Heating up the AdS_4 Quantum Spectral Curve

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2306.09883, 2306XX.XXXX w J. Minahan & C. Thull

(2211.07810 w A.Cavaglià, N. Gromov & P. Ryan)

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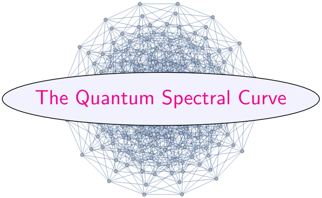
Introduction

- Main topic of this talk: [Gromov, Kazakov,Leurent,Volin '13'14]



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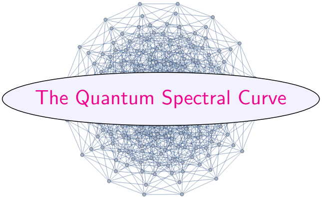

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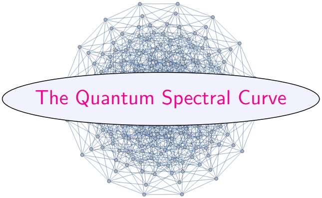

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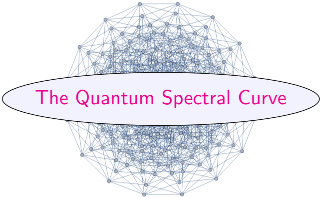

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- The QSC also has applications beyond N $= 4$ and the spectral problem.
- Focus of today: The Hagedorn temperature in AdS_4/CFT_3 using the QSC.

Outline of the Talk

1. QSC for AdS_5, AdS_4 and AdS_3

2. The Hagedorn temperature from the AdS_4 QSC

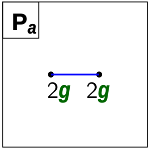
3. Technical details or Solving the AdS_4 QSC

4. Conclusions and outlook

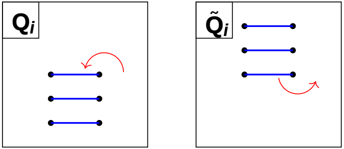
QSC for AdS_5, AdS_4 and AdS_3

What is the QSC?

- The N $= 4$ QSC is based on $\mathrm{psu}_{2;2j4}$. It is a collection of 256 Q-functions, functions of 1 complex parameter $\boldsymbol{u}$. Among them: $\mathbf{P}_{a}(\boldsymbol{u})$ and $\mathbf{Q}_{i}(\boldsymbol{u})$



$$\boldsymbol{g} = \frac{\mathrm{p}\_}{4}$$



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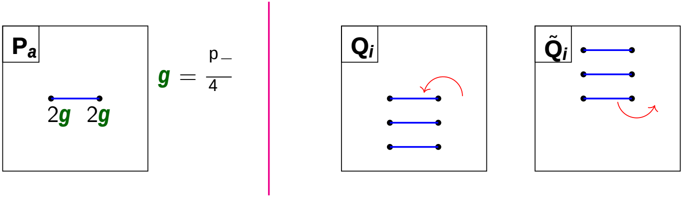

- Other Q-functions are obtained from QQ-relations. Example:

$$\boldsymbol{Q}^{+}_{a\mathrm{j}i} \quad \boldsymbol{Q}_{a\mathrm{j}i} = \mathbf{P}_a \mathbf{Q}_i \,; \qquad \boldsymbol{f}^{[n]} = \boldsymbol{f}(\boldsymbol{u} + \mathrm{i}\frac{\boldsymbol{n}}{2}); \quad \boldsymbol{f} \quad = \boldsymbol{f}^{[\ \ 1]}$$

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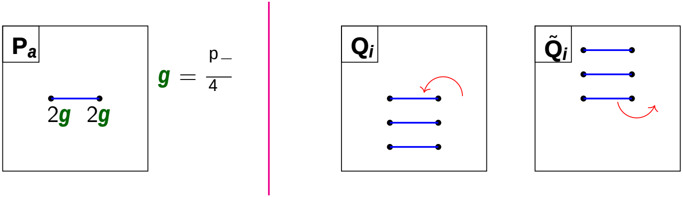


- Other Q-functions are obtained from QQ-relations. Example:

$$Q^+_{a|i} - Q^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i \, ; \qquad f^{[n]} = f(u + i\frac{n}{2}); \quad f^{\pm} = f^{[\pm 1]}$$

- Large u encode quantum numbers.

so_6 quantum numbers

$$\mathbf{P}_a \underset{u \to \infty}{\simeq} A_a u^{-\hat{M}_a} \qquad \Delta = \Delta^{(0)} +$$

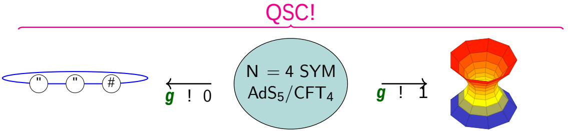
$$\mathbf{Q}_i \underset{u \to \infty}{\simeq} B_i u^{\hat{M}_i - 1}$$

$su_{2,2}$ quantum numbers

The spectral problem

- The QSC allows computations both at strong and weak coupling

[Marboe,Volin'14,'18,Gromov,Levkovich-Maslyuk,Sizov'15,]



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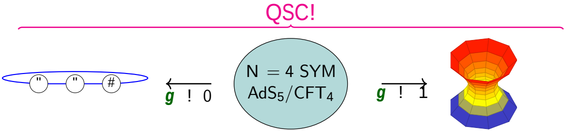
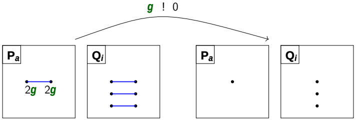


- Weak coupling: Perturbation around a non-compact spin chain.



Functions: $u, \frac{1}{u}, \eta_s = \sum_{n=0}^{\infty} \frac{1}{(u+in)^s}, \ldots$

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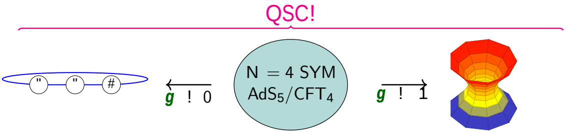

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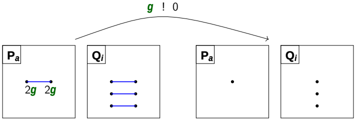


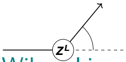
Functions: $u, \frac{1}{u}, \eta_s = \sum_{n=0}^{\infty} \frac{1}{(u+in)^s}, \ldots$

- Strong coupling: Have to resort to numerical methods!

Variations of QSC

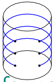
- There exist a plethora of deformations of the $\mathcal{N} = 4$ QSC.

[Gromov,Levkovich-Maslyuk '15]



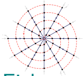
Wilson Lines

[Klabbers,van Tongeren '17]



- deformations

[Gromov, Kazakov,Korchemsky,Negro,Sizov '17]



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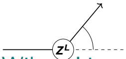

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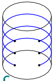

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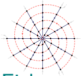


Fishnets

- For AdS/CFT currently only two other curves on the market

ABJM
AdS_4 CP^3

[Cavaglià,Fioravanti,Gromov,Tateo 14']

AdS_3/CFT_2

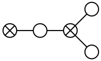
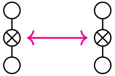
[SE,Volin'21]
[Cavaglià,Gromov,Stefański,Torrielli,21']

Conjecture: AdS_3 S^3 T^4 with RR-flux

(TBA was constructed [Frolov,Sfondrini '21])

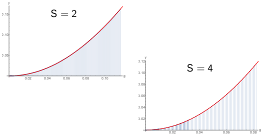
Agree? Open question.

Status of low-dimensional QSC

	AdS_4	AdS_3
Status:	Derived from TBA	Conjectured
Algebraic Structure:	 $osp_{6|4}$	 $psu_{1,1|2}$ $psu_{1,1|2}$
Analytic Structure	Quadratic cuts	No quadratic cuts
"sl_2" Weak Coupling	✓ [Bombardelli,Cavaglià,Conti,Tateo '18] [Anselmetti,Bombardelli,Cavaglià,Tateo '15]	✓ [Cavaglià,SE,Gromov,Ryan '22]
Strong Coupling Numerics	✓ [Bombardelli,Cavaglià,Conti,Tateo '18]	?

Example of explicit results

- The AdS_3 QSC was solved in an "sl_2"-sector, AdS_3 analogue of $\mathrm{tr}\, \boldsymbol{Z} D^S \boldsymbol{Z}$ [Cavaglià,SE,Gromov,Ryan'22]



$$
{}_{S=2} = 12\,\boldsymbol{g}^2 + \frac{864}{35}\boldsymbol{g}^3 + (\quad 48 \qquad \frac{576}{7\quad{}^{2}})\boldsymbol{g}^4
+ \qquad \frac{405504}{875\quad{}^{3}} \qquad \frac{51552}{143} \qquad \boldsymbol{g}^5
+ (444 \qquad \frac{70665216}{4375\quad{}^{4}} + \frac{230121984}{175175\quad{}^{2}})\boldsymbol{g}^6
+ \qquad \frac{16896}{35} \quad {}^{3} \qquad \frac{4965482496}{21875\quad{}^{5}}
+ \frac{6791453184}{875875\quad{}^{3}} + \frac{1102677696}{146965} \qquad \boldsymbol{g}^7 + O(\boldsymbol{g}^8)
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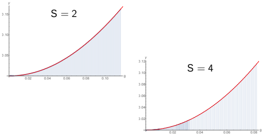

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- Fitting $S = 2; 4; 6; 8$ it was found that

$$
{}_S = 8 \boldsymbol{S}_1(S)\, g^2 + \frac{384}{35} \boldsymbol{S}_1(S)^2\, g^3 + \quad {}^{N=4}_{(4)} \quad \frac{512}{21\ \ 2} \boldsymbol{S}_1(S)^3 \quad g^4 + O(g^5)
$$

$$
\boldsymbol{S}_1(S) = \textstyle\sum_{n=1}^{S} \frac{1}{n}
$$

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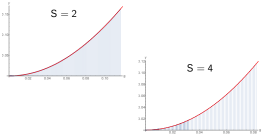

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- Features

g^{odd} $\quad \underbrace{\frac{1}{a}}_{\text{Unexpected}} \quad \underbrace{\frac{384}{35} S_1(S)^2 g^3}_{\text{Early wrapping!}}$

The Hagedorn temperature from the AdS_4 QSC

Beyond the spectral problem: Hagedorn

- The Hagedorn temperature, $\boldsymbol{T_H}$, is the temperature for which the partition function diverges:

$$\lim_{T \,!\, T_H} e \quad \mathrm{D} = 1\,; \qquad\qquad = \frac{1}{T}\,:$$

The dilation operator

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- Goal: Compute the Hagedorn temperature for ABJM (in the large $\boldsymbol{N}$ limit) using QSC following the N $= 4$ computation of Harmark and Wilhelm [Harmark,Wilhelm 17',18',21'].

Hagedorn temperature for free $\mathcal{N} = 4$ SYM

- We start from $g = 0$ [Sundborg '99].

  $\mathcal{N} = 4$ Fields: $\{F \; ; \; _i ; \; \bar{}^{i} ; \; _I\}$ $\quad z_{\mathcal{O}} = \mathrm{tr}_{\mathcal{O}}\, e^{\; D_0} = \mathrm{tr}_{\mathcal{O}}\, y^{D_0};$

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- Let us calculate z :

$$z \;(y) = \frac{6\,y}{(1 \quad y)^4} \; 1 \quad y^2 \; = \frac{6y(1+y)}{(1 \quad y)^3}$$

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- Single-trace partition function, (Euler's $\ $-function, $\omega = e^{2\ \ i}$)

$$Z = \sum_{\text{single-trace } O} y^{D_0(O)} = \underbrace{\sum_{n=1}^{\infty} \frac{\ (n)}{n} \log\ 1\ \ \ z(\omega^{n+1} y^n)}_{\text{diverges for } z=1}\ \ z\,:$$

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$$T_H^{(0)} = \frac{1}{2\log\; 2 + \mathrm{p}\,\overline{3}}:$$

Hagedorn from QSC

- Finite coupling using TBA and QSC explained in [Harmark,Wilhelm 17',18',21']. They gave the following recipe:

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- Twisted asymptotics

$$\mathbf{Q}_i \simeq_{u\to\infty} e^{\sigma_i \pi u} e^{\sigma_i \frac{\mathrm{i}u}{2T_H}} u^{\tilde{M}_i}, \quad \tilde{M}_i \in \mathbb{Z} \qquad \sigma_i = \{1, 1, -1, -1\}.$$

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- Done!

$\mathbf{AdS_4}$ in a nutshell

- We now turn to ABJM. Symmetry algebra $\mathfrak{osp}_{6j4}$. Two basic representations

$$\underbrace{(\ {}_i;\ {}^i)}_{\text{Particle A } (\boldsymbol{N};\bar{\boldsymbol{N}})} \qquad \underbrace{(\bar{\ }^i;\bar{\ }_i)}_{\text{Particle B } (\bar{\boldsymbol{N}};\boldsymbol{N})}$$

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- Single-trace operators:

$$\mathrm{O} = \mathrm{tr}\, W_A W_B W_A ::: \qquad W_A \ 2 \ \mathrm{f}\ \ _i;\ ^i \mathrm{g}; W_B \ 2 \ \mathrm{f}\bar{\ }^i;\bar{\ }_i \mathrm{g}:$$

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- Single-trace operators:

$$\mathcal{O} = \operatorname{tr} \boldsymbol{W_A W_B W_A} \dots \qquad \boldsymbol{W_A} \in \{\ _i ;\ ^i\} ; \boldsymbol{W_B} \in \{\bar{\ }^i ;\ \bar{\ }_i\} .$$

- Singelton partition functions

$$\boldsymbol{z_A} = \boldsymbol{z_B} = \frac{4\sqrt{\boldsymbol{y}}}{(1 - \sqrt{\boldsymbol{y}})^2} ;$$

$$\boldsymbol{z_A}(\boldsymbol{y}_{\boldsymbol{H}}^{(0)})\, \boldsymbol{z_B}(\boldsymbol{y}_{\boldsymbol{H}}^{(0)}) = 1 \implies \boldsymbol{T}_{\boldsymbol{H}}^{(0)} = \frac{1}{4 \log\left(1 + \sqrt{2}\right)} .$$

What I will explain

- To Do:
  - Identify $\boldsymbol{z}$ in the QSC and twist the curve appropriately.
  - Solve analytically at weak coupling.
  - Go to strong coupling using numerics.
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- First: The outcome.

Weak coupling results

- Use $\boldsymbol{h}$ for integrability coupling constant. Conjecture [Gromov,Sizov '14]

$$= \frac{\sinh(2\ \ \boldsymbol{h})}{2}\ {}_3\boldsymbol{F}_2\quad \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; 1; \frac{3}{2};\quad \sinh^2(2\ \ \boldsymbol{h})$$

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$$T_H^{(0)} = \frac{1}{4} \frac{1}{\log\ 1 + \sqrt{2}}; \qquad T_H^{(1)} = \frac{\sqrt{2}\ \ \ 1}{\log\ 1 + \sqrt{2}};$$

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- Agrees with tree-level and, up to a factor 2, with h^2 calculation in [Papathanasiou,Spradlin 09']

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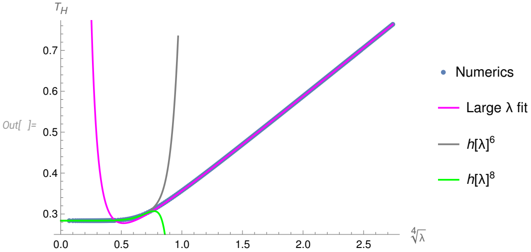

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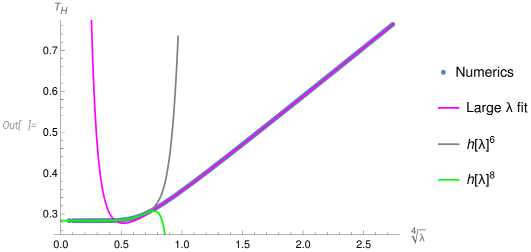

- Fitting the curve:

$$\boldsymbol{T_H} = \frac{\lambda^{\frac{1}{4}}}{2^{\frac{5}{4}}\sqrt{\pi}} + \frac{3}{8\pi} - \frac{(0.0308 \pm 0.0004)}{\lambda^{\frac{1}{4}}} + \frac{0.046 \pm 0.002}{\lambda^{\frac{1}{2}}} + \dots$$

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$$T_H^{AdS_{d+1}} = \frac{1}{2\sqrt{2\lambda}} + \frac{d}{8}$$

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Conjecture ⟶

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- First principle derivation? Validity beyond $d = 3, 4$?

Technical details or Solving the AdS_4 QSC

Basics of $\mathfrak{osp}_{6\mathrm{j}4}$ Q-systems

- Intuition: The structure of the Q-system should reflect the underlying $\mathfrak{osp}_{6\mathrm{j}4}$ symmetry algebra.

$$\det \boldsymbol{Q}_{a\mathrm{j}i} = \det \boldsymbol{Q}^{a}{}_{i} = \quad 1$$

$$\mathbf{P}_{A} = \mathrm{Q}_{A\mathrm{j};}\,; \qquad \mathbf{Q}_{I} = \mathrm{Q}_{;\mathrm{j}I}\,; \qquad \boldsymbol{Q}_{a\mathrm{j}i} \quad \boldsymbol{Q}^{a}{}_{\mathrm{j}i}\,:$$

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- $\boldsymbol{Q}_{a|i}$ and $\boldsymbol{Q}^{a}{}_{|i}$ are basic. Construct other functions from them

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Same procedure using ${}_{AB}, \Sigma_{IJ}$ to find $\mathrm{Q}_{A|I}, \mathrm{Q}_{AB|I}, \mathrm{Q}_{A|IJ}, \mathrm{Q}_{AB|IJ}$.

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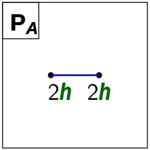
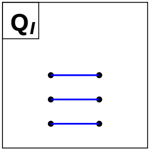
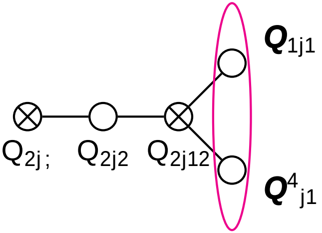
- $\mathbf{P}_A$ and $\mathbf{Q}_I$ have the same structure as in AdS_5





Bethe equations

- Bethe equations follows naturally from the bilinear expressions.
- Example:



$$\left.\frac{Q^{+}_{2|2}}{Q_{2|2}}\right|_{P_2=0} = 1\,;$$

$$\left.\frac{Q^{[2]}_{2|2}}{Q^{[-2]}_{2|2}}\,\frac{P^{-}_{2}\,Q^{-}_{2|12}}{P^{+}_{2}\,Q^{+}_{2|12}}\right|_{\mathcal{Q}_{2|2}=0} = 1\,;$$

$$\left.\frac{Q^{+}_{1|1}\,(Q^{4}{}_{|1})^{+}\,Q^{-}_{2|2}}{Q^{-}_{1|1}\,(Q^{4}{}_{|1})^{-}\,Q^{+}_{2|2}}\right|_{\mathcal{Q}_{2|12}=0} = 1\,;$$

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- We need to nd characters from QSC, this means twisting.

Twisting the curve

- Let us turn o the coupling $h = 0$ and take a twisted ansatz

$$\begin{cases} Q_{aji} = A_{aji}\, x^{i u \omega_a}\, y^{i u_i}; \\ Q^{a}_{ji} = A^{a}_{ji}\, x^{i u \omega_a}\, y^{i u_i}; \end{cases} \qquad \omega = \frac{1}{2}\begin{pmatrix} + + + \\ + \\ + \\ + \end{pmatrix}; \qquad = \frac{1}{2}\begin{pmatrix} + + \\ + \\ + \end{pmatrix}:$$

$$x = \{x_1; x_2; x_5\} \quad y = \{y_1; y_2\}$$

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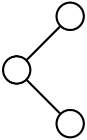
- Trick to solve the Q-system: Consistency equations



$$4 \wedge 4 = 6$$
$$\overline{4} \wedge \overline{4} = 6$$

$$Q_{A|IJ} = {}^{ij}_{IJ} \, \overline{}^{ab}_{A} \, Q^{+}_{a|i} \, Q_{b|j}$$
$$= {}^{ij}_{IJ} \, (\;_A)_{ab} \, (Q^{a}{}_{|i})^{+} \, (Q^{b}{}_{|j})$$

Characters from Q-functions

- To nd the partition functions we construct bilinears again!
  - Compact spin chains:

$$\frac{1}{2} \;{}^{ij}\, Q^{[2]}_{aji} \,(Q^{a}{}_{jj})^{[\quad 2]} = x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_3 + \frac{1}{x_3} \quad y_1 \quad \frac{1}{y_1} \quad y_2 \quad \frac{1}{y_2}$$

  - Non-compact: $(y_1 = y_2)$

$$Q^{[1]}_{aj4}\,(Q^{a}{}_{j1})^{[\quad 1]} = \frac{1}{(1 \quad y^2)^3}\left(y(1 \quad y^2)_{\;4} \quad 2y^2(1 \quad y^2)_{\;\overline{4}}\right) \qquad (3.1)$$

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- Simplest case: Set $y_1{}^{i u} = y_2{}^{i u} = e^{u} y^{\;\;i u}; x_a = 1$.

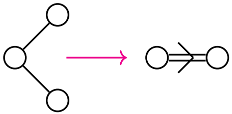
$$Q_{aji} \simeq \begin{pmatrix} e^{\;\;u} y^{i u} & u & 1 & e^{u} y^{\;\;i u} \\ e^{\;\;u} y^{i u} u^2 & u^3 & u^2 & e^{u} y^{\;\;i u} u^2 \\ e^{\;\;u} y^{i u} u & u^2 & u & e^{u} y^{\;\;i u} u \\ e^{\;\;u} y^{i u} u^3 & u^4 & u^3 & e^{u} y^{\;\;i u} u^3 \end{pmatrix} ; \qquad y = e^{\;\;\frac{1}{2T}} :$$

Turning on coupling

- We are now in the symmetric sector

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$Q^a_{ji} = \quad {}^{ab} Q_{bjj} \, K^j_i$$



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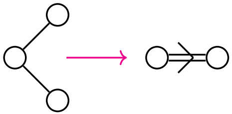
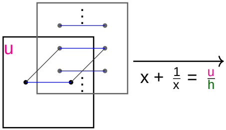


- While Q_{aji} are "basic" P_A have simpler analytic properties. Can parameterise P_A using Zhukovsky $x + \frac{1}{x} = \frac{u}{h}$

Parameters to $x!$

$$P_A \, / \, \mathrm{P}\; {}^{1}_{n=}\; M_a \frac{c_{A;n}}{x^n}$$



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$$Q_{aji} = Q^{(0)}_{aji} + h^2 Q^{(0)}_{ajj} (b^{j}{}_{i})^{+} + O(h^4)\,;$$

Then one nds

$$b^{j}{}_{i} \quad \underbrace{(\ \ 1)^{\frac{i(i\ \ 1)}{2} + \frac{j(j\ \ 2)}{2}} (b^{j}{}_{i})^{[2]}}_{\text{Di erent from } N\ \ = 4} = \quad \underbrace{K^{j}_{k}\ {}^{kl}((Q^{(0)})^{a}{}_{jl})^{+} P^{(1)}_{ab}\ {}^{bc}(Q^{(0)}_{cji})}_{\text{Known}}$$

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- b^{i}_{j} will be given in terms of

$$u\,; \qquad \frac{1}{u}\,; \qquad {}^{t}_{s} = \overset{1}{\underset{n=0}{X}} \frac{t^{n}}{(u + i\, n)^{s}}\,; \qquad t = 1\,; y \quad {}^{1}; y \quad {}^{2} \qquad (3.2)$$

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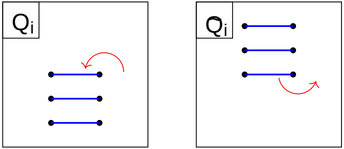
- This implies that $T_H^{(n)}$ in the end will be written in terms of $T_H^{(0)}; e^{\frac{1}{2T_H^{(0)}}}$ and

$$ \mathrm{Li}_{s_1;\dots;s_k}(t_1;\dots;t_k) = \sum_{0<n_1<\dots<n_k} \frac{t_1^{n_1}\cdots t_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}} \qquad (3.4) $$

- Still many free parameters around, we need to x them!

Gluing conditions

- To nd T_H we need to also consider $\tilde{Q}_i$. Recall:



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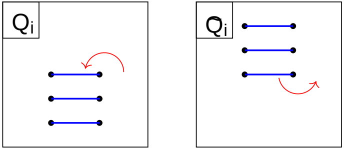

- Construct a lower-halfplane analytic $\tilde{Q}_I$ using parity:

$$\tilde{Q}_I(u) = \begin{pmatrix} e^{2 u} & 0 & & 0 & 0 \\ 0 & e^{2 u} & 0 & & 0 \\ 0 & 0 & e^{\ 2 u} & 0 & 0 \\ 0 & 0 & 0 & e^{\ 2 u} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} {}_I{}^J Q_J(\ u) \qquad (3.5)$$

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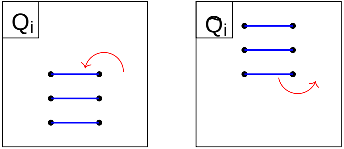

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- Zeros are xed by asymptotics and parity. Example:

$$Q_3 \simeq_{u\to i\infty} e^{\frac{|u|}{2T}} \qquad \tilde{Q}_3 \simeq_{u\to -i\infty} e^{\frac{|u|}{2T}} \quad (3.6)$$

Quantisation and results

- Finally we can now fix all coefficients by demanding that

$$\left.\left(Q_l + \tilde{Q}_l\right)\right|_{u\simeq 0} = \text{regular}, \qquad \left.\left(\frac{Q_l}{\sqrt{u-2\hbar}} - \frac{\tilde{Q}_l}{\sqrt{u+2\hbar}}\right)\right|_{u\simeq 0} = \text{regular}.$$

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- This fixes T_H! For example

$$T_H^{(0)} = \frac{1}{4 \log\left(1 + \sqrt{2}\right)} \simeq 0.2836481643\ldots$$

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- We computed up to h^8, rather long expressions, numerically

$$T_H^{(2)} = 2.542811207\ldots \qquad T_H^{(3)} = 21.77821058\ldots$$

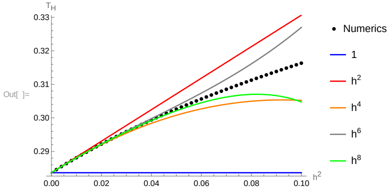
$$T_H^{(4)} = 222.2996920\ldots$$

Explicit T_H^4

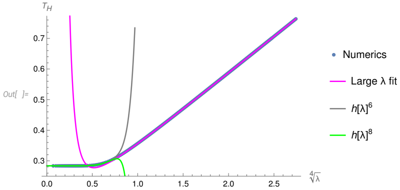
$$
\begin{aligned}
T_H^{(3)} = \frac{4}{3} \quad & 48\mathrm{Li}_{1;1}\left(\frac{1}{(1+\sqrt{2})^2}; \frac{1}{(1+\sqrt{2})^2}\right) \quad 48\mathrm{Li}_{1;1}\left(3+2\sqrt{2}; \frac{1}{(1+\sqrt{2})^2}\right) \\
& + 12\sqrt{2}\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^4}\right) \quad 45\sqrt{2}\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) + 84\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) \\
& + 36\sqrt{2}\mathrm{Li}_3\left(\frac{1}{(1+\sqrt{2})^2}\right) + 24\mathrm{Li}_3\left(\frac{1}{(1+\sqrt{2})^2}\right) \\
& + 48\sqrt{2}\log\left(1+\sqrt{2}\right) \quad \mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^2}\right)^2 \\
& + \mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^2}\right) \quad 48\sqrt{2}\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) + 18 \quad 4 \quad 5\sqrt{2} \quad \log\left(1+\sqrt{2}\right) \\
& + \frac{\left(12\sqrt{2} \quad \mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right)\right)^2}{\log\left(1+\sqrt{2}\right)} + 48\sqrt{2}\log\left(1+\sqrt{2}\right) \mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^4}\right) \\
& + 24\sqrt{2}\log\left(1+\sqrt{2}\right) \mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) + 16\log\left(1+\sqrt{2}\right) \mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) \\
& + \frac{18\sqrt{2}\mathrm{Li}_4\left(\frac{1}{(1+\sqrt{2})^2}\right)}{\log\left(1+\sqrt{2}\right)} + \frac{12\mathrm{Li}_4\left(\frac{1}{(1+\sqrt{2})^2}\right)}{\log\left(1+\sqrt{2}\right)} + 45\sqrt{2} \quad 66 + 35\sqrt{2}\log\left(1+\sqrt{2}\right) \\
& \quad 52\log\left(1+\sqrt{2}\right)
\end{aligned}
$$

Numerics

- Numerics: Use the N $= 4$ algorithm of [Gromov,Levkovich-Maslyuk,Sizov '15].
- Procedure: Minimise the gluing condition.
- We can verify weak-coupling



- And go to strong coupling



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- $\boldsymbol{T}_{\boldsymbol{H}}^{(1)}$ depends on $\boldsymbol{D}_2$:

$$\frac{\boldsymbol{T}_{\boldsymbol{H}}^{(1)}}{\boldsymbol{T}_{\boldsymbol{H}}^{(0)}} \propto \langle \boldsymbol{D}_2 \rangle_{\mathrm{V}_A \quad \mathrm{V}_B \quad \mathrm{V}_A} \Big|_{\boldsymbol{y} = \boldsymbol{y}_H^{(0)}} \tag{3.7}$$

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- $T_H^{(1)}$ depends on D_2:

$$\frac{T_H^{(1)}}{T_H^{(0)}} \propto \langle D_2 \rangle_{V_A \quad V_B \quad V_A} \Big|_{y = y_H^{(0)}} \tag{3.7}$$

- Computation of $\langle D_2 \rangle$

$$\langle D_2 \rangle = \sum_{j=0}^{\infty} M_j \quad _j\,; \qquad V_A \quad V_B \quad V_A = \quad_{j=0}^{1} V_j\,; \tag{3.8}$$

$_j$ obtainable from [Dolan '08] and M_j from [Papathanasiou ,Spradlin '09]

Turning on fugacities, Part 2

- Result:

$$\frac{T_H^{(1)}}{T_H^{(0)}} = \frac{4(y_H^{(0)})^2}{(1+y_H^{(0)})^2(1-y_H^{(0)})^5}\prod_{a=1}^{2}\frac{(1+x_a)(y_H^{(0)}+x_a)(1+y_H^{(0)}x_a)}{x_a^{\frac{3}{2}}}$$

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- Twisting R-symmetry doesn't change the gluing matrix. Everything keeps working!
- Slow to compute with undetermined fugacities... For numerical values we find a perfect match!

Conclusions and outlook

Conclusions

- The Quantum Spectral Curve is useful not only for AdS_5 but beyond.
- Obtained weak coupling expansion for $\boldsymbol{T_H}$ in AdS_4 up to $O(\boldsymbol{h}^8)$
- Numerical prediction for strong coupling expansion + Conjecture for exact expression:

$$\boldsymbol{T_H} = \frac{\quad}{2}^{\frac{1}{4}} \frac{1}{2\sqrt{\quad}} + \frac{3}{8} + \frac{3 \quad 6\log(2)}{8 \quad ^{3/2}} \frac{\quad}{2}^{\frac{1}{4}} + \frac{165}{512 \quad ^2} \frac{\quad}{2}^{\frac{1}{2}} + O(\quad ^{\frac{3}{4}})$$

- Included R-symmetry fugacities and matched to order $O(\boldsymbol{h}^2)$ (Also works in AdS_5)

Outlook

- AdS_3 using QSC or TBA? Inclusion of NSNS-flux?
- Strong coupling calculations with additional fugacities. (Work in progress)
- Twisting the ABJM curve should be useful for
  - Wilson lines [Correa, Giraldo-Rivera,Lagares,'23]
  - Study various deformations ; ::: [Chen,Liu,Wu,'16]
- More general ABJM questions: Structure constants?

  [Basso,Georgoudis,Klemenchuk Sueiro '22;Bercini,Homrich,Vieira '22]

Thank you

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