Ground state energy and TBA of the twisted $AdS_3 \times S^3 \times T^4$ superstring

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Wrapping interactions and Ground state

An important further progress in Gauge/Gravity integrability is to exactly solve $AdS_3 \times S^3 \times \mathcal{M}^4$ superstring model, which represents one side of the long standing AdS_3/CFT_2 problem. In the present work, we address **finite-size effects** and generic **Non-BPS Ground State Energy** of the $AdS_3 \times S^3 \times T^4$ superstring, which in general can be equipped with a mixed flux (RR and NSNS 3-form fluxes). It is also known that finite-size corrections are related to **wrapping interactions** that come from the dynamics of the virtual particles (on finite volume). At the same time finite-size spectrum of the lightcone superstring sigma model can be addressed by Thermodynamic Bethe Ansatz. From the investigation of AdS_3 string hypothesis and S-matrix, it is possible to obtain the corresponding Bethe-Yang system and construct TBA for the Mirror model, which constitutes double Wick-rotation of the initial superstring model (Fig.1, 2).

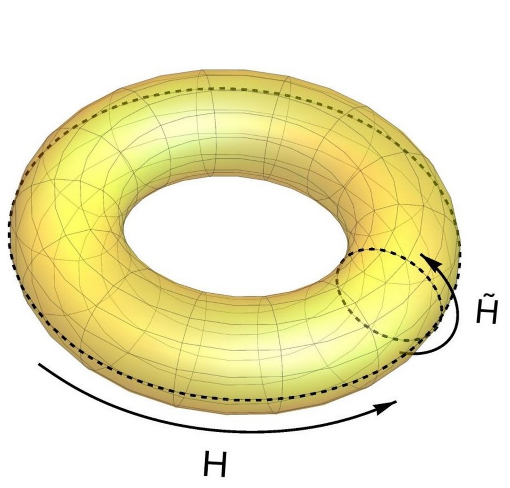


Fig.1: A theory on a torus can be described by the partition function $\mathcal{Z}$ on the circle at finite temperature. Evolution of such theory can be given through either of the cycles, where $p \to -i\tilde{H}$, $H \to i\tilde{p}$.

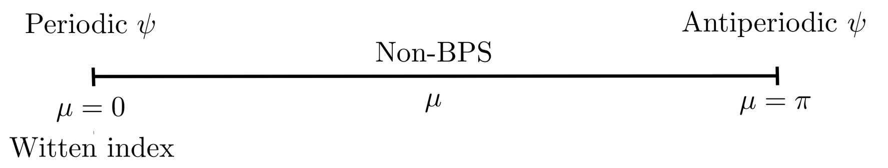
More specifically, it can be shown that mirror partition function $\widetilde{\mathcal{Z}}$ agrees with the initial one

$$\widetilde{\mathcal{Z}} = \sum_k \langle \widetilde{\psi}_k | e^{-L\widetilde{H}} | \widetilde{\psi}_k \rangle = \int \mathcal{D}\widetilde{p}\,\mathcal{D}x\, e^{\int_0^R \mathrm{d}\tau \int_0^L \mathrm{d}\sigma (ipx' - \widetilde{H})} \qquad \widetilde{\mathcal{Z}}(L,R) = \mathcal{Z}(L,R) \tag{1}$$

where in the original model the size is L and R is the inverse temperature, which are swapped in the mirror model. Hence in the infinite volume limit one establishes relation between GSE (original) and bulk free energy (mirror)

$$R \to +\infty : \qquad E(L) = L\widetilde{f}(L) \tag{2}$$

where $\widetilde{f}$ at temperature $1/L$ can be obtained from the Mirror TBA. Fundamentally the AdS_3 MTBA depends on mirror momentum $\widetilde{p}$ (rapidity) and a set of Y-functions: Left/Right chirality particles described by $Y_Q/\bar{Y}_Q$ functions, massless excitations by $Y_0^{(\dot\alpha)}$ and auxiliary ones by $Y_\pm^{(\alpha)}$. As indicated above, by considering non-BPS vacua one must introduce **twist** dependence that would allow parametric flow between even and odd winding sectors



(a) GSE twist interpolation between supervacuum and Non-BPS sector

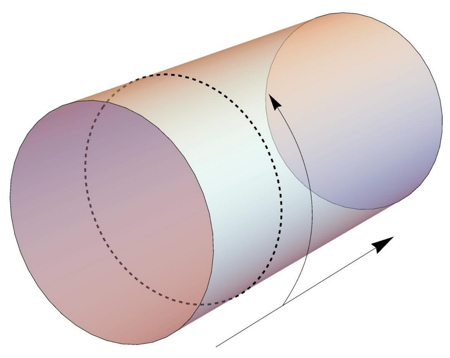


Fig.2: Space and time are interchanged in the Mirror Model $\sigma \to \widetilde{\tau} = -i\sigma$ and $\tau \to \widetilde{\sigma} = i\tau$, and it forms the mirror counterpart of the decompactifying zero-temperature limit, which corresponds to the theory in infinite volume and finite temperature.

In what follows, we shall address finite-size effects and GSE of $AdS_3 \times S^3 \times T^4$ and independently prove it from **Mirror TBA**, **Generalised Lüscher formalism** and **Lightcone superstring** [1].

AdS_3 Mirror TBA

$AdS_3 \times S^3 \times T^4$ GSE From the $AdS_3 \times S^3 \times T^4$ mirror TBA [2, 3] the ground state energy receives contributions from both chiral massive sectors as well as massless excitations

$$E(\mu,h,L) = -\sum_{Q=1}^{+\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi}\frac{d\widetilde{p}^Q}{du} \log\left[(1+Y_Q)\left(1+\overline{Y}_Q\right)\right] - \sum_{\dot\alpha=1}^{N_0} \int_{|u|>2} \frac{du}{2\pi}\frac{d\widetilde{p}^0}{du} \log\left[1+Y_0^{(\dot\alpha)}\right] \tag{3}$$

with twist μ, string tension $h = \frac{\sqrt{\lambda}}{2\pi}$ and lightcone momentum L. In the temporal gauge L is also identified with the charge J associated to the $U(1)$ isometry of S^3 and gets quantised.

Y-ansatz We begin from an analytic structure of the TBA system and its solvability. For instance, the equation for left particles

$$\begin{aligned} -\log Y_Q &= L\widetilde{\mathcal{E}}_Q - \log\left(1+Y_{Q'}\right) \star K^{Q'Q}_{\mathfrak{sl}(2)} - \log\left(1+\overline{Y}_{Q'}\right) \star \widetilde{K}^{Q'Q}_{\mathfrak{su}(2)} - \sum_{\dot\alpha=1,2} \log\left(1+Y_0^{(\dot\alpha)}\right) \check{\star} K^{0Q} \\ &- \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right) \hat{\star} K_+^{yQ} - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right) \hat{\star} K_-^{yQ}, \qquad \widetilde{\mathcal{E}}_Q = 2\,\mathrm{ASh}\left(\frac{\sqrt{(\widetilde{p}^Q)^2 + Q^2}}{2h}\right) \end{aligned} \tag{4}$$

allows to evaluate the leading contributing terms when perturbed in μ, where $\mu_\alpha = (-1)^\alpha \mu$, $\alpha = \{1,2\}$ and K^{ab} are kernels in the appropriate mirror particle sector. The massive mirror energy is spanned by the Q-particle mirror momentum $\widetilde{p}_Q$ or can be also given through Zhukovsky coordinates $\widetilde{\mathcal{E}}_Q = \log x^-/x^+$, and fused shifts $x^\pm = x\left(u \pm iQ/h\right)$ (real particles possess rapidity $\widetilde{p}(u)$, $u \in \mathbb{R}$, whereas massive bound states are on the u-plane with complex momentum and energy). Right $\bar{Y}_Q$ and massless Y_0 equations admit similar analysis.

On the other hand, the coupled system on $\mathbf{y}_\pm$ particles should be taken separately since all the terms appear to contribute at the same level

$$\log Y_+^{(\alpha)} = \log\left(1+\overline{Y}_Q\right) \star K_-^{Qy} - \log\left(1+Y_Q\right) \star K_+^{Qy} - \sum_{\dot\alpha=1,2} \log\left(1+Y_0^{(\dot\alpha)}\right) \check{\star} K^{0y}\,. \tag{5}$$

In this regard, we first consider Y-functions perturbed in μ. Analytic structure and closure of TBA system shows that contribution of each Y-function begins at $\mathcal{O}[\mu^2]$, more detailed in [1].

Mirror TBA solution space

Small μ For the case of small twist, additional analysis on convolutions of massless particles is required, after which TBA can be solved and results in

$$Y_{\{Q,\overline{Q}\}} \approx \mu^2 \left[\frac{x_Q^+}{x_Q^-}\right]^L \qquad Y_0^{(\dot\alpha)} \approx \mu^2 \left[\frac{x_0^+}{x_0^-}\right]^L \qquad \widetilde{\mathcal{E}}_Q = \log\frac{x_Q^-}{x_Q^+}\,. \tag{6}$$

One can now obtain the GSE at μ^2 and arbitrary L in the following form

$$E(\mu,h,L) \approx -\mu^2 \left[\sum_{Q=1}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\widetilde{p}^Q 2e^{-L\widetilde{\mathcal{E}}_Q} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\widetilde{p}^Q 2e^{-L\widetilde{\mathcal{E}}_0}\right] = -\frac{\mu^2}{\pi}\left[\mathcal{I} + \frac{8hL}{4L^2-1}\right], \tag{7}$$

where the massless integral in (7) is analytic for $L > \frac{1}{2}$ and the massive term $\mathcal{I}$ can be transformed into

$$\mathcal{I} = L\sum_{k=0}^{+\infty}(-1)^k 4^{k+L} \frac{\Gamma\left(k+L-\frac{1}{2}\right)\Gamma\left(k+L+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma(k+2L+1)} h^{2k+2L}\zeta(2k+2L-1)\,, \quad \mathcal{I}_{\text{conv}} : L > 1,\ |h| \le \frac{1}{2} \tag{8}$$

Large L Solving TBA in the large L case and arbitrary twist, one acquires

$$E(\mu,h,L) \approx -\frac{4}{\pi}\sin^2\left(\frac{\mu}{2}\right)\left[\sum_{Q=1}^{+\infty}\int_{-\infty}^{+\infty} d\widetilde{p}^Q e^{-L\widetilde{\mathcal{E}}_Q} + \int_{-\infty}^{+\infty} d\widetilde{p}^Q e^{-L\widetilde{\mathcal{E}}_0}\right] = -\frac{4}{\pi}\sin^2\left(\frac{\mu}{2}\right)\left[\mathcal{I} + \frac{8hL}{4L^2-1}\right] \tag{9}$$

which corresponds to replacing the twist factor in (7) by $\mu^2 \to 4\sin^2\left(\frac{\mu}{2}\right)$. This fact nontrivially follows from resolution of the TBA by considering $Y \sim e^{-L\widetilde{\mathcal{E}}_x}$ type ansatz for massless and massive particles. Clearly, (9) recovers (7) for $\mu \ll 1$.

Generalised Lüscher formalism

From the perspective of finite-size corrections to GSE and investigation of multiplet representations, it become possible to consider the Lüscher formalism by introducing massless **deformation** and **twisting**

$$\begin{aligned} E(\mu,h,L) &= -2\sum_{Q=1}^{+\infty}\int_{-\infty}^{\infty}\frac{du}{2\pi}\frac{d\widetilde{p}^Q}{du}e^{-L\widetilde{\mathcal{E}}_Q}\mathfrak{F}_Q - n_0\int_{|u|>2}\frac{du}{2\pi}\frac{d\widetilde{p}^0}{du}e^{-L\widetilde{\mathcal{E}}_0}\mathfrak{F}_0 + \mathcal{O}\left[e^{-2L\widetilde{\mathcal{E}}_x}\right] \\ &= -\frac{4}{\pi}\sin^2\left(\frac{\mu}{2}\right)\int_{-\infty}^{+\infty}\sum_{Q=1}^{+\infty} d\widetilde{p}^Q\, e^{-L\widetilde{\mathcal{E}}_Q} + -\frac{4}{\pi}n_0\sin^2\left(\frac{\mu}{2}\right)\frac{8hL}{4L^2-1} + \mathcal{O}\left[e^{-2L\widetilde{\mathcal{E}}_Q}\right] \end{aligned} \tag{10}$$

$$\mathfrak{F}_x = \mathrm{Tr}_x e^{i(\pi+\mu)F} = \left(1-e^{-i\mu}\right)\left(1-e^{i\mu}\right) = 4\sin^2\left(\frac{\mu}{2}\right) \tag{11}$$

where F plays a role of the fermion number operator, Tr_x is over appropriate $x = \{Q/\overline{Q}/0\}$ 2-dim representations X_i, $X = X_{x_1} \otimes X_{x_2}$ That is in complete agreement with (9) and (7).

$AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ GSE

In relation to the lightcone string, it is important to analyse GSE when the string tension h and size L become large, but the ratio $\mathcal{J} = L/h$ is fixed. Hence we derive it for both AdS_5 and AdS_3 superstrings

$$E_{AdS_3}(\mathcal{J} \ll 1) = -\frac{\mu^2}{\pi}\left(\frac{n_0-1}{\mathcal{J}} - \frac{\pi}{\mathcal{J}^2}\right) \qquad E_{AdS_5}(\mathcal{J} \ll 1) = -\frac{6\mu^2}{\mathcal{J}^4}$$

$$E_{AdS_3}(\mathcal{J} \gg 1) = -\frac{\mu^2}{\pi}\left(\frac{n_0}{\mathcal{J}} - \sqrt{\frac{2\pi}{\mathcal{J}}}\,e^{-\mathcal{J}}\right) \qquad E_{AdS_5}(\mathcal{J} \gg 1) = -2\,\frac{\mu^2}{\pi}\frac{\sqrt{2\pi}}{\sqrt{\mathcal{J}}}e^{-\mathcal{J}}$$

where one can notice that at **large $\mathcal{J}$ massless modes** start to **dominate** in the $AdS_3 \times S^3 \times T^4$ case and appear to be new behaviour.

$AdS_3 \times S^3 \times \mathcal{M}^4$ lightcone superstring

From the lightcone perspective, it is possible to construct string sigma models with **twisted boundary conditions** on the fields, which are charged under $\mathfrak{su}(2)_\bullet$ and $\mathfrak{su}(2)_\circ$. In the context of the $AdS_3 \times S^3 \times \mathcal{M}^4$ lightcone superstring one can consider generic boso-fermionic twisting

$$\phi_k(\tau,\sigma+\mathcal{J}) = e^{-i\mu_k^{\mathrm{B}}}\phi_k(\tau,\sigma) \quad \text{and} \quad \psi_{k,\alpha}(\tau,\sigma+\mathcal{J}) = e^{-i\mu_{k,\alpha}^{\mathrm{F}}}\psi_{k,\alpha}(\tau,\sigma)\,, \quad \alpha = 1,2 \tag{12}$$

The dispersion relation for the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring with the mixed flux can be given by

$$\begin{aligned} E = &\sum_{n=-\infty}^{\infty}\sqrt{2\rho q(n+\widetilde{\mu}_B)m_{B,i} + \rho^2(n+\widetilde{\mu}_B)^2 + m_{B,i}^2} \\ &- \sum_{n=-\infty}^{\infty}\sqrt{2\rho q(n+\widetilde{\mu}_F)m_{F,i} + \rho^2(n+\widetilde{\mu}_F)^2 + m_{F,i}^2} \end{aligned} \qquad \text{with} \quad \hat{q}^2 + q^2 = 1\,,\ \rho = 2\pi/\mathcal{J} \tag{13}$$

where m denotes boson/fermion masses, $\hat{q}$ and q are accordingly RR and NSNS flux parameters and $\mathcal{J} = L/h$. Generically for ***N* complex** massless and massive bosons and fermions, we derive for GSE

$$\begin{aligned} E^{N_{B,F}} = &\sum_{i=1}^{N_0^B}\left(\frac{|\mu_{0,B,i}|}{\mathcal{J}} - \frac{\mu_{0,B,i}^2}{2\pi\mathcal{J}} - \frac{\pi}{3\mathcal{J}}\right) - \sum_{i=1}^{N_0^F}\left(\frac{|\mu_{0,F,i}|}{\mathcal{J}} - \frac{\mu_{0,F,i}^2}{2\pi\mathcal{J}} - \frac{\pi}{3\mathcal{J}}\right) \\ &- \frac{2\hat{q}}{\pi}\sum_{w=1}^{+\infty}\left[\sum_{l=1}^{N_B}\frac{m_{B,l}}{w}K_1\left(\mathcal{J}wm_{B,l}\,\hat{q}\right)\cos\left(\mathcal{J}w\,m_{B,l}\sqrt{1-\hat{q}^2} + w\mu_{B,l}\right)\right. \\ &\left. - \sum_{n=1}^{N_F}\frac{m_{F,n}}{w}K_1\left(\mathcal{J}wm_{F,n}\,\hat{q}\right)\cos\left(\mathcal{J}w\,m_{F,n}\sqrt{1-\hat{q}^2} + w\mu_{F,n}\right)\right], \end{aligned} \tag{14}$$

which after taking mass limits reduces to $\boldsymbol{\mathcal{M}^4 = T^4}$, which contains **two particles in each sector**. To establish mapping with TBA, it appears useful to consider large $\mathcal{J}$ regime and (14) becomes

$$E = \frac{|\mu_1-\mu_2| + |\mu_1+\mu_2| - 2\mu_2}{\mathcal{J}} - \frac{\mu_1^2}{\pi\mathcal{J}} - \frac{4}{\pi}\sin^2\left(\frac{\mu}{2}\right)\sqrt{\hat{q}\frac{2\pi}{\mathcal{J}}}\cos\left(\mathcal{J}\sqrt{1-\hat{q}^2}\right)e^{-\mathcal{J}\hat{q}} + \ldots \tag{15}$$

$AdS_3 \times S^3 \times T^4$ mixed flux: GSE proposal from TBA

For this purpose we derive the mixed flux mirror energy in terms of mirror momentum $\widetilde{p}$ and bound state number Q

$$\widetilde{\mathcal{E}}_Q = \frac{1}{h}\sqrt{\widetilde{p}^2 + \hat{q}^2Q^2} + iq\frac{Q}{h} \tag{16}$$

By making **mixed flux TBA proposal** and recalling (9), we find for the $AdS_3 \times S^3 \times T^4$ GSE

$$\begin{aligned} E &= -\frac{\mu^2}{2\pi}\int_{-\infty}^{+\infty}\sum_{Q=1}^{+\infty}d\widetilde{p}^Q\left(e^{-L\widetilde{\mathcal{E}}_Q} + e^{-L\overline{\widetilde{\mathcal{E}}}_Q}\right) - \frac{n_0\mu^2}{2\pi}\int_{-\infty}^{+\infty}d\widetilde{p}^0 e^{-L\widetilde{\mathcal{E}}_0} \\ &= -\frac{\mu^2}{\pi\mathcal{J}} - \frac{\mu^2}{\pi}\sqrt{\hat{q}\frac{2\pi}{\mathcal{J}}}\cos\left(\mathcal{J}\sqrt{1-\hat{q}^2}\right)e^{-\mathcal{J}\hat{q}} + \mathcal{O}\left[\mathcal{J}^{-\frac{3}{2}}\cos(q\mathcal{J})e^{-\mathcal{J}\hat{q}}\right] \end{aligned} \tag{17}$$

that is in full agreement with (15) for large $\mathcal{J}$ and small twist ($\mu_{1,2} = \mu$). As can be noticed $n_0 = 1$, which corresponds to a single Y_0-function in the AdS_3 TBA. More details, including near-BMN, various regimes along with AdS_5 and other results can be found in [1].

Discussion and further directions

As it could be noticed from the generalised Lüscher derivation, near-BMN and mixed flux case there is extra **massless freedom** denoted by $\boldsymbol{n_0}$, which we have found to be 1. It is in contrary to the initial AdS_3 Mirror TBA proposal [2] and could be an indication for **distinct massless *S*-matrix** structure or **limit order inconsistency** and it is necessary to study the TBA with fixed $\mathcal{J}$ and large L, h. Important to note that at the **NLO massive/massless kernels** start to contribute and it is crucial to find resolution of the TBA due to arising nontrivial dressing dependence. In this context, **higher order Lüscher formalism** would appear useful.

It would be interesting to derive the **excited TBA** system by contour deformation method and address resolution in string tension regimes. It is also important to further investigate recent proposals of ***AdS*$_3$ RR Quantum Spectral Curve** to compute GSE (including twisted string) and identify Y-Q system transitions.

References

[1] Sergey Frolov, Anton Pribytok, and Alessandro Sfondrini. Ground state energy of twisted $AdS_3 \times S^3 \times T^4$ superstring and the TBA. 5 2023.

[2] Sergey Frolov and Alessandro Sfondrini. Mirror thermodynamic Bethe ansatz for AdS3/CFT2. *JHEP*, 03:138, 2022.

[3] Sergey Frolov and Alessandro Sfondrini. New dressing factors for AdS3/CFT2. *JHEP*, 04:162, 2022.