Spin-s Rational Q-system

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Abstract

Bethe ansatz equations for spin-s XXX spin chain (XXX$_s$ chain) with $s \geq 1$ are significantly more difficult to analyze than the spin-$\frac{1}{2}$ case, due to the presence of repeated roots. As a result, it is challenging to derive extra conditions for the Bethe roots to be physical and study the related completeness problem.

In our paper[1], we propose the rational Q-system for the XXX$_s$ chain, which gives all and only physical solutions of the Bethe ansatz equations(BAEs), and the extra conditions for solutions to be physical. Then we prove the completeness of spin-s Bethe ansatz rigorously.

I Introduction

The BAEs for the XXX$_s$ chain of length L with M magnons read

$$\left(\frac{u_j + is}{u_j - is}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad j = 1, \dots, M. \tag{1}$$

There are two types of solutions which need special care. One is the solutions with *repeated roots* and the other are the so-called *singular solutions*. The repeated solutions can take the following form

$$\{\underbrace{u_1, \dots, u_1}_{n_1}, \underbrace{u_2, \dots, u_2}_{n_2}, \dots, \underbrace{u_M, \dots, u_M}_{n_M}\} \tag{2}$$

A generic singular solution can take the following form

$$\{(is)^{m_0}, (i(s-1))^{m_1}, \dots, (-is)^{m_{2s}}, u_1, u_2, \dots, u_{M_r}\} \tag{3}$$

where $\{u_1, \dots, u_{M_r}\}$ are the regular roots. Not all of them are physical, which can be derived by algebraic Bethe ansatz. To get physical solutions, we need other constraints besides BAEs.

The TQ-relation method can eliminate non-physical repeated solutions automatically. Baxter's TQ-relation reads

$$\tau(u)Q(u) = (u + \tfrac{i}{2})^L Q(u - i) + (u - \tfrac{i}{2})^L Q(u + i)\,, \tag{4}$$

where $\tau(u)$, $Q(u)$ are polynomials. The zeros of $Q(u)$ are the Bethe roots $\{u_k\}$. The polynomiality of $\tau(u)$ imposes extra constraints for the Bethe roots. The Q-system method can eliminate non-physical singular solutions automatically.

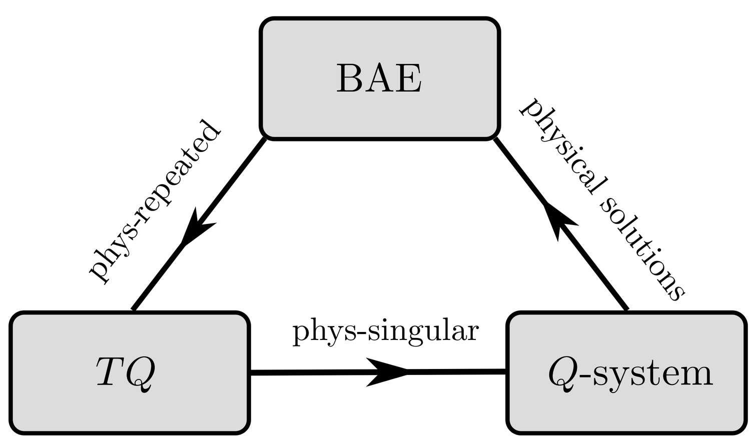

Figure 1: Relations between BAE, TQ and rational Q-system.

II Spin-s rational Q-system

To construct the rational Q-system, we need the following ingredients

1. A Young diagram on which we define Q-functions;
2. QQ-relations which relates the four Q-functions defined on each box;
3. Boundary conditions which specify the Q-functions partially or completely on the left and upper boundary of the Young diagram.

Young diagram The Young diagram has two rows and is given by $(2sL - M, M)$. On each node, we define a function $Q_{a,b}(u)$,

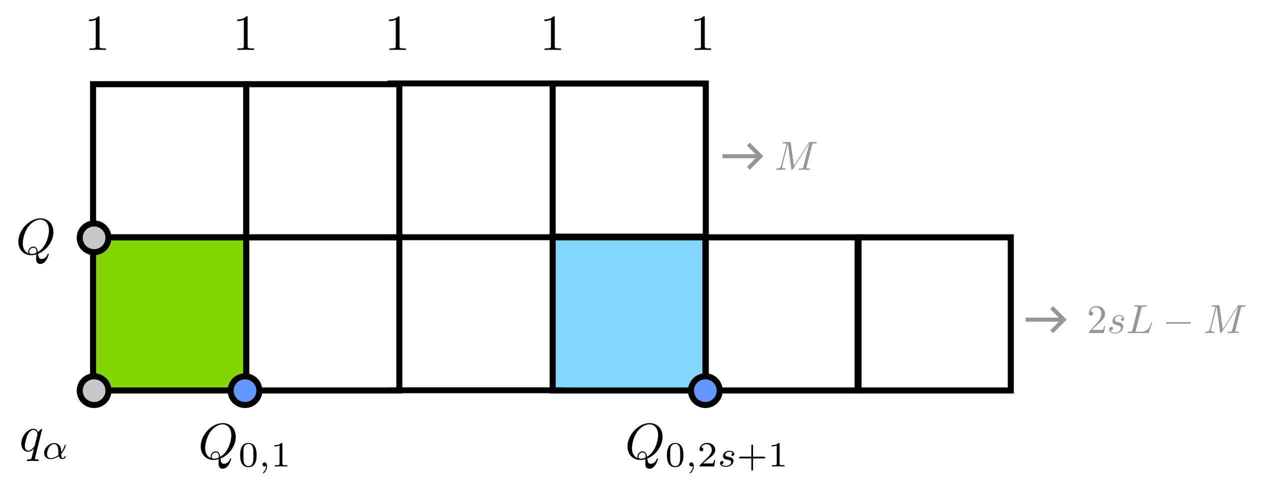

Figure 2: Rational Q-system for XXX$_s$ spin chain.

QQ-relations

$$Q_{a,b+1}Q_{a+1,b} = Q^+_{a,b}Q^-_{a+1,b+1} - Q^-_{a,b}Q^+_{a+1,b+1}\,. \tag{5}$$

Boundary conditions

$$Q_{2,b} = 1, \qquad b = 0, 1, \dots, M\,, \tag{6}$$
$$Q_{1,0} = Q(u),$$
$$Q_{0,0} = q_\alpha(u).$$

Solve the Q-system We make the usual ansatz

$$Q(u) = u^M + \sum_{k=0}^{M-1} c_k u^k\,. \tag{7}$$

We then require $Q_{0,1}/\alpha(u)$ and $Q_{0,2s+1}$ to be polynomials.

Here, $F^{\pm}(u) \equiv F(u \pm \frac{i}{2})$, $q_\alpha(u) \equiv \prod_{k=0}^{2s-1}\left(u - i(s - k - \frac{1}{2})\right)^L$, and $\alpha(u) \equiv \prod_{k=1}^{2s-1}\left(u - i(s-k)\right)^L$. c_ks are undetermined constants and we will obtain them by solving the Q-system.

III Completeness of spin-s Bethe ansatz

All physical Bethe vectors span the Hilbert space. The expected number of physical solutions for a spin-s L-site chain with M magnons is given by[2]

$$\mathcal{N}_s(L, M) = c_s(L, M) - c_s(L, M-1), \tag{8}$$

with

$$c_s(L; M) = \sum_{j=0}^{\lfloor \frac{L+M-1}{2s+1} \rfloor} (-1)^j \binom{L}{j} \binom{L + M - 1 - (2s+1)j}{L-1}. \tag{9}$$

Numerical Checks.

	$\mathcal{N}_{\text{phys}}$
$L=3, M=1, s=1$	2
$L=3, M=2, s=1$	3
$L=3, M=3, s=1$	1
$L=6, M=4, s=1$	40
$L=6, M=5, s=1$	36
$L=6, M=6, s=1$	15
$L=9, M=1, s=1$	8
$L=9, M=2, s=1$	36
$L=9, M=3, s=1$	111
$L=9, M=4, s=1$	258
$L=12, M=3, s=1$	274
$L=12, M=4, s=1$	869

	$\mathcal{N}_{\text{phys}}$
$L=8, M=4, s=\frac{3}{2}$	202
$L=8, M=6, s=\frac{3}{2}$	700
$L=9, M=4, s=\frac{3}{2}$	321
$L=10, M=4, s=\frac{3}{2}$	485
$L=6, M=5, s=2$	120
$L=6, M=6, s=2$	180
$L=7, M=5, s=2$	245
$L=7, M=6, s=2$	420

Table 1: $\mathcal{N}_{\text{phys}}$ denotes the number of all physical solutions obtained from the Q-system of a spin-s L-site chain with M magnons.

Rigorously Proof.

In [3], authors prove if the difference equation (4) about $Q(u)$ has two polynomial solutions $Q(u)$, $P(u)$ such that $deg(Q) < deg(P)$, then the completeness of spin-s Bethe ansatz is valid. In our paper [1], we prove for XXX$_s$ chain, the above premise is valid.

Please read our paper for more details.

IV Conclusions

1. We propose the rational Q-system for spin-s XXX spin chain, which gives all and only physical solutions. Furthermore, the additional constraints for singular and repeated solutions are derived (see [1]).
2. The completeness of spin-s Bethe ansatz is tested numerically and proved rigorously.

References

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