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Mathematical Structures in Massive Operator Matrix Elements

Elliptics & Beyond 2023

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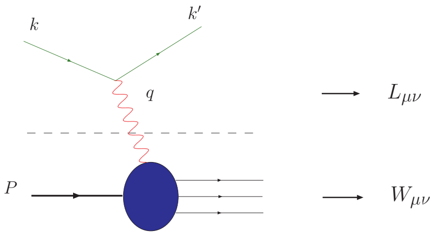
Theory of Deep Inelastic Scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{em}(\xi), J_\nu^{em}(\xi) \right] | P \rangle$$
$$= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2)$$
$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot P S^\sigma - q \cdot S P^\sigma \right)}{(q \cdot P)^2} g_2(x, Q^2)$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Factorization of the Structure Functions

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At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x)\otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x-yz) f(y) g(z)\,.$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x)\,.$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right).$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)+\mathcal{O}\left(\frac{m^2}{Q^2}\right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle$$

$$O_q^{\rm S}=i^{N-1}{\rm S}\left[\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\dots D_{\mu_N}\psi\right]-\text{trace terms},$$
$$O_g^{\rm S}=2i^{N-2}{\rm SSp}\left[F^a_{\mu_1\alpha}D_{\mu_2}\dots F^{\alpha,a}_{\mu_N}\right]-\text{trace terms}$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME Calculations

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Leading Order: [Witten (1976); Babcock, Sivers, Wolfram (1978); Shifman, Vainshtein, Zakharov (1978); Leveille, Weiler (1979); Glück, Reya (1979); Glück, Hoffmann, Reya (1982)]

Next-to-Leading Order:

full m dependence (numeric) [Laenen, van Neerven, Riemersma, Smith (1993)]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven (1996)]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein (2007)]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein (2008, 2009)]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments (using `MATAD` [Steinhauser (2000)]):
  - F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein (2009)]
  - transversity: $N = 1...13$
  - Two masses $m_1 \neq m_2 \rightarrow$ Moments N = 2,4,6 [Blümlein, Wißbrock (2011)]
- Analytic solutions for $A_{qq,Q}^{NS}$, $A_{qg,Q}$, $A_{gq,Q}$, $A_{qq,Q}^{PS}$, A_{Qq}^{PS} [Blümlein et al (2010-2023)] , with recent extension to polarized scattering.
- Analytic two mass solutions for $A_{qq,Q}^{NS}$, $A_{qg,Q}$, $A_{gq,Q}$, $A_{qq,Q}^{PS}$, A_{Qq}^{PS}, $A_{gg,Q}$ [Blümlein et al (2017-2020)] , with recent extension to polarized scattering.

The Wilson Coefficients at Large Q^2

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$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]$$

$$\begin{aligned}L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.\\ &\quad\left.+ A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]\end{aligned}$$

$$\begin{aligned}H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]\\ &\quad+ a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right]\end{aligned}$$

$$\begin{aligned}H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]\\ &\quad+ a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]\\ &\quad+ a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.\\ &\quad\left.+ A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]\end{aligned}$$

The Variable Flavor Number Scheme

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- Matching conditions for parton distribution functions:

$$f_k(N_F+2)+f_{\overline{k}}(N_F+2)=A_{qq,Q}^{NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$f_Q(N_F+2)+f_{\overline{Q}}(N_F+2)=A_{Qq}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$\Sigma(N_F+2)=\left[A_{qq,Q}^{NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qq}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F)\,,$$

$$G(N_F+2)=A_{gq,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,.$$

Massive Operator Matrix Elements

Technical aspects

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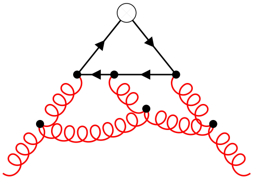
- The diagrams are given by propagators with operator insertions..
- To deal with the operators we can resum them into propagator structures:

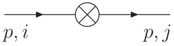
$$(\Delta.k)^N \to \sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t\,\Delta.k}$$

$$\sum_{j=0}^{N} (\Delta.k_1)^j (\Delta.k_2)^{N-j} \to \sum_{N\geq 0, j\leq N}^{\infty} t^N (\Delta.k_1)^j (\Delta.k_2)^{N-j} = \frac{1}{[1 - t\,\Delta.k_1][1 - t\,\Delta.k_2]}$$

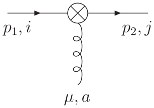


- With the linear propagators we use IBP reductions.
- We can derive a system of differential equations in t.

Operators induce additional Feynman rules, e.g.:



$$\delta^{ij} \not{\Delta} \gamma_{\pm} (\Delta \cdot p)^{N-1}$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}$$

Relation between the different spaces

$$\hat{f}(t) = \sum_{N=1}^{\infty} t^N \tilde{f}(N)$$

$$\updownarrow$$

$$\tilde{f}(N) = \int_0^1 \mathrm{d}x \, x^{N-1} f(x)$$

$$\updownarrow$$

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-N} \tilde{f}(N) \mathrm{d}N$$

- $\hat{f}(t) \to \tilde{f}(N)$ and $\hat{f}(x) \to \tilde{f}(N)$: calculable via recurrence equations
- $\tilde{f}(N) \to f(x)$: calculable via differential equations
- algorithms implemented in public packages `Sigma` [Scheider ('07-)] and `HarmonicSums` [Ablinger et al. ('10-)]

but: algorithmic solution only possible if recurrences or differential equations factorize to first order

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Are $\hat{f}(t)$ and $f(x)$ directly connected?

Inverse Mellin transform via analytic continuation

[based on: Behring, Blümlein, Schönwald (JHEP (2023))]

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$$\hat{f}(t) = \sum_{N=1}^{\infty} \tilde{f}(N) t^N = \sum_{N=1}^{\infty} \int_0^1 \mathrm{d}x' \; t^N {x'}^{N-1} f(x') = \int_0^1 \mathrm{d}x' \; \frac{t}{1 - tx'} f(x')$$

Setting $t = \frac{1}{x}$ we obtain:

$$\hat{f}\left(\frac{1}{x}\right) = \int_0^1 \mathrm{d}x' \frac{f(x')}{x - x'}$$

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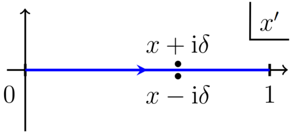

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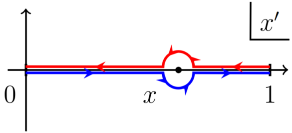
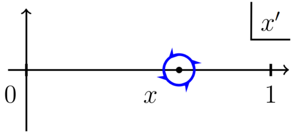


Therefore:

$$f(x) = \frac{i}{2\pi} \lim_{\delta \to 0} \oint_{|x - x'| = \delta} \frac{f(x')}{x - x'} = \frac{i}{2\pi} \operatorname*{Disc}_x \hat{f}\left(\frac{1}{x}\right)$$



Inverse Mellin transform via analytic continuation

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The discussion before used some implicit assumptions.
The x-space representation

1. has no $(-1)^N$ term.
2. is regular and has now contributions from distributions.
3. has a support only on $x \in (0, 1)$.

Inverse Mellin transform via analytic continuation

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For physical examples:

$$\tilde{f}(N) = \int_0^1 \mathrm{d}x\, x^{N-1} \left[f(x) + (-1)^N g(x) + \left(f_\delta + (-1)^N g_\delta\right) \delta(1-x)\right] + \int_0^1 \mathrm{d}x\, \frac{x^{N-1} - 1}{1-x}, \left[f_+(x) + (-1)^N g_+(x)\right]$$

Inverse Mellin transform via analytic continuation

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All of this can be lifted, but the discussion is more involved.

First order factorizable sector – The function spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on ${}_2F_1$ functions

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras

All other ones stem from 1st order factorizable equations.

First order factorizable sector – $A_{gg,Q}$ as an example

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- $A_{gg,Q}$ is an important build block for the variable flavor number scheme.
- We find much more involved analytical structures than in the massless case:
  - Binomially weighted sums in Mellin space, e.g.:

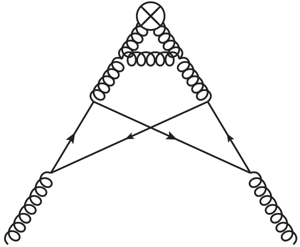
$$\mathrm{BS}_3(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}\left(2\tau_1\right)!}{\left(\tau_1!\right)^2 \tau_1}, \qquad \mathrm{BS}_8(N) = \sum_{\tau_1=1}^{N} \frac{\sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}\left(\tau_2!\right)^2}{\left(2\tau_2\right)!\tau_2^2}}{\tau_1}$$

  - Iterated integrals over square root valued letters in x-space, i.e. over the alphabet:

$$\left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \sqrt{x(1-x)}\right\}$$

  - The (inverse) Mellin transformations can be calculated analytically with `HarmonicSums`:

$$\mathbf{M}^{-1}[\mathrm{BS}_8(N)](x) = \left[-\frac{4\left(1-\sqrt{1-x}\right)}{1-x} + \left(\frac{2(1-\ln(2))}{1-x} + \frac{\mathrm{H}_0(x)}{\sqrt{1-x}}\right)\mathrm{H}_1(x) - \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}} + \frac{\mathrm{H}_1(x)}{2(1-x)}\int_0^x \frac{\mathrm{H}_0(x)}{\sqrt{1-x}}\mathrm{d}x - \frac{1}{2(1-x)}\int_0^x \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}}\mathrm{d}x\right]_+$$



Small and Large x Limits of $a_{gg,Q}^{(3)}$

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- We considered unpolarized and polarized scattering.
- The analytic results allow to obtain the small and large x expansion analytically.
- Despite the iterated integrals over square roots only well known constants occur in both expansions.
- We provide deep expansions (up to 50th order) for easy numerical evaluation.
- The x-space of some diagrams has been obtained via analytic continuation from t-space.

$$
\begin{aligned}
&a_{gg,Q}^{x\to 0}(x) \propto \\
&\frac{1}{x}\Biggl\{\ln(x)\Biggl[C_A^2 T_F\left(-\frac{11488}{81}+\frac{224\zeta_2}{27}+\frac{256\zeta_3}{3}\right)+C_A C_F T_F\left(-\frac{15040}{243}-\frac{1408\zeta_2}{27}\right.\\
&\left.-\frac{1088\zeta_3}{9}\right)\Biggr]+C_A T_F^2\left[\frac{112016}{729}+\frac{1288}{27}\zeta_2+\frac{1120}{27}\zeta_3+\left(\frac{108256}{729}+\frac{368\zeta_2}{27}-\frac{448\zeta_3}{27}\right)\right.\\
&\left.\times N_F\right]+C_F\left[T_F^2\left(-\frac{107488}{729}-\frac{656}{27}\zeta_2+\frac{3904}{27}\zeta_3+\left(\frac{116800}{729}+\frac{224\zeta_2}{27}-\frac{1792\zeta_3}{27}\right)N_F\right)\right.\\
&\left.+C_A T_F\left(-\frac{5538448}{3645}+\frac{1664\mathrm{B}_4}{3}-\frac{43024\zeta_4}{9}+\frac{12208}{27}\zeta_2+\frac{211504}{45}\zeta_3\right)\right]\\
&+C_A^2 T_F\left(-\frac{4849484}{3645}-\frac{352\mathrm{B}_4}{3}+\frac{11056\zeta_4}{9}-\frac{1088}{81}\zeta_2-\frac{84764}{135}\zeta_3\right)\\
&+C_F^2 T_F\left(\frac{10048}{5}-640\mathrm{B}_4+\frac{51104\zeta_4}{9}-\frac{10096}{9}\zeta_2-\frac{280016}{45}\zeta_3\right)\Biggr\}\\
&+\left[-\frac{4}{3}C_F C_A T_F+\frac{2}{15}C_F^2 T_F\right]\ln^5(x)+\left[-\frac{40}{27}C_A^2 T_F+\frac{4}{9}C_F^2 T_F+C_F\left(-\frac{296}{27}C_A T_F\right.\right.\\
&\left.\left.+\left(\frac{28}{27}+\frac{56}{27}N_F\right)T_F^2\right)\right]\ln^4(x)+\left[\frac{112}{81}C_A(1+2N_F)T_F^2+C_F\left(\left(\frac{1016}{81}+\frac{496}{81}N_F\right)T_F^2\right.\right.\\
&\left.\left.+C_A T_F\left(-\frac{10372}{81}-\frac{328\zeta_2}{9}\right)\right)+C_F^2 T_F\left[-\frac{2}{3}+\frac{4\zeta_2}{9}\right]+C_A^2 T_F\left[-\frac{1672}{81}+8\zeta_2\right]\right]\ln^3(x)\\
&+\left[\frac{8}{81}C_A(155+118N_F)T_F^2+C_F\left[T_F^2\left(-\frac{32}{81}+N_F\left(\frac{3872}{81}-\frac{16\zeta_2}{9}\right)+\frac{232\zeta_2}{9}\right)\right.\right.\\
&\left.+C_A T_F\left(-\frac{70304}{81}-\frac{680\zeta_2}{9}+\frac{80\zeta_3}{3}\right)\right]+C_A^2 T_F\left[\frac{4684}{81}+\frac{20\zeta_2}{3}\right]+C_F^2 T_F\left[56\right.\\
&\left.\left.+\frac{8\zeta_2}{3}-40\zeta_3\right]\right]\ln^2(x)+\left[C_F\left[T_F^2\left(\frac{140992}{243}+N_F\left(\frac{182528}{243}-\frac{400\zeta_2}{27}-\frac{640\zeta_3}{9}\right)\right.\right.\right.\\
&\left.\left.-\frac{728}{27}\zeta_2-\frac{224}{9}\zeta_3\right)+C_A T_F\left(-\frac{514952}{243}+\frac{152\zeta_4}{3}-\frac{21140\zeta_2}{27}-\frac{2576\zeta_3}{9}\right)\right]\\
&+C_A T_F^2\left[\frac{184}{27}+N_F\left(\frac{656}{27}-\frac{32\zeta_2}{27}\right)+\frac{464\zeta_2}{27}\right]+C_A^2 T_F\left[-\frac{42476}{81}-92\zeta_4+\frac{4504\zeta_2}{27}\right.\\
&\left.\left.+\frac{64\zeta_3}{3}\right]+C_F^2 T_F\left[-\frac{1036}{3}-\frac{976\zeta_4}{3}-\frac{58\zeta_2}{3}+\frac{416\zeta_3}{3}\right]\right]\ln(x),
\end{aligned}
$$

Small and Large x Limits of $a_{gg,Q}^{(3)}$

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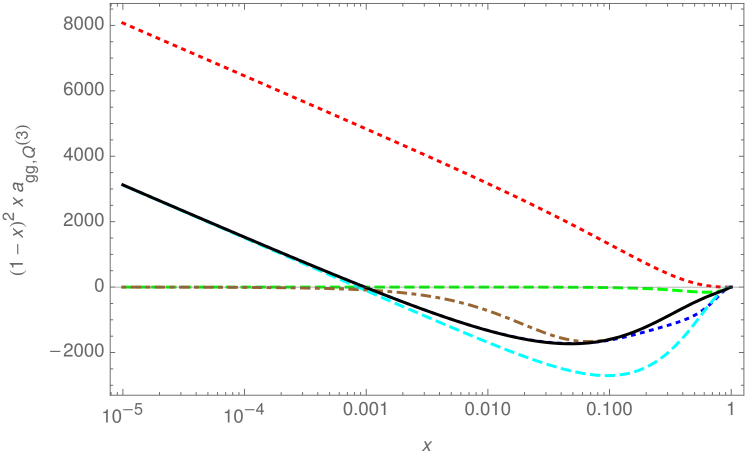
- We considered unpolarized and polarized scattering.
- The analytic results allow to obtain the small and large x expansion analytically.
- Despite the iterated integrals over square roots only well known constants occur in both expansions.
- We provide deep expansions (up to 50th order) for easy numerical evaluation.
- The x-space of some diagrams has been obtained via analytic continuation from t-space.

$$
\begin{aligned}
a_{gg,Q}^{(3),x\to 1}(x) \propto\, & a_{gg,Q,\delta}^{(3)}\delta(1-x)+a_{gg,Q,\text{plus}}^{(3)}(x)+\Bigg[-\frac{32}{27}C_A T_F^2(17+12N_F)+C_A C_F T_F\left(56-\frac{32\zeta_2}{3}\right) \\
&+C_A^2 T_F\left(\frac{9238}{81}-\frac{104\zeta_2}{9}+16\zeta_3\right)\Bigg]\ln(1-x)+\Bigg[-\frac{8}{27}C_A T_F^2(7+8N_F) \\
&+C_A^2 T_F\left(\frac{314}{27}-\frac{4\zeta_2}{3}\right)\Bigg]\ln^2(1-x)+\frac{32}{27}C_A^2 T_F \ln^3(1-x). \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
(\Delta)a_{gg,Q,\delta}^{(3)} = T_F\Bigg\{ & C_F\Bigg[C_A\left(\frac{16541}{162}-\frac{64\mathrm{B}_4}{3}+\frac{128\zeta_4}{3}+52\zeta_2-\frac{2617\zeta_3}{12}\right)+T_F\Bigg(-\frac{1478}{81} \\
&+N_F\left(-\frac{1942}{81}-\frac{20\zeta_2}{3}\right)-\frac{88\zeta_2}{3}-7\zeta_3\Bigg)\Bigg]+C_A^2\Bigg[\frac{34315}{324}+\frac{32\mathrm{B}_4}{3}-\frac{3778\zeta_4}{27} \\
&+\frac{992}{27}\zeta_2+\left(\frac{20435}{216}+24\zeta_2\right)\zeta_3-\frac{304}{9}\zeta_5\Bigg]+C_A T_F\Bigg[\frac{2587}{135}+N_F\left(-\frac{178}{9}+\frac{196\zeta_2}{27}\right) \\
&+\frac{572\zeta_2}{27}-\frac{291\zeta_3}{10}\Bigg]+C_F^2\left[\frac{274}{9}+\frac{95\zeta_3}{3}\right]+\frac{64}{27}T_F^2\zeta_3\Bigg\}, \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
(\Delta)a_{gg,Q,\text{plus}}^{(3)} = \frac{T_F}{1-x}\Bigg\{ & C_A T_F\left[\frac{35168}{729}+N_F\left(\frac{55552}{729}+\frac{160\zeta_2}{27}-\frac{448\zeta_3}{27}\right)+\frac{560}{27}\zeta_2+\frac{1120}{27}\zeta_3\right] \\
&+C_A^2\left[-\frac{32564}{729}-\frac{32\mathrm{B}_4}{3}+104\zeta_4-\frac{3248\zeta_2}{81}-\frac{1796\zeta_3}{27}\right]+C_A C_F\Bigg[-\frac{6152}{27}+\frac{64\mathrm{B}_4}{3} \\
&-96\zeta_4-40\zeta_2+\frac{1208\zeta_3}{9}\Bigg]\Bigg\}. \qquad (4.7)
\end{aligned}
$$



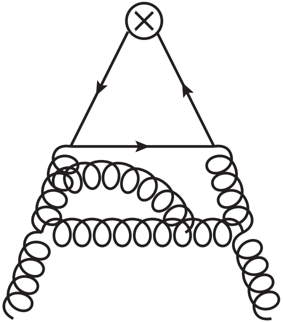
The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$, BFKL limit; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution.

Elliptic Structures in A_{Qg}

First order factorizable contributions

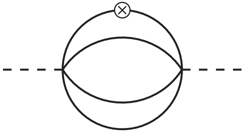
- 468 out of 666 master integrals solved analytically.
- 1009 out of 1233 contributing Feynman diagrams solved.
- Solved via the method of large moments [Blümlein, Schneider (Phys.Lett.B (2017))] : N_F-term, ζ_2, ζ_4 and B_4 terms
- Inverse Mellin transform calculated via analytic continuation of the t-space.
- Alphabet:

$$\left\{\frac{1}{t}, \frac{1}{1+t}, \frac{1}{1-t}, \frac{\sqrt{4+t}}{t}, \frac{\sqrt{4-t}}{t}, \frac{\sqrt{4+t}}{1+t}, \frac{\sqrt{4-t}}{1+t}, \dots\right\}$$



The underlying elliptic sector

$$\frac{d}{dt}\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & -\frac{1}{1-t} & 0 \\ 0 & -\frac{1}{t(1-t)} & -\frac{2}{1-t} \\ 0 & \frac{2}{t(8+t)} & \frac{1}{8+t} \end{bmatrix}\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} + \begin{bmatrix} R_1(t,\varepsilon) \\ R_2(t,\varepsilon) \\ R_3(t,\varepsilon) \end{bmatrix} + O(\varepsilon),$$



$$R_1(t,\varepsilon) = \frac{1}{t(1-t)\varepsilon^3}\left[16 - \frac{68}{3}\varepsilon + \left(\frac{59}{3} + 6\zeta_2\right)\varepsilon^2 + \left(-\frac{65}{12} - \frac{17}{2}\zeta_2 + 2\zeta_3\right)\varepsilon^3\right] + O(\varepsilon),$$

$$R_2(t,\varepsilon) = \frac{1}{t(1-t)\varepsilon^3}\left[8 - \frac{16}{3}\varepsilon + \left(\frac{4}{3} + 3\zeta_2\right)\varepsilon^2 + \left(\frac{14}{3} - 2\zeta_2 + \zeta_3\right)\varepsilon^3\right] + O(\varepsilon),$$

$$R_3(t,\varepsilon) = \frac{1}{12t(8+t)\varepsilon^3}\left[-192 + 8\varepsilon - 8(4 + 9\zeta_2)\varepsilon^2 + (68 + 3\zeta_2 - 24\zeta_3)\varepsilon^3\right] + O(\varepsilon).$$

Homogenous solutions I

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After decoupling for $F_1(t)$ we find the differential equation

$$f_1^{(3)}(t) - \frac{2(4+5t)}{t(1-t)(8+t)} f_1^{(2)}(t) + \frac{4}{t(1-t)(8+t)} f_1^{(1)}(t) = 0$$

with $F_1(t) = f_1(t)/t$ and

We the methods of [Immamoglu, van Hoeij (J.Symb.Comput.(2017))] implemented in `Maple` we find solutions for $f_1^{(1)}(t)$:

$$g_1(t) = \frac{t^2(8+t)^2}{(4-t)^4}\, {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3}\\ 2\end{matrix}; z(t)\right],$$

$$g_2(t) = \frac{t^2(8+t)^2}{(4-t)^4}\, {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3}\\ 2\end{matrix}; 1 - z(t)\right]$$

with

$$z(t) = \frac{27t^2}{(4-t)^3}$$

Other representations

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- A similar solution was found for the analytic calculation of the ρ parameter at 3-loop order: [Ablinger, Blümlein, De Freitas, van Hoeij, Imamoglu (J.Math.Phys.(2018))]

$$\psi_{1a}^{(0)}(x) = \frac{x^2(x^2-1)(x^2-9)^2}{(x^2+3)^4} {}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{5}{3}\\2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$
$$\sim -(x-1)(x-3)(x+3)^2\sqrt{\frac{x+1}{9-3x}}\mathrm{K}\left(-\frac{16x^3}{(x+1)(x-3)^3}\right)$$
$$+(x^2+3)(x-3)^2\sqrt{\frac{x+1}{9-3x}}\mathrm{E}\left(-\frac{16x^3}{(x+1)(x-3)^3}\right)$$

- In [Abreu, Becchetti, Duhr, Marzucca (JHEP (2022))] it was shown that a representation in terms of eMPLs and iterated Eisenstein integrals exists.

Homogeneous solutions II

- When decoupling for F_3 first, we find:

$$F_1'(t) + \frac{1}{t} F_1(t) = 0, \quad g_0 = \frac{1}{t}$$

$$F_3''(t) + \frac{(2-t)}{(1-t)t} F_3'(t) + \frac{2+t}{(1-t)t(8+t)} F_3(t) = 0,$$

with

$$\begin{aligned}
g_1(t) &= \frac{2}{(1-t)^{2/3}(8+t)^{1/3}} {}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{4}{3} \\ 2\end{matrix}; -\frac{27t}{(1-t)^2(8+t)}\right], \\
g_2(t) &= \frac{9\sqrt{3}\Gamma^2(1/3)}{8\pi} \frac{1}{(1-t)^{2/3}(8+t)^{1/3}} {}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{4}{3} \\ \frac{2}{3}\end{matrix}; 1+\frac{27t}{(1-t)^2(8+t)}\right], \\
W(t) &= \frac{1-t}{t^2}
\end{aligned} \tag{1}$$

Full solution

- Once the homogenous solutions are found, we can obtain the full solution by variation of constants.
- E.g. we find:

$$F_3(t) = \frac{1}{\epsilon^2}\left[\frac{10}{3} - \frac{t}{6}\right] + \frac{1}{\epsilon}\left[-\frac{31}{6} + \frac{3t}{8} - \left(\frac{1}{3} - \frac{1}{6t} - \frac{t}{6}\right) H_1(t)\right] + \left[\frac{3}{4}\ln(2) g_1(t) + \frac{1}{12}(10 + \pi(-3i + \sqrt{3})) g_1(t) \right.$$
$$- \frac{g_2(t)}{3} + \frac{25}{54}\left[g_1(t)G(13;t) - g_2(t)G(7;t)\right] + \frac{28}{27}\left[g_2(t)G(8;t) - g_1(t)G(14;t)\right]$$
$$\left. + \frac{1}{3}\left[g_1(t)G(16;t) - g_2(t)G(10;t)\right]\right]\zeta_2 + \ldots$$

with the alphabet:

$$A = \{1, 2, \ldots, 17\} = \left\{\frac{1}{t}, \frac{1}{1-t}, \frac{1}{8+t}, g_1, g_2, \frac{g_1}{t}, \frac{g_1}{1-t}, \frac{g_1}{8+t}, \frac{g_1'}{t}, \frac{g_1'}{1-t}, \frac{g_1'}{8+t}, \frac{g_2}{t}, \frac{g_2}{1-t}, \frac{g_2}{8+t}, \frac{g_2'}{t}, \frac{g_2'}{1-t}, \frac{g_2'}{8+t}, t\,g_1, t\,g_2\right\}$$

$G(w_1, \vec{w}; t) = \int_0^t \mathrm{d}t'\, A_{w_1}(t') G(\vec{w}; t')$, with the usual regularization at $t = 0$ understood implicitly

Analytic continuation

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General idea:

- Evaluate a series expansion around a potential singular point, e.g. $t = 1^-$.

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Prositives:

- We find exact integral representations.
- The boundary conditions for the new region are 'analytic'.
- We can extract 'analytic' series expansions.

Negatives:

- The letters of the iterated integrals have more singularities than we expect from the physical amplitude.
- We have to introduce a new set of constants for each step in the analytic continuation.
- At high weight the constants can be hard to evaluate numerically.

Solutions

- After the analytic continuation we find exact integral representations.
- For numerical evaluation it is often simpler to consider expansions. We considered expansions around $x = 0, 1/2, 1$ to find precise results for $x \in (0,1)$.
- The accuracy of the expansion coefficients depends on the numerical evaluation of the integral representations, e.g.

$$\int_0^1 \frac{g_1(t)}{8+t} \mathrm{Li}_2(t) \mathrm{d}t = 0.06619...$$

- Around $x = 0$ we can use `PSLQ` to reconstruct the analytic expansions:

$$F_1^{(0)}(x) = \frac{1}{x}\left(-\frac{1}{6} - \frac{3}{4}\ln(x)\right) + \frac{11}{4} - \frac{3}{4}\zeta_2 + \frac{29}{6}\ln(x) + \frac{5}{4}\ln^2(x)$$
$$+ x\left(-\frac{113}{16} - \frac{27}{8}\zeta_2 + 5\zeta_3 + \left[\frac{83}{24} + \frac{3}{2}\zeta_2\right]\ln(x) - \frac{3}{8}\ln^2(x) - \frac{5}{6}\ln^3(x)\right) + ...$$

Summary and Outlook

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Summary

- Massive operator matrix elements are important for the interretation of DIS precision data, the determination of parton distribution functions, and therefore LHC phenomenology.
- All 1st order factorizing cases have been calculated.
- At 3-loop order the OME A_{Qg} depends on two elliptic sectors.
- We proposed a method how to obtain the x-space representation directly from the analytic results in the resummation variable t.
- The master integrals can be expressed as iterated integrals over kernels which depend on Gauss-Hypergeometric ${}_2F_1$ functions.

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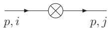
Outlook

- 192 master integrals depend on the elliptic sectors via the inhomogenous terms.
- Functional relations between the different iterated integrals (and their special values) have to be studied further.

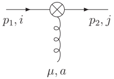
Backup

Calculation of the 3-loop Operator Matrix Elements

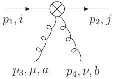
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



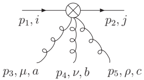
$$\delta^{ij}\not\Delta\gamma_\pm(\Delta\cdot p)^{N-1}\,,\quad N\geq 1$$



$$g t^a_{ji}\Delta^\mu\not\Delta\gamma_\pm\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\,,\quad N\geq 2$$



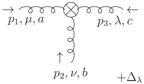
$$g^2\Delta^\mu\Delta^\nu\not\Delta\gamma_\pm\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}$$
$$\Big[(t^at^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^bt^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\,,$$
$$N\geq 3$$



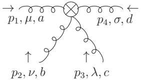
$$g^3\Delta_\mu\Delta_\nu\Delta_\rho\not\Delta\gamma_\pm\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}$$
$$\Big[\ (t^at^bt^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^at^ct^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^bt^at^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^bt^ct^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}$$
$$+(t^ct^at^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^ct^bt^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\,,$$
$$N\geq 4$$

$$\gamma_+=1\,,\quad\gamma_-=\gamma_5\,.$$

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\,,\quad N\geq 2$$



$$-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg($$
$$\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}$$
$$+\Delta_\lambda\Big[\Delta\cdot p_1p_{2,\mu}\Delta_\nu+\Delta\cdot p_2p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]$$
$$\times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}$$
$$+\left\{{p_1\to p_2\to p_3\to p_1 \atop \mu\to\nu\to\lambda\to\mu}\right\}+\left\{{p_1\to p_3\to p_2\to p_1 \atop \mu\to\lambda\to\nu\to\mu}\right\}\Bigg)\,,\quad N\geq 2$$



$$g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$
$$+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\,,$$
$$O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}$$
$$+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}$$
$$-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}$$
$$+[\Delta\cdot p_1\Delta\cdot p_4g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1p_{4,\mu}\Delta_\sigma]$$
$$\times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}$$
$$-\left\{{p_1\leftrightarrow p_2 \atop \mu\leftrightarrow\nu}\right\}-\left\{{p_3\leftrightarrow p_4 \atop \lambda\leftrightarrow\sigma}\right\}+\left\{{p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4 \atop \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma}\right\}\,,\quad N\geq 2$$

N Space Evaluation

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$$\begin{aligned}
\mathrm{BS}_8(N) - \mathrm{BS}_8(N-1) &= \frac{1}{N}\mathrm{BS}_4(N), \\
\mathrm{BS}_4(N) &= \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}\left(\tau_1!\right)^2}{\left(2\tau_1\right)!\tau_1^2} \\
\mathrm{BS}_8(N) &\propto -7\zeta_3 + \left[+3(\ln(N)+\gamma_E) + \frac{3}{2N} - \frac{1}{4N^2} + \frac{1}{40N^4} - \frac{1}{84N^6} + \frac{1}{80N^8} - \frac{1}{44N^{10}}\right]\zeta_2 \\
&\quad + \sqrt{\frac{\pi}{N}}\left[4 - \frac{23}{18N} + \frac{1163}{2400N^2} - \frac{64177}{564480N^3} - \frac{237829}{7741440N^4} + \frac{5982083}{166526976N^5}\right. \\
&\quad + \frac{5577806159}{438593126400N^6} - \frac{12013850977}{377864847360N^7} - \frac{1042694885077}{90766080737280N^8} \\
&\quad \left. + \frac{6663445693908281}{127863697547722752N^9} + \frac{23651830282693133}{1363413316298342400N^{10}}\right] \qquad (2)
\end{aligned}$$

Representations of the OME

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- The logarithmic parts of $(\Delta)A_{gg}^{(3)}$ have been computed before [Behring et al., (2014)], [Blümlein et al. (2021)].
- ***N* space**
  - Recursions available for all building blocks: $N \to N + 1$.
  - Asymptotic representations available.
  - Contour integral around the singularities of the problem at the non-positive real axis.
- ***x* space**
  - All constants occurring in the transition $t \to x$ can be calculated in terms of ζ-values.
  - This can be proven analytically by first rationalizing and then calculating the obtained cyclotomic harmonic polylogarithms.
  - Separate the $\delta(1 - x)$ and $+$-function terms first.
  - Series representations to 50 terms around $x = 0$ and $x = 1$ can be derived for the regular part analytically (12 digits).
  - The accuracy can be easily enlarged, if needed.

Example of the analytic continuation

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$$\hat{f}_1(t) = H_{0,0,1}(t) = \text{Li}_3(t) \, ,$$
$$\hat{f}_1\left(t = \frac{1}{x}\right) = -2\zeta_2 H_0(x) + \frac{1}{6} H_0^3(x) + H_{0,0,1}(x) + \frac{i\pi}{2} H_0^2(x) \, ,$$
$$\hat{f}_1\left(t = -\frac{1}{x}\right) = \zeta_2 H_0(x) + \frac{1}{6} H_0^3(x) - H_{0,0,-1}(x) \, ,$$
$$f_1(x) = \frac{1}{2} H_0^2 \, ,$$
$$\tilde{f}_1(N) = \frac{1}{N^3}$$