Topological CFT defects on the lattice

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Thanks to:
Hubert Saleur, Ananda Roy, Jesper Lykke Jacobsen, Madhav Sinha, Fei Yan, Jonathan Belletête, Thiago Silva Tavares

Setting: 2D bulk CFT
Symmetry: $Vir \times Vir$,
generated by modes of T and $\bar{T}$.

Lattice description
Critical 2D lattice model on the cylinder $\rightarrow$ periodic spin chain in the Hamiltonian limit.

- Low energy eigenstates of H identified with conformal primaries and descendants
- Can make a lattice discretization of the Virasoro generators ("Koo-Saleur generators")

Lattice description useful e.g. for non-unitary CFT.

Goal: lattice description of topological defects

Boundaries and interfaces in 2D

1D defects in 2D CFTs

CFT 1 | CFT 2

(Here: CFT 1 = CFT 2)

Conformal defects

- $T - \bar{T}$ continuous across defect.
- Fixed points of RG flow.
- Show presence of internal symmetries, and order-disorder dualities of Kramers–Wannier type.
- Can relate a defect to a boundary by "folding", the condition on $T - \bar{T}$ guarantees conformal b.c.

Topological defects $\subset$ conformal defects

- $T, \bar{T}$ *independently* continuous across the defect
- Can deform continuously without changing the partition function, and without changing correlation functions as long as it does not cross the operator insertions (hence the name)

Simplest example: defects in diagonal minimal models

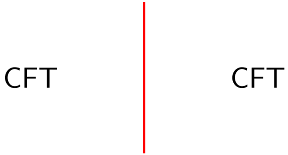
Defects labelled the same way as primary fields,
by Kac labels r, s.

Example: Ising model: $r \in \{1,2\}$, $s \in \{1,2,3\}$:

Kac table indices	Dimension	Primary field	Name
$(1,1)$ or $(2,3)$	0	$\mathbf{1}$	Identity
$(1,2)$ or $(2,2)$	$\frac{1}{16}$	σ	Spin
$(2,1)$ or $(1,3)$	$\frac{1}{2}$	ϵ	Energy

$$\mathcal{H} = \mathcal{R}_0 \otimes \bar{\mathcal{R}}_0 \oplus \mathcal{R}_{\frac{1}{16}} \otimes \bar{\mathcal{R}}_{\frac{1}{16}} \oplus \mathcal{R}_{\frac{1}{2}} \otimes \bar{\mathcal{R}}_{\frac{1}{2}}$$

$$Z(q,\bar{q}) = \chi_0(q)\bar{\chi}_0(\bar{q}) + \chi_{\frac{1}{16}}(q)\bar{\chi}_{\frac{1}{16}}(\bar{q}) + \chi_{\frac{1}{2}}(q)\bar{\chi}_{\frac{1}{2}}(\bar{q})$$



Direct channel defects (extending in time):
The defect acts on Z by changing the Virasoro modules though fusing:

$$Z_{x/defect} = \sum_{ij} N_{ij}^{x} \chi_i(\tilde{q}) \bar{\chi}_j(\bar{\tilde{q}})$$

Defects also fuse with each others, obeying fusion rules:

$$D_a \times D_b = \sum_c N_{abc} D_c$$

CFT

CFT

Crossed channel defects (extending in space):
The defect acts as an operator, whose eigenvalues depend on the Virasoro module:

$$Z_{x/tw} = \sum_j \frac{S_{xj}}{S_{1j}} \chi_j(q) \bar{\chi}_j(\bar{q})$$

Fusion of the defects is now seen as products of operators.

Want a lattice realization of the defects

- Well controlled, e.g. can put on computer
- Useful when other approaches fail (e.g. due to non-unitarity)

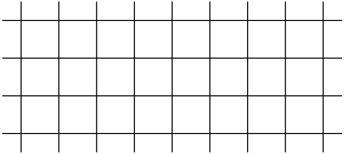
In particular: Integrable lattice models based on the Temperley-Lieb (TL) algebra

- Describe all minimal model CFTs (RSOS lattice models)
- Also non-unitary CFTs, e.g. through 6-vertex model, loop model
- Can use integrability and TL for the construction

Quick overview of TL based lattice models

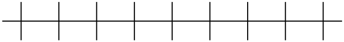
Row to row transfer matrix $\mathbf{T}(u)$:

$\mathbf{T}\uparrow$



with

$\mathbf{T} =$



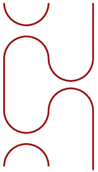
and where $R(u) =$  is in terms of the Temperley-Lieb algebra.

More specifically,

$$R_j(u) = \sin(\gamma - u)\mathbf{1} + \sin(u)e_j$$

with γ depending on the model ($q = e^{i\gamma}$)
and e_j fulfilling the Temperley-Lieb relations:

$$e_j^2 = de_j, \quad e_j e_{j\pm 1} e_j = e_j$$

Pictorially: $e_j =$  with  $= d$   $=$ 

(technically: affine TL since periodic system – also have translation generator)

Hamiltonian limit

$$H = \mathbf{T}(0)^{-1} \frac{\partial \mathbf{T}}{\partial u}\bigg|_{u=0}$$

We obtain the Temperley-Lieb Hamiltonian

$$H = -\sum_j e_j$$

Different representations of TL give different models: XXZ, RSOS, Ising, Potts...

General idea:
Introduce impurities (spectral parameter inhomogeneities) to realize the defects.

Summary of results:

- Construction of all (r, s) defects in Temperley-Lieb based models. Previously, only $(1, s)$ defects were realized on the lattice.
- Checks include computing modified spectrum in the direct channel, eigenvalues of defect operators in the crossed channel, fusion of defects and entanglement entropy in the presence of defects.

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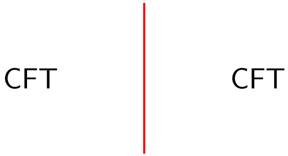
Part II: Using impurities
to realize defects on the lattice

Construction and results.



Direct channel defects (extending in time):
Impurity Hamiltonians, where the spectral parameter has different values on some sites. The defect acts on Z by changing the Virasoro modules though fusing $\Rightarrow$ modified values for the conformal dimensions. **We can compute the effect on the spectrum.**

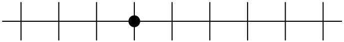
CFT

CFT

Crossed channel defects (extending in space):
Operators D made by a transfer matrix with a different value of the spectral paramenter throughout, or products of such transfer matrices. A defect acts as an operator on the state space, and **we can compute its eigenvalues.**

Direct channel, single impurity

One of the R-matrices has a different spectral parameter v_j

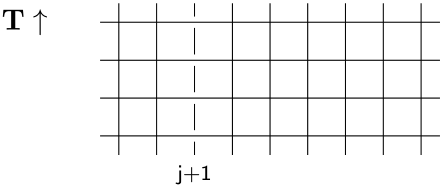
$\mathbf{T}(u) =$ 

In particular, we can tune this value such that we obtain a generator of the braid group:

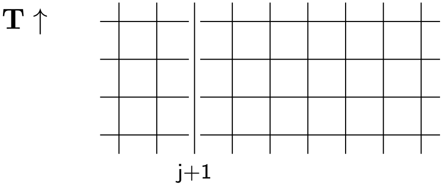
$$g_j = (-q)^{1/2}\mathbf{1} + (-q)^{-1/2} e_j$$
$$g_j^{-1} = (-q)^{-1/2}\mathbf{1} + (-q)^{1/2} e_j$$

Taking $v_j = iv$, $v \to \infty$ we obtain $R_j = g_j$.
Taking $v_j = iv$, $v \to -\infty$ we obtain $R_j = g_j^{-1}$.

$v \to -\infty$:



$v \to \infty$:

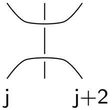


$$H = -\sin\gamma \mathbf{T}^{-1}(0)\frac{\partial}{\partial u}\mathbf{T}(u)\Big|_{u=0}$$

with one impurity $u = iv, v \to -\infty$:

$$H = -\sum_{k\neq j,j+1} E_k - g_j E_{j+1} g_j^{-1}$$

with the modified interaction

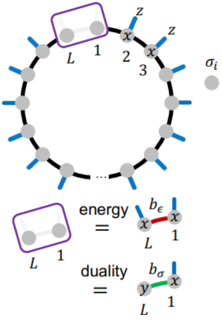


This defect is clearly topological, as the line decouples from the rest and can be continuously deformed.

Ising representation of the Temperley-Lieb algebra

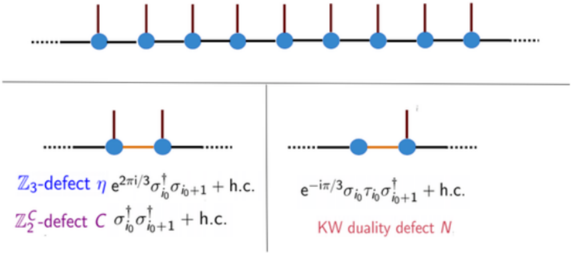
$$H = -\frac{1}{\sqrt{2}}\sum_{k\neq j}(\mathbf{1}+\sigma_k^z) \\ -\frac{1}{\sqrt{2}}\sum_{k\neq j}(\mathbf{1}+\sigma_k^x\sigma_{k+1}^x) - \frac{1}{\sqrt{2}}(\mathbf{1}+\sigma_j^y\sigma_{j+1}^x)$$

This is the Ising duality defect.



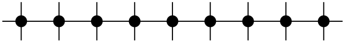
3 state Potts representation of the Temperley-Lieb algebra

Obtain Potts analog of the Ising duality defect (absence of the transverse field term and a modified nearest neighbor interaction)

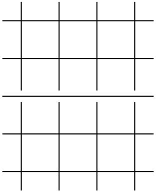


Crossed channel: impurities along the entire row

All of the R-matrices have the modified spectral parameter

$$\mathbf{T}(u) =$$  .

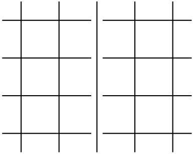
$T(u)$ with $u \to \pm i\infty$ gives a "hoop operator"



Modular invariance:
same Z in direct and crossed

$$Z_{x/defect} = \sum_{ij} N_{ij}^{x} \chi_i(\tilde{q}) \bar{\chi}_j(\bar{\tilde{q}}) \qquad\qquad Z_{x/tw} = \sum_j \frac{S_{xj}}{S_{1j}} \chi_j(q) \bar{\chi}_j(\bar{q})$$

Modular invariance
already on the lattice



Modular invariance:
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$$Z_{x/defect} = \sum_{ij} N_{ij}^{x} \chi_i(\tilde{q}) \bar{\chi}_j(\bar{\tilde{q}})$$

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Modular invariance
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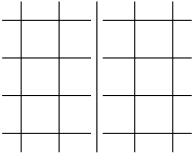
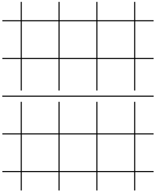


Modular invariance
already on the lattice



$$\mathrm{Tr}(T_d T_d T_d \ldots.) = \mathrm{Tr}(T T \tilde{T} T T \ldots)$$

In terms of Kac labels: $\pm i\infty$ corresponds to a defect of type $(1, 2)$.

Example: in Ising, the ground state conformal dimension goes from $h_{1,1} + h_{1,1} = 0$ to $h_{1,1} + h_{1,2} = 1/16$.

Can obtain all defects of type $(1, s)$ this way, through fusion.

- Type $(1, s)$ known since a while:
  Aasen, Fendley and Mong
  2016 and 2020,
  Belletête et al 2020...
- These are the ones that are topological
  on the lattice, thanks to the braiding.
  (If time permits: further discussion
  in terms of the center of TL).
- **Would like to obtain $(r, 1)$ type defects as well.**
  We can then construct all defects (r, s) through fusion.

Fusion examples (Diagonal M.M.):

$$D_{1,2} \times D_{1,2} = D_{1,1} + D_{1,3}$$

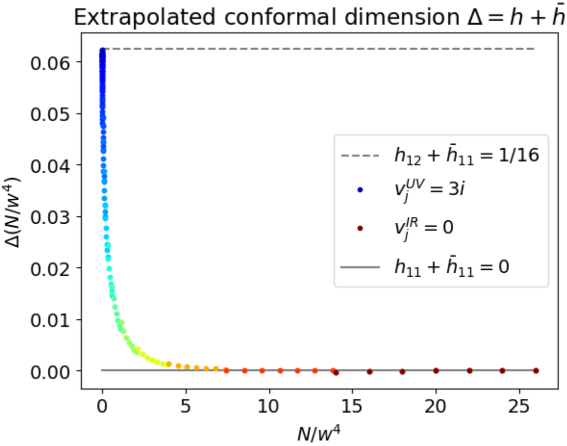
$$D_{1,2} \times D_{2,1} = D_{2,2}$$

Defect flow as a means to obtain r-type defects

At $v_j \to \pm i\infty$ we have a defect of type $(1,2)$.
At $v_j = 0$ we have no defect, or equivalently the identity defect $(1,1)$.

What happens in between?

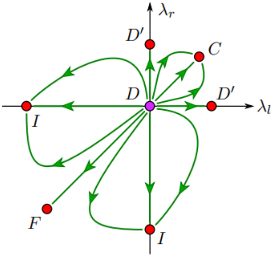
Ising model example, $v_j = iv, w = \exp(v)$.
GS energy, flow from $(1,2)$ to $(1,1)$



Effective system size governed by $h_{1,3}$. Interpret the spectral parameter inhomogeneity in terms of local perturbation of the CFT by $\phi_{1,3}$.

Defect flows in minimal models

Márton Kormosa, Ingo Runkel, Gérard M. T. Watts, arXiv: 0907.1497

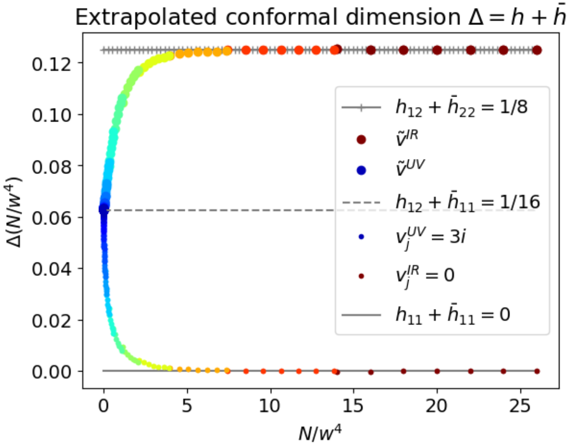


$$D = (1,2)$$
$$I = (1,2)$$
$$D' = (2,1)$$

$$D_{12}(\lambda_l \phi + \lambda_r \bar{\phi})$$

The (1,2)-defect perturbed by $\lambda_l \phi + \lambda_r \bar{\phi}$ with $\phi = \phi_{h_{(1,3)},0}$ and $\bar{\phi} = \phi_{0,h_{(1,3)}}$.

By changing the impurity we can change the flow, reaching $D' = (2,1)$ instead of $I = (1,1)$.



r-type defects generally no longer topological on the lattice, R-matrix is no longer forming a braid generator. Can only be topological in the continuum.

Another way to check the construction: entanglement entropy

Contributions: central charge c for bulk,
g for boundary/defect.

For a subsystem located symmetrically around the defect:

$$\mathcal{S}_D(r) = \frac{c}{3} \ln \left[\frac{L}{\pi} \sin \frac{\pi r}{L} \right] + S_0 + \ln g_D,$$

(S_0 non-universal but independent of defect.)

Compare entanglement entropy with and without the defect $\Rightarrow$ measure g_D

Ising:

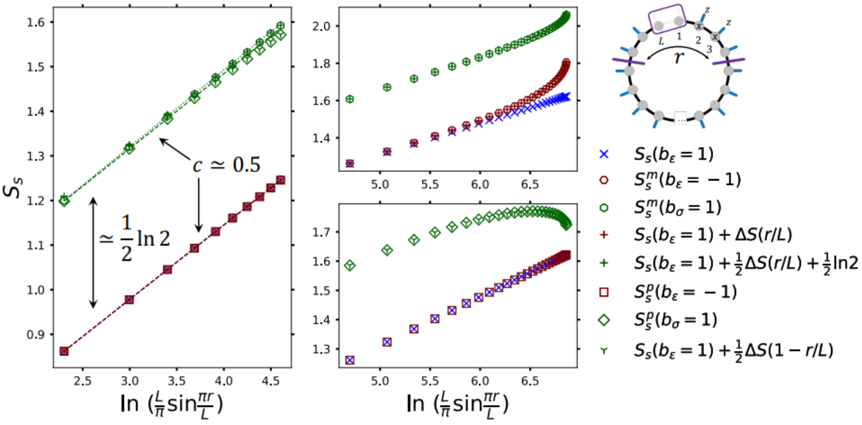
$$g_{(1,2)} = \sqrt{2}$$

Potts:

$$g_{(1,2)} = \sqrt{3}, \quad g_{(2,1)} = \frac{1+\sqrt{5}}{2}$$

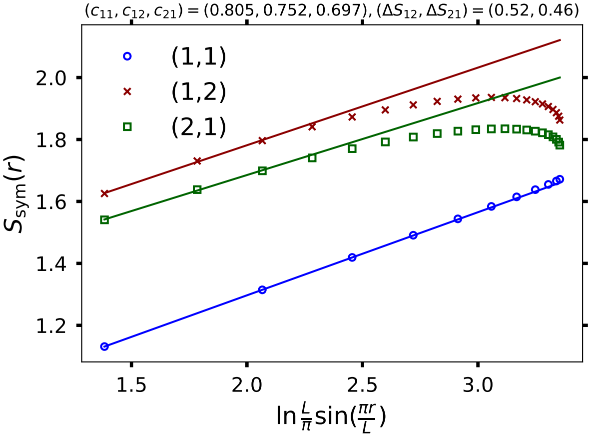
Ising model, (1,2) defect

(Ananda Roy, Hubert Saleur, arXiv 2111.04534)



Potts model, (1,2) and (2,1) defects

(Madhav Sinha, LGS, Fei Yan, Ananda Roy, Hubert Saleur, in preparation)



Expected offsets for (12) and (13) are $\ln(\sqrt{3}) \approx 0.549$ and $\ln(\frac{1+\sqrt{5}}{2}) \sim 0.481$ (Work in progress!)

Further notes on being topological on the lattice

$T, \bar{T}$ being continuous across the defect translates to the condition

$$[D, L_n] = [D, \bar{L}_n] = 0 \quad \forall n$$

in the crossed channel, where L_n are the Virasoro generators.

Hoop operators (defects of type $(1, s)$) generate the full center of the TL algebra.

Meanwhile, the lattice discretizations of the Virasoro generators are *also* constructed from TL generators. (Koo-Saleur generators.)

$$\mathscr{L}_n[N] = \frac{N}{4\pi}\left[-\frac{\gamma}{\pi\sin\gamma}\sum_{j=1}^{N} e^{inj2\pi/N}\left(e_j - e_\infty + \frac{i\gamma}{\pi\sin\gamma}[e_j, e_{j+1}]\right)\right] + \frac{c}{24}\delta_{n,0}$$

Conclusion: Any s-type defect will obey

$$[D, L_n] = [D, \bar{L}_n] = 0 \quad \forall n$$

already on the lattice.

r-type defects will generally not be topological on the lattice, only in the continuum.

Can check that the r-type defects commute with Koo-Saleur generators in the continuum limit. (Future work.)

Summary

- Lattice realization of topological defects through impurities (spectral parameter inhomogeneities)
- s-type defects straightforward to construct, topological on the lattice
- Can flow from s-type defects to r-type defects. These are only topological in the continuum limit
- Can compute entanglement entropy in the presence of s- and r-type defects

Future directions

- Confirming that r-type defect construction really is topological in the continuum limit
- Mixed fusion of defects in the direct and crossed channel

Thank you!