Classical Affine Bialgebras for AdS/CFT Integrability

Egor Im, Niklas Beisert

Institut für Theoretische Physik Eidgenössische Technische Hochschule Zürich

ETH zürich

Motivation

The *integrability* feature of the AdS_5 superstrings AdS/CFT dual to $\mathcal{N} = 4$ sYM is attributed to a particular *Yangian* $\mathrm{Y}(\mathfrak{psu}(2|2))$. However, this Yangian is of non-standard kind. Moreover, the known additional symmetries (i.e. the *secret* and *Lorentz boost* symmetry) do not fit within the Yangian description, rendering the quantum group incomplete. Therefore, it is desirable to obtain a quantum algebra that could capture all the symmetries and whose *universal R-matrix* would evaluate (in some representation) to the worldsheet (or magnon) *S-matrix*.

The analysis is complicated by the non-standard and non-simple nature of the underlying algebras. In the current work [1, 2] we investigate (at the classical level) the possibility to realise the bialgebras of interest starting from some standard constructions developed for the semisimple Lie algebras in order to circumfere the aforementioned complications, thus performing a first step towards a complete algebraic description of the AdS/CFT integrability.

Bialgebra

The goal of this work is to show that the following sequence of algebras:

$$\widehat{\mathfrak{sl}(2)} \times \widehat{\mathfrak{d}(2,1;\alpha)} \xrightarrow{\text{contr.}} \widehat{\mathfrak{psu}(2|2)}_{m.e.} \xrightarrow{\text{reduct.}} \widehat{\mathfrak{u}(2|2)} \tag{1}$$

holds together with its (quasi-triangular) coalgebra structure, giving rise to the classical limit of the AdS/CFT symmetry (bi)algebra. The key steps in this procedure are the **contraction** and **reduction**.

Morally, it is enough to consider the simplified setting:

$$\mathfrak{sl}(2) \times \mathfrak{sl}(2) \xrightarrow{\text{contr.}} \mathfrak{iso}(2,1) \xrightarrow{\text{reduct.}} \mathfrak{u}(1) \times \mathbb{R}, \tag{2}$$

since the lifting to the full case is mostly straightforward.

The coalgebra structures of the algebras in the sequence (2) are induced by the *classical r-matrix* that satisfies the CYBE

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \overset{!}{=} 0. \tag{3}$$

The benefit of this construction is the possibility to obtain the classical r-matrix of AdS/CFT starting from the most standard Yang's parametric r-matrix

$$r(u_1, u_2) = \frac{-c_{ab}\mathrm{J}^a \otimes \mathrm{J}^b}{u_1 - u_2} \simeq c_{ab} \sum_{k=0}^{\infty} \mathrm{J}^a_k \otimes \mathrm{J}^b_{-k-1}, \tag{4}$$

where c_{ab} is the inverse of the Killing form. In the second equality the corresponding loop algebra form of the r-matrix is given.

Contraction

The 3D Poincaré algebra $\mathfrak{iso}(2,1)$ can be obtained from the AdS algebra $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2)_1 \times \mathfrak{sl}(2)_2$ as a **contraction** limit:

$$\mathrm{L}^a = \mathrm{M}^a_1 + \mathrm{M}^a_2, \qquad \mathrm{P}^a = \epsilon\bar{m}\mathrm{M}^a_1. \tag{5}$$

This procedure is lifted to the coalgebra structure analogously. We start with the parametric r-matrix of the $\mathfrak{so}(2,2)$ (in the $\mathfrak{sl}(2)$ basis)

$$r_{\mathfrak{so}(2,2)}(u_1, u_2) = -\frac{\nu_1 \mathrm{M}^2_1}{u_{1;1} - u_{1;2}} - \frac{\nu_2 \mathrm{M}^2_2}{u_{2;1} - u_{2;2}} + \xi(\mathrm{M}^0_1 + \mathrm{M}^0_2) \wedge \mathrm{M}^+_2, \tag{6}$$

where M^a_i are the generators of $\mathfrak{sl}(2)_i$ and M^2_i are quadratic invariants. The last twist term in (6) is needed for consistent **reduction**.

The **contraction** prescription (5) together with an appropriated choice of ν_i, $u_{i;j}$ and ξ as functions of ϵ leads to the r-matrix of $\mathfrak{iso}(2,1)$

$$r_{\mathfrak{iso}(2,1)}(u_1, v_1; u_2, v_2) = -\frac{2\nu \mathrm{L}\cdot\mathrm{P}}{u_1 - u_2} - \frac{\nu' \mathrm{P}^2}{u_1 - u_2} + \frac{\nu(v_1 - v_2)\mathrm{P}^2}{(u_1 - u_2)^2} + \xi' \mathrm{L}^0 \wedge \mathrm{P}^+, \tag{7}$$

which can be viewed as a particular evaluation representation of the loop r-matrix

$$r_{\mathfrak{iso}(2,1)} = c_{ab} \sum_{k=0}^{\infty} \Big(\nu(\mathrm{L}^a_k \otimes \mathrm{P}^b_{-k-1} + \mathrm{P}^a_k \otimes \mathrm{L}^b_{-k-1}) + \nu' \mathrm{P}^a_k \otimes \mathrm{P}^b_{-k-1}\Big) + \xi' \mathrm{L}^0_0 \wedge \mathrm{P}^+_0. \tag{8}$$

Reduction

The **reduction** of the 3D Poincaré algebra essentially consists of fixing a particular direction in $\mathbb{R}^{2,1}$:

$$p^\pm = \mathrm{e}^{\pm\mathring{i}\beta} u^{-1} p^0. \tag{9}$$

This is achieved by choosing a Lorentz generators that leaves the chosen direction invariant

$$\mathrm{L} = \beta^{-1} u \mathrm{L}^0 - \tfrac{1}{2}\mathrm{e}^{-\mathring{i}\alpha}\mathrm{L}^+ - \tfrac{1}{2}\mathrm{e}^{+\mathring{i}\alpha}\mathrm{L}^-, \tag{10}$$

and dividing out the ideal spanned by the orthogonal directions:

$$\mathrm{P}^0 \equiv \beta^{-1} u \mathrm{P}, \qquad \mathrm{P}^\pm \equiv \mathrm{e}^{\pm\mathring{i}\alpha}\mathrm{P}. \tag{11}$$

As far as the r-matrix is concerned, the **reduction** is applied straightforwardly to (8) yielding

$$r_{\mathfrak{u}(1)\times\mathbb{R}} = \sum_{n=0}^{\infty} \nu[-\mathrm{L}_n \otimes \mathrm{P}_{-n-1} - \mathrm{P}_n \otimes \mathrm{L}_{-n-1}] + \nu'\big[\mathrm{P}_n \otimes \mathrm{P}_{-n-1} - \beta^{-2}\mathrm{P}_{n+1} \otimes \mathrm{P}_{-n}\big]. \tag{12}$$

The parameter ξ' is now fixed by the consistency of the reduciton.

Representations

The algebra $\mathfrak{so}(2,2)$ has an infinite-dimensional representation of normalisable fields on $\mathrm{AdS}^{2,1}$ of fixed mass and spin. This can be related to the two copies of *principle series representation* for each copy of $\mathfrak{sl}(2)_i$:

$$\mathrm{J}^0_i|k\rangle_{\gamma_i,\chi_i} = (k + \chi_i)|k\rangle_{\gamma_i,\chi_i}, \tag{13}$$

$$\mathrm{J}^+_i|k\rangle_{\gamma_i,\chi_i} = \theta_{i,k}(k + \chi_i + \gamma_i + \tfrac{1}{2})|k+1\rangle_{\gamma_i,\chi_i}, \tag{14}$$

$$\mathrm{J}^-_i|k\rangle_{\gamma_i,\chi_i} = \theta^{-1}_{i,k-1}(k + \chi_i - \gamma_i - \tfrac{1}{2})|k-1\rangle_{\gamma_i,\chi_i}. \tag{15}$$

The parameters $\theta_{i,k}$, χ_i and γ_i are constrained by unitarity conditions and related to the mass μ and spin s of the corresponding fields.

The **contraction** of this representation is given by

1. contraction of the states:

$$|p,\phi\rangle_{m,s} := \sum_k \mathrm{e}^{-\mathring{i}(k_1-k_2)\phi}|k_1, k_2\rangle_{2m/\bar{m}/\epsilon, s}, \qquad k_{1,2} := \frac{e_m(p) - m}{\epsilon\bar{m}} \pm k, \tag{16}$$

2. contraction of the representation parameters:

$$:= \mu = \frac{2m}{\epsilon\bar{m}} + \mathcal{O}(\epsilon^0), \tag{17}$$

3. contraction of generators according to (5) at the level of representations.

Altogether this yields the field representation of the 3D Poincaré algebra of mass m and spin s:

$$\mathrm{L}^0|p,\phi\rangle_{m,s} = \left(\mathring{i}\frac{\partial}{\partial\phi} + s\right)|p,\phi\rangle_{m,s}, \tag{18}$$

$$\mathrm{L}^\pm|p,\phi\rangle_{m,s} = \mathrm{e}^{\pm\mathring{i}\phi}\left(\pm e_m(p)\frac{\partial}{\partial p} + \mathring{i}\frac{e_m(p)}{p}\frac{\partial}{\partial\phi} + \frac{sp}{e_m(p) + m}\right)|p,\phi\rangle_{m,s}, \tag{19}$$

$$\mathrm{P}^0|p,\phi\rangle_{m,s} = e_m(p)|p,\phi\rangle_{m,s}, \tag{20}$$

$$\mathrm{P}^\pm|p,\phi\rangle_{m,s} = \mathrm{e}^{\pm\mathring{i}\phi} p|p,\phi\rangle_{m,s}. \tag{21}$$

Finally, we perform the **reduction** of the representation by fixing the particular direction (9), obtaining the representation of the target algebra:

$$\mathrm{L}_n|u,v\rangle_{m,s} = u^n\frac{sm}{p(u)}|u,v\rangle_{m,s} + vu^{n-1}\big((n+1)u^2\beta^{-2} - n\big)p(u)|u,v\rangle_{m,s}, \tag{22}$$

$$\mathrm{P}_n|u,v\rangle_{m,s} = u^n p(u)|u,v\rangle_{m,s}. \tag{23}$$

Superalgebra

The lift to the full superalgebra case is straightforward. After applying the **contraction** and **reduction** to the loop extension of the semisimple $\mathfrak{sl}(2) \times \mathfrak{d}(2,1;\epsilon)$ one obtains a deformation of

$$\mathfrak{u}(2,2)[u, u^{-1}] \supset [\mathfrak{u}(1) \times \mathbb{R}][u, u^{-1}]. \tag{24}$$

Importantly, the deformed loop algebra exhibits loop mixing in algebra relations with supercharges rendering the classical loop superalgebra relevant to the AdS/CFT integrability.

The additional part of the r-matrix is not affected by the contraction and reduction, thus the coalgebra structure is simply obtained by adding the contributions from the supercharges and residual $\mathfrak{su}(2)_{\mathrm{R,L}}$ within $\mathfrak{psu}(2|2)$

$$r_{\mathfrak{u}(2|2)} = r_{\mathfrak{u}(1)\times\mathbb{R}} + \nu\bar{m}\frac{\mathrm{Q}^2 - \mathrm{J}^2_{\mathrm{L}} + \mathrm{J}^2_{\mathrm{R}}}{u_1 - u_2}. \tag{25}$$

Evaluated on the fundamental representation of $\mathfrak{psu}(2|2)$ and the infinite dimensional representation of $\mathfrak{u}(1) \times \mathbb{R}$ from above one obtains the tree-level S-matrix of AdS/CFT.

Affine Extension

The whole construction can be repeated in the presence of the affine extension [2]. Whereas the **reduction** does not affect the derivations, their **contraction**

$$\mathrm{D_L} = \mathrm{D}_1 + \mathrm{D}_2, \qquad \mathrm{D_P} = \bar{m}\epsilon\mathrm{D}_1, \tag{26}$$

produces additional symmetry generators corresponding to the Lorentz boost symmetry.

Moreover, the presence of derivations has an important implication: it implements additional purely algebraic constraints on formerly undetermined phase factor

$$r_{\text{phase}} \simeq (u - v)\mathrm{P} \otimes \mathrm{P}. \tag{27}$$

Trigonometric Case

The procedure described in this work can be lifted to the trigonometric case. It merely amounts to a different choice of the distinguished direction (9). In this way one obtains the classical bialgebra of the 1D Hubbard model. Since the latter is related to the η-deformed AdS_5 superstrings by means of the Drienfel'd twist, our construction should generalise to this case as well.

References

[1] N. Beisert and E. Im, *"Classical Lie Bialgebras for AdS/CFT Integrability by Contraction and Reduction"*, `arxiv:2210.11150`.

[2] N. Beisert and E. Im, *"Affine Classical Lie Bialgebras for AdS/CFT Integrability"*, in preparation.