Flux deformed Neumann-Rosochatius integrable model for strings in different near horizon brane geometries

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Integrable Neumann-Rosochatius Model

- One of the earliest 1d rational Liouville integrable model.
- Describes **constrained harmonic oscillator motion on a unit sphere with a combined effect of inverse square centrifugal potential**.
- Integrable extension of classical Neumann model.
- **Lagrangian :** $L = \frac{1}{2}\sum_{i=1}^{N}\left[\dot{x}_i^2 + \frac{v_i^2}{x_i^2} - \omega_i^2 x_i^2\right] - \frac{\Lambda}{2}\left(\sum_{i=1}^{N} x_i^2 - 1\right)$
- **Hamiltonian :** $H = \frac{1}{2}\sum_{i=1}^{2}\left[\dot{x}_i^2 - \frac{v_i^2}{x_i^2} + \omega_i^2 x_i^2\right]$
- **Uhlenbeck constants of motion** : $I_i = x_i^2 + \sum_{j\neq i}\frac{1}{\omega_i^2-\omega_j^2}\left[\left(x_i x_j' - x_j x_i'\right)^2 + v_i^2\frac{x_j^2}{x_i^2} + v_j^2\frac{x_i^2}{x_j^2}\right]$, $\sum_{i=1}^{N-1} I_i = 1$, $(N-1)$ number of I's, $\left\{I_i, I_j\right\} = 0$

Significance of NR model in studying string-sigma model

***One can reproduce an equivalent 1D NR integrable version of various 2D string-sigma models.**

*Formulates a large class of string sigma-model solutions by using a family of general NR ansatz.

*Solutions are derived from the corresponding integrable EOM's of the model.

*Solutions generally match with some limiting cases of holographically well-established spectrum.

Specific choices of near horizon brane geometries as target spaces

Target space backgrounds :

1. $AdS_4 \times \mathbb{CP}^3$ with pure 2-form NSNS holonomy around $\mathbb{CP}^1$
   (Near horizon decoupling limit of $|$M-N$|$ fractional number of parallel M2 branes situated at $C^4/\mathbf{Z_k}$ singularity alongside N number of parallel M2 branes moving freely)
2. $AdS_3 \times S^3 \times \mathcal{M}^4$, $\mathcal{M}^4 = T^4$ or $S^3 \times S^1$ with finite flux.
   (Near horizon decoupling limit of $D1 - D5$ brane systems)
3. Near horizon limit of the intersection of two stacks of parallel NS5 brane, known as I-brane.

Natural probe strings :

1. Fundamental string and (p,q)-type string.

NR embeddings for constructing string sigma models : $W_a(\tau,\sigma) = R r_a(\tau,\sigma) e^{i\Phi_a(\tau,\sigma)}$

Rotating string: $t = \kappa\tau,\ \ r_a(\tau,\sigma) = r_a(\zeta),\ \ \Phi_a(\tau,\sigma) = \omega_a\tau + f_a(\zeta),\ \ \zeta = \alpha\sigma + \beta\tau$,

Pulsating string: $t = \tau,\ \ r_a = r_a(\tau),\ \ \Phi_a(\tau,\sigma) = m_a\tau + f_a(\tau)$

NR construction for fundamental string in ABJ dual

- **Lagrangian** and **Uhlenbeck Integrals of Motion** for rotating string ansatz:

$$L_{NR} = \sum_{a=1}^{4}[(\alpha^2-\beta^2)r_a'^2 - \frac{1}{(\alpha^2-\beta^2)}\frac{C_a}{r_a^2} - \omega_a^2 r_a^2] + \sum_{a=1}^{4}\alpha\omega_a r_a r_a' - 2\Lambda(\sum_{a=1}^{4}{r_a}^2 - 1) - 2\Lambda_0\sum_{a=1}^{4}({r_a}^2\omega_a) + \sum_{a=1}^{4}\frac{1}{(\alpha^2-\beta^2)}(\beta\omega_a + \Lambda_1)r_a^2.$$

$$I_a = \alpha^2 r_a^2 + (\alpha^2-\beta^2)^2\sum_{b\neq a}\frac{(r_a' r_b - r_a r_b')^2}{(\omega_a^2-\omega_b^2)} + \sum_{b\neq a}\frac{1}{(\omega_a^2-\omega_b^2)}\left(\frac{C_a r_b}{r_a} + \frac{C_b r_a}{r_b}\right)^2 + \frac{\alpha^2}{4}\sum_{b\neq a}\left(\frac{\omega_a+\omega_b}{\omega_a-\omega_b}\right)r_a^2 r_b^2$$

- **Lagrangian** and **Uhlenbeck integrals motion** for pulsating string ansatz:

$$L_{NR} = \frac{\dot{z_0}^2}{4} - \frac{C_0}{4z_0^2} - \tilde{\Lambda}\left(z_0^2+1\right) + \sum_{a=1}^{4}\left(\dot{r}_a^2 + \frac{C_a}{r_a^2}\right) + \sum_{a=1}^{4} r_a\dot{r}_a m_a + \sum_{a=1}^{4}\left(r_a^2-1\right)\left(\sum_{a=1}^{4} C_a\right)^2 - \sum_{a=1}^{4} m_a^2 r_a^2 + \Lambda\left(\sum_{a=1}^{4} r_a^2 - 1\right) + \Lambda_2\sum_{a=1}^{4} m_a r_a^2$$

$$I_a = z_0^2 + r_a^2\sum_{b\neq a}\left[\frac{(\dot{r}_a r_b - r_a\dot{r}_b)^2}{m_a^2-m_b^2} + \frac{1}{m_a^2-m_b^2}\left(\frac{C_a r_b}{r_a} + \frac{C_b r_a}{r_b}\right)^2 + 2\frac{\left(r_a\dot{r}_a r_b^2 - r_b\dot{r}_b r_a^2\right)}{m_a+m_b} + \frac{1}{4}\left(\frac{m_a-m_b}{m_a+m_b}\right)r_a^2 r_b^2\right]$$

The system forms NR-like structure along with flux and geometrical deformations

NR construction for (m,n)-string in $AdS_3 \times S^3 \times T^4$ with flux

$$Q = \frac{mq+n\sqrt{1-q^2}}{\sqrt{(m-n\chi)^2+n^2e^{-2\phi}}},\ \tau_{(m,n)} = T_{D1}\sqrt{(m-n\chi)^2 + n^2e^{-2\phi}}$$

- **Lagrangian** and **Uhlenbeck integrals of motion** for rotating string ansatz:

$$L_{NR} = \frac{(\alpha^2-\beta^2)}{2}\sum_{a=1}^{2} r_a'^2 - \frac{1}{2(\alpha^2-\beta^2)}\sum_{a=1}^{2}\left(\frac{C_a^2+Q^2\alpha^4 r_2^4\omega_b^2}{r_a^2}\right) + \left[\frac{\alpha^2}{2(\alpha^2-\beta^2)}\sum_{a=1}^{2}\left(\omega_a^2 r_a^2 + 2C_a Q\omega_b r_2^2\epsilon_{ba}\right)\right] + \frac{A}{2}\left(r_1^2+r_2^2\right) - \frac{Q\alpha^2 r_2^2}{2(\alpha^2-\beta^2)}\left[\frac{\omega_2 C_1}{r_1^2} - \frac{C_2\omega_1}{r_2^2} + \frac{Q\alpha^2(\omega_1^2 r_1^2+\omega_2^2 r_2^2)}{r_1^2}\right]$$

$$\bar{I}_1 = \frac{\alpha^2-\beta^2}{\omega_1^2-\omega_2^2}\left(r_1 r_2' - r_1' r_2\right)^2 + \frac{2}{\omega_1^2-\omega_2^2}\left[\left(\frac{C_1-Q\alpha^2 r_2^2\omega_2}{r_1}\right)^2 + \left(\frac{C_2+Q\alpha^2 r_2^2\omega_1}{r_2}\right)^2\right] - \frac{2\alpha^2}{\omega_1^2-\omega_2^2}\left[\left(1+\frac{2Q^2\alpha^2 r_2^2}{r_1^2}\right)\left(\omega_1^2 r_1^2+\omega_2^2 r_2^2\right) + 2Qr_2^2\left(\frac{C_1\omega_2}{r_1^2} - \frac{C_2\omega_1}{r_2^2}\right)\right] + \frac{1}{\omega_1^2-\omega_2^2}\left(\frac{C_1^2}{r_1^2} + \frac{C_2^2 r_1^2}{r_2^2}\right)$$

- **Lagrangian** and **Uhlenbeck integrals of motion** for pulsating string ansatz:

$$L_{NR} = \frac{1}{2}\left(\dot{z}_0^2 + \dot{r}_1^2 + \dot{r}_2^2\right) - \frac{1}{2}\frac{\left(C_1-Qm_2r_2^2\right)^2}{r_1^2} - \frac{1}{2}\frac{\left(C_2+Qm_1r_2^2\right)^2}{r_2^2} - \frac{C_0^2}{8z_0^2} + \frac{1}{2}\left(m_1^2 r_1^2 + m_2^2 r_2^2\right) + \frac{A}{2}\left(r_1^2+r_2^2-1\right) + \frac{\tilde{A}}{2}z_0^2 + \frac{1}{2}Qr_2^2\left(\frac{m_1C_2}{r_2^2} - \frac{m_2C_1}{r_1^2} + \frac{Qm_1^2r_2^2}{r_2^2} + \frac{Qm_2^2r_2^2}{r_1^2}\right)$$

$$\bar{I}_a = \frac{1}{m_1^2-m_2^2}\left(r_1\dot{r}_2 - \dot{r}_1 r_2\right)^2 + \frac{2}{m_1^2-m_2^2}\left[\left(\frac{C_1-Qr_2^2m_2}{r_1}\right)^2 + \left(\frac{C_2+Qr_2^2m_1}{r_2}\right)^2\right] - \frac{2}{m_1^2-m_2^2}\left[\left(1+\frac{2Q^2r_2^2}{r_1^2}\right)\left(m_1^2r_1^2+m_2^2r_2^2\right) + 2Qr_2^2\left(\frac{C_1m_2}{r_1^2} - \frac{C_2m_1}{r_2^2}\right)\right] + \frac{1}{m_1^2-m_2^2}\left(\frac{C_1^2}{r_1^2} + \frac{C_2^2r_1^2}{r_2^2}\right)$$

The system attains NR-like structure along with flux deformation

NR construction for F1-string in $AdS_3 \times S^3 \times S^3 \times S^1$ with flux

Relative spherical geometries: $\alpha \equiv \cos^2\varphi = \frac{R_3^2}{R_1^2} = 1 - \frac{R_3^2}{R_2^2},\ \ \frac{R_3^2}{R_2^2} \equiv \sin^2\varphi$

Relative flux parameters (for pure NSNS): $\alpha = \frac{b_1^2}{b_0^2},\ \ 1-\alpha = \frac{b_2^2}{b_0^2}$

*Similar structure of Lagrangian and Uhlenbeck integrals of motion.

* Fundamental string tension T but effective flux parameters b_0, $\frac{b_0}{\sqrt{\alpha}}$ and $\frac{b_0}{\sqrt{1-\alpha}}$ for AdS_3, S_1^3 and S_2^3 respectively.

* I_i's follow the constraints $\bar{I}_1 - \bar{I}_0 = -\alpha_1^2(1-b_0^2)$, $\bar{I}_2 + \bar{I}_3 = \alpha_1^2(1-\frac{b_0^2}{\alpha})$ and $\bar{I}_4 + \bar{I}_5 = \alpha_1^2(1-\frac{b_2^2}{1-\alpha})$ for AdS_3, S_1^3 and S_2^3 respectively.

Spiky solutions for string rotating with angular momenta along $S^3 \times S^1$:

*Rotating spiky-type string solutions from NR construction with pure RR case (b_0, b_1, b_2=0).

*Different nature of spiky strings, spikes not ending in cusps.

Number of spikes: $N = \frac{\pi}{\Delta\phi}$, $\Delta\phi$= Angle between valley and spikes.

*Rounded off spikes are at $y = y_2$, $y = \frac{1}{1-2r_2^2}$ (due to extra J along S^1)

NR construction for F1-string in near horizon limit of I brane

I-brane background: $ds^2 = -dx_0^2 + dx_1^2 + 1 + \frac{k_1 l_s^2}{\sum_{i=2}^{5}(y_i)^2}\sum_{i=2}^{5} dy_i^2 + 1 + \frac{k_2 l_s^2}{\sum_{j=2}^{5}(z_j)^2}\sum_{j=6}^{9} dz_j^2,$

Near Horizon limit: $\frac{k_1 l_s^2}{y^2} \gg 1,\ \ \frac{k_1 l_s^2}{z^2} \gg 1.$

Coordinate transformation for near horizon limit:

$k_1 = k_2 = N,\ \ r_1 = \ln\frac{\rho_1}{\sqrt{N}},\ \ r_2 = \ln\frac{\rho_2}{\sqrt{N}},\ \ x_0 = \sqrt{N}t,\ \ x_1 = \sqrt{N}y$

*Coordinates along the directions of intersection of the branes to be localized at $y = 0$ and $r_1 = r_2 =$ const.

*Metric: $ds^2 = N\left(-dt^2 + d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + \cos^2\theta_1 d\psi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2 + \cos^2\theta_2 d\psi_2^2\right)$

NR Lagrangian assumes similar form as $S^3 \times S^3$, spheres having equal radii.

Energy spectra produced from the solutions of the NR structures

- **For string rotating in $\mathbb{CP}^3$ with pairs of equal and opposite angular momenta**

$$\mathcal{E} - \mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4\sin^2\frac{p}{2}} - \frac{32\sin^4\frac{p}{2}}{\sqrt{\mathcal{J}_2^2+4\sin^2\frac{p}{2}}}\exp\left[-2\frac{(\mathcal{J}_1+\mathcal{J}_2+\sqrt{\mathcal{J}_2^2+4\sin^2\frac{p}{2}})}{\mathcal{J}_2^2+4\sin^4\frac{p}{2}}\sqrt{\mathcal{J}_2^2+4\sin^2\frac{p}{2}}\sin^2\frac{p}{2}\right] - \frac{1}{4}\left(\mathcal{J}_2^2+4\sin^2\frac{p}{2}\right)\left(\exp\left[-2\frac{(\mathcal{J}_1+\mathcal{J}_2+\sqrt{\mathcal{J}_2^2+4\sin^2\frac{p}{2}})}{\mathcal{J}_2^2+4\sin^4\frac{p}{2}}\sqrt{\mathcal{J}_2^2+4\sin^2\frac{p}{2}}\sin^2\frac{p}{2}\right] - 1\right)$$

  ***Flux-dependent dispersion relation for dyonic giant magnon in the** $R_t \times S^3$
  *Matches upto leading order with the spectrum of SU(2)× SU(2) sector of integrable SU(4) spin chain with alternate fundamental and anti-fundamental representation.

- **For string pulsating in $\mathbb{CP}^3$ with pairs of equal and opposite angular momenta**
  Small energy expansion in terms of pulsation number $\mathcal{N}$

$$\mathcal{E} = \mathcal{M} + K(m_a)\frac{5\pi m_a}{32}\mathcal{J}_a^2 + \mathcal{O}[\mathcal{J}_a]^4,\ \ \mathcal{M} = \frac{\pi}{16\sqrt{m_a}} + \sqrt{\mathcal{N}}, K(m_a) = \frac{117\sqrt{m_a}}{32} - \mathcal{N}\left(11m_a^{\frac{1}{2}} + \frac{63\pi}{256}\right).$$

  *Matches up to leading order with antiferromagnetic XXX_1 spin chain description.

- **Constant radii solutions for (m,n) string rotating in S^3 with pairs of equal and opposite angular momenta** : $E^2 = J^2 - 4\pi\tau_{(m,n)}L^2Q\bar{m}J + 2\pi^2\tau_{(m,n)}^2L^4Q^2\bar{m}^2$
  *For large J this matches with vacuum state of integrable Heisenberg XXX spin chain.

- **For string pulsating in S^3 with pairs of equal and opposite angular momenta**
  Small energy expansion in terms of pulsation number $\mathcal{N}$
  $\mathcal{E}^2 = (25.6704 + 1.8333\mathcal{N}) + (24.7749 + 1.2526\mathcal{N})\mathcal{J}_1 + (6.0259 + 0.0082\mathcal{N})\mathcal{J}_1^2 + \mathcal{O}[\mathcal{J}_1^3]$
  *$m = 1, n = 0,\ m_a \to 1,\ \mathcal{J}_a \to 0$: Leading term matches with that obtained with pulsation in ABJ background.

- **For large J rotating string in I-brane background**
  $E = \frac{\kappa J}{\omega}\left[1 + \frac{qm}{\omega} + \left(1+\frac{m^2}{\omega^2}\right)q^2 + \mathcal{O}(q^3)\right]$, $0 < q < 1$, vacuum BMN solution up to leading order

- **For large E and small J pulsating string in I-brane background**

$$\mathcal{E} = \left\{\frac{3\mathcal{J}^2\left(1+q^2\right) - m^2N^2q^2\left(3+4q^2\right) - 3\mathcal{J}mNq\left(1+6q^2\right)}{24q\sqrt{1-q^2}}\right\}\frac{1}{\mathcal{N}} - \left\{\frac{48m^2N^2q(1-q^2)^{\frac{3}{2}}}{3\mathcal{J}^2(1+q^2) - m^2N^2q^2(3+4q^2) - 3\mathcal{J}mNq(1+6q^2)}\right\}\mathcal{N} + \ldots.$$

  *Matches with pulsating string states in $R \times T^{1,1}$ nonintegrable spacetime.

Summary and future ideas

- **Solving NR integrable model can be used as competent equipment to extract some generic class of holographically reasonable string states in any prescribed near horizon branes that have either spherical or AdS geometries, even in the presence of flux-deformations.**
- **Spiky solutions without cusps can be speculated as NR model solutions.**
- **Emergence of NR integrability in the family of backgrounds with higher supersymmetry, like I-brane.**

Similar appearance of NR for deformed backgrounds probed by non-relativistic strings......

(In near future)

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