new function required by **subsector contributions**



topsector is not elliptic, but a next-to-top sector is

new functions through elliptic sector and subsector contributions

[Bonciani Duca Frellesvig Henn Moriello Smirnov `16]



topsector contains two coupled elliptic curves

[Duhr Klemm Nega Tancredi `22]

Summary and Outlook

- Proposal of a procedure conjectured to derive generalized canonical differential equations for **any** Feynman integral family with any number of scales almost algorithmically

- Critical step: splitting of Wronskian (utilize property of **unipotence**)

- Still, many open questions remain:
  - Can the **integrand analysis** be generalised further beyond the polylogarithmic case? Would help to find a good initial basis.
  - How does the procedure work in **more complicated cases**? Are further steps needed?
  - What are the **properties** of the **new functions**? What's the resulting **function space**? Are there **non-trivial relations** among the resulting iterated integrals?
  - How do these new functions contribute to an **actual physics problem**?

Thank you for your attention!