Exact integrated correlators in $\mathcal{N} = 4$ super-Yang-Mills

Daniele Dorigoni

Joint work w/ Congkao Wen
Michael B. Green
and Haitian Xie

ETH zürich

Integrability in Gauge and String Theory 2023

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~~Integrability~~
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Outline

- Integrated correlators in $\mathcal{N} = 4$ SYM
- Exact integrated correlators from Susy loc.
- GNO Duality (E/M duality)
- Large-N perturbative and non-
- Holographic interpretation
- Laplace-Difference equations

What do we want?
Why do we want it?
How do we get it?

What type of correlators

$\mathcal{N} = 4$ correlators

Amongst the many fields of $\mathcal{N} = 4$ we have:

Φ_I I=1,…,6 Adjoint scalar

$\mathcal{O}_2(x,Y) = \mathrm{Tr}(\Phi_I \Phi_J) Y^I Y^J$ $\Delta = 2$

Null polarisation vectors for

1/2 BPS operators, super-conformal primaries in the stress-energy tensor super-multiplet ([0,2,0])

$\mathcal{N} = 4$ correlators

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$$\mathcal{O}_2(x, Y) = \text{Tr}(\Phi_I \Phi_J) Y^I Y^J \qquad \Delta = 2$$

1/2 BPS operators, super-conformal primaries in the stress-energy tensor super-multiplet

We consider the (NON-PROTECTED!) 4pt function

$$\langle \mathcal{O}_2(x_1, Y_1) ... \mathcal{O}_2(x_4, Y_4) \rangle = \frac{1}{|x_{12}|^4 |x_{34}|^4} \mathcal{I}(Y) \mathcal{T}_{G_N}(u, v)$$

$\mathcal{N} = 4$ correlators

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Fixed by super-conformal symmetry

$\mathcal{N} = 4$ correlators

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1/2 BPS operators, super-conformal primaries in the stress-energy tensor super-multiplet

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Non-trivial function of u,v, coupling, gauge group G_N

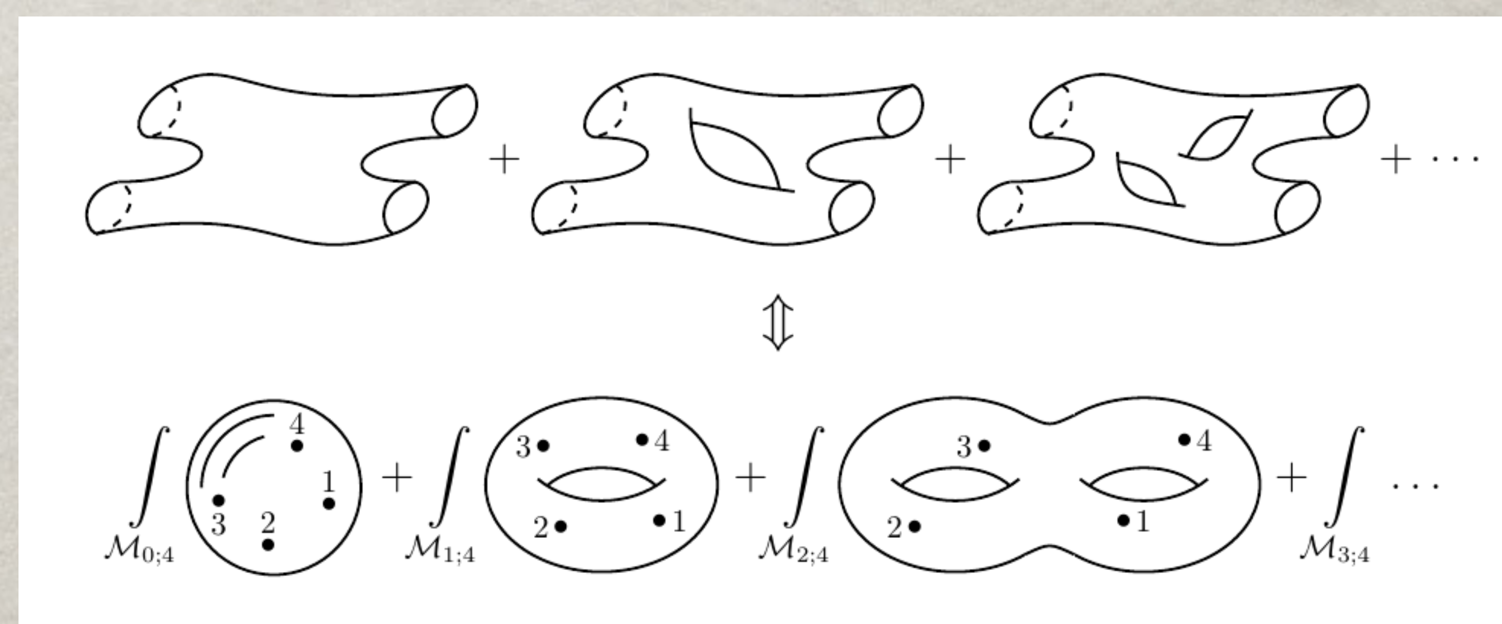
Why do we care?

stress-energy tensor ≃ gravitons

$$\langle \mathcal{O}_2(x_1, Y_1)...\mathcal{O}_2(x_4, Y_4)\rangle = \frac{1}{|x_{12}|^4|x_{34}|^4}\mathcal{I}(Y)\mathcal{T}_{G_N}(u,v)$$

Fun part = hard part

$$\langle T_{\mu_1\nu_1}(x_1)...T_{\mu_4\nu_4}(x_4)\rangle$$



4-graviton amplitude in IIB (see later)

How to get a handle on

$\mathcal{N} = 4$ Integrated correlators

Pestun' Susy localisation on S^4 for $\mathcal{N} = 2^*$
massive deformation, m, of $\mathcal{N} = 4$
S^4 partition function $Z_{G_N}(m; \tau, \bar{\tau})$

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

with $SL(2,\mathbb{Z})$ invariant Laplacian $\Delta_\tau = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$

G_N Gauge group (more on this later)

$\mathcal{N} = 4$ Integrated correlators

Pestun' Susy localisation on S^4 for $\mathcal{N} = 2^*$
massive deformation, m, of $\mathcal{N} = 4$
S^4 partition function $Z_{G_N}(m; \tau, \bar{\tau})$

$$\mathcal{C}_{G_N}(\tau, \bar{\tau}) = \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_{G_N}(m; \tau, \bar{\tau})|_{m=0}$$

[Binder,Chester,Pufu,Wang]

$$= \int \prod_{i=1}^{4} dx_i \mu(\{x_i\}) \langle \mathcal{O}_2(x_1) ... \mathcal{O}_2(x_4) \rangle$$

$$\tilde{\mathcal{C}}_{G_N}(\tau, \bar{\tau}) = \partial_m^4 \log Z_{G_N}(m; \tau, \bar{\tau})|_{m=0}$$

[Chester,Pufu]

$$= \int \prod_{i=1}^{4} dx_i \tilde{\mu}(\{x_i\}) \langle \mathcal{O}_2(x_1) ... \mathcal{O}_2(x_4) \rangle$$

$\mathcal{N} = 4$ Integrated correlators

$$\mathcal{C}_{G_N}(\tau,\bar\tau) = \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_{G_N}(m;\tau,\bar\tau)|_{m=0}$$

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[Binder,Chester,Pufu,Wang]

Very specific measures fixed by susy!

[Chester,Pufu]

$$\tilde{\mathcal{C}}_{G_N}(\tau,\bar\tau) = \partial_m^4 \log Z_{G_N}(m;\tau,\bar\tau)|_{m=0}$$

$$= \int \prod_{i=1}^{4} dx_i \tilde\mu(\{x_i\}) \langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

Why Integrated?

In this talk we will focus entirely on:

$$\mathcal{C}_{G_N}(\tau,\bar\tau) = \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_{G_N}(m;\tau,\bar\tau)|_{m=0}$$
$$= \int \prod_{i=1}^{4} dx_i \mu(\{x_i\}) \langle \mathcal{O}_2(x_1) \ldots \mathcal{O}_2(x_4)\rangle$$

[Binder, Chester, Pufu, Wang]

We found an exact formula valid for all G_N
and all values of the coupling τ and GNO co/in-variant

What did we find?

$$\mathcal{C}_{G_N}(\tau,\bar\tau) = \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_{G_N}(m;\tau,\bar\tau)|_{m=0}$$

$$= \int \prod_{i=1}^{4} dx_i \mu(\{x_i\}) \langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

For $G_N = SU(N), SO(2N), SO(2N+1), USp(2N)$

$$\mathcal{C}_{G_N}(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty \left(B^1_{G_N}(t) e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} + B^2_{G_N}(t) e^{-t\pi \frac{|m+2n\tau|^2}{2\tau_2}} \right) dt$$

[DD,Green,Wen]

Using key-results of [Chester, Green, Pufu, Wang, Wen] and [Alday,Chester,Hansen]

$B^{(1,2)}_{G_N}(t)$ are rational functions of t (back in a sec)

Exact Integrated correlators

For $G_N = SU(N), SO(2N), SO(2N+1), USp(2N)$

$$\mathcal{C}_{G_N}(\tau,\bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty \left(B^1_{G_N}(t) e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} + B^2_{G_N}(t) e^{-t\pi \frac{|m+2n\tau|^2}{2\tau_2}} \right) dt$$

[DD,Green,Wen]

Using key-results of [Chester, Green, Pufu, Wang, Wen] and [Alday,Chester,Hansen]

First example of a non-perturbative and non-protected quantity in $\mathcal{N}=4$ valid

- for all values of N
- for all classical gauge groups
- for all values of the complex coupling AND
- Montonen-Olive (GNO) duality covariant!

Other Integrated correlators

[Paul,Perlmutter,Raj - Brown,Wen, Xie]

$$\mathcal{C}_{N,p}(\tau,\bar{\tau}) = \int \prod_{i=1}^{4} dx_i \nu(\{x_i\}) \langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_p(x_3)\mathcal{O}_p(x_4)\rangle$$

super-conformal primaries $\Delta = p$

[Pufu, Rodriguez, Wang]

$$I_N(\tau,\bar{\tau}) = \partial_m^2 \langle W \rangle|_{m=0} = \int \prod_{i=1}^{2} dx_i \tilde{\nu}(\{x_i\}) \langle W \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\rangle$$

1/2-BPS fundamental Wilson loop along great circle

Before The Nitty Gritty, any questions?

SUSY Loc

Pestun's $\mathcal{N} = 2^*$ S^4 partition fct:

$$Z_{G_N}(m,\tau,\bar{\tau}) = \int V_{G_N}(a) e^{-2\pi\tau_2\langle a,a\rangle} \hat{Z}^{pert}_{G_N}(m,a) |\hat{Z}^{inst}_{G_N}(m,\tau,a)|^2 \, d^r a$$

1-Loop Nekrasov

- $G_N = SU(N), SO(N), USp(2N)$ Classical gauge groups
- a parametrizes r-dim Cartan subalgebra, real variables
- $V_{G_N}(a)$ Vandermonde determinant, e.g. $V_{SU(N)}(a) = \prod_{i<j} (a_i - a_j)^2$
- $\langle a, a \rangle$ Killing form
- $\tau = \tau_1 + i\tau_2 = \dfrac{\theta}{2\pi} + \dfrac{4\pi i}{g_{YM}^2}$; also $\pi \,\mathrm{Im}\, \tau = \pi\, \tau_2 = y$; $m \in \mathbb{R}_{\geq 0}$

SUSY Loc

Pestun's $\mathcal{N} = 2^*$ S^4 partition fct:

$$Z_{G_N}(m,\tau,\bar\tau) = \int V_{G_N}(a) e^{-2\pi\tau_2 \langle a,a\rangle} \hat{Z}^{pert}_{G_N}(m,a) |\hat{Z}^{inst}_{G_N}(m,\tau,a)|^2 \, d^r a$$

We want to compute:

$$\mathcal{C}_{G_N}(\tau,\bar\tau) = \frac{1}{4}\Delta_\tau \partial_{m^2} \log\, Z_{G_N}(m,\tau,\bar\tau)\Big|_{m=0}$$

with $SL(2,\mathbb{Z})$ invariant Laplacian $\Delta_\tau = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$

Goddard-Nuyts-Olive/Montonen-Olive duality
is the key player!

GNO Duality

Consider $\mathcal{N} = 4$ SYM with gauge group G_N and coupling

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

Montonen-Olive E/M duality generalizes to

$$T : (G_N, \tau) \to (G_N, \tau + 1)\,,$$
$$\hat{S} : (G_N, \tau) \to ({}^L G_N, -\frac{1}{r\tau})\,,$$

With r the ratio of the length squared of long and short roots of the Lie algebra of G_N

GNO Duality

G_N	${}^L G_N$
$U(N)$	$U(N)$
$SU(N)$	$PSU(N) = SU(N)/\mathbb{Z}_N$
$Spin(2N)$	$SO(2N)/\mathbb{Z}_2$
$Sp(N) = USp(2N)$	$SO(2N+1)$
$Spin(2N+1)$	$Sp(N)/\mathbb{Z}_2 = USp(2N)/\mathbb{Z}_2$
G_2	G_2
F_4	F_4
$E_{r=6,7,8}$	$E_r/\mathbb{Z}_{9-r}$

$$T \; : \; (G_N, \tau) \to (G_N, \tau + 1)\,,$$
$$\hat{S} \; : \; (G_N, \tau) \to ({}^L G_N, -\frac{1}{r\tau})\,,$$

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To appear shortly [DD, Vallarino]

GNO Duality

We consider only insertions of local operators
And only classical groups

$\mathfrak{g}_N$	${}^L\mathfrak{g}_N$
$su(N)$	$su(N)$
$so(2N)$	$so(2N)$
$usp(2N)$	$so(2N+1)$
$so(2N+1)$	$usp(2N)$

$$T \,:\, (G_N, \tau) \to (G_N, \tau + 1)\,,$$
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$$T \,:\, (G_N, \tau) \to (G_N, \tau+1)\,,$$
$$\hat{S} \,:\, (G_N, \tau) \to ({}^L G_N, -\frac{1}{r\tau})\,,$$

For $SU(N)$ and $SO(2N)$ we have r=1

$\hat{S} = S$ and T generate standard $SL(2,\mathbb{Z})$

Montonen-Olive self-duality

$$\tau \to \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Exact Integrated correlators

GNO Duality:

For $G_N = SU(N), SO(2N)$ we find:

$$\mathcal{C}_{G_N}(\tau,\bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty dt \left(B^1_{G_N}(t) e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} \right)$$

[DD, Green, Wen]

Hence $\mathcal{C}_{G_N}(\gamma\cdot\tau,\gamma\cdot\bar{\tau}) = \mathcal{C}_{G_N}(\tau,\bar{\tau})$ for $\gamma \in SL(2,\mathbb{Z})$

i.e. it is a non-holomorphic modular invariant function.

Spectral decomposition and ensemble averages

[Collier, Perlmutter]

GNO Duality

$\mathfrak{g}_N$	${}^L\mathfrak{g}_N$
$su(N)$	$su(N)$
$so(2N)$	$so(2N)$
$usp(2N)$	$so(2N+1)$
$so(2N+1)$	$usp(2N)$

$$T \,:\, (G_N,\tau) \to (G_N,\tau+1)\,,$$
$$\hat{S} \,:\, (G_N,\tau) \to ({}^LG_N,-\frac{1}{r\tau})\,,$$

For $SO(2N+1)$ and $USp(2N)$ we have r=2 hence under

$\hat{S}\cdot\tau = -\dfrac{1}{2\tau}$ we expect $SO(2N+1)$ to go into $USp(2N)$

T and $\hat{S}T\hat{S}$ generate self-duality group $\Gamma_0(2)$ i.e. $c \equiv 0 \bmod 2$

$$\hat{S}T\hat{S} : (G_N,\tau) \to (G_N, \frac{\tau}{1-2\tau})$$

Exact Integrated correlators

GNO Duality:

For $G_N = SO(2N+1), USp(2N)$ we find that:

$$B^1_{SO(2N+1)}(t) = B^2_{USp(2N)}(t)$$

$$B^2_{SO(2N+1)}(t) = B^1_{USp(2N)}(t)$$

$\hat{S} \cdot \tau = -\dfrac{1}{2\tau}$ we expect $SO(2N+1)$ to go into $USp(2N)$

Which implies:

$$\mathcal{C}_{SO(2N+1)}(\tau, \bar{\tau}) = \mathcal{C}_{USp(2N)}(-1/(2\tau), -1/(2\bar{\tau}))$$

$$\mathcal{C}_{G_N}(\tau, \bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty dt \left(B^1_{G_N}(t) e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} + B^2_{G_N}(t) e^{-t\pi \frac{|m+2n\tau|^2}{2\tau_2}} \right)$$

[DD,Green,Wen]

Exact Integrated correlators

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[DD,Green,Wen]

Exact Integrated correlators

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[DD, Green, Wen]

And $\mathcal{C}_{G_N}(\gamma \cdot \tau, \gamma \cdot \bar{\tau}) = \mathcal{C}_{G_N}(\tau, \bar{\tau})$ for $\gamma \in \Gamma_0(2)$

Exact Results for SU(N)

For $G_N = SU(N)$ we have $B^2_{SU(N)}(t) = 0$, $B^1_{SU(N)}(t) = B_{SU(N)}(t)$

$$B_{SU(N)}(t) = \sum_{s=2}^{\infty} b_{SU(N)}(s) \frac{t^{s-1}}{\Gamma(s)} = \frac{\mathcal{Q}_{SU(N)}(t)}{(t+1)^{2N+1}}$$

$$\mathcal{Q}_{SU(N)}(t) = -\frac{1}{4} N(N-1)(1-t)^{N-1}(1+t)^{N+1}$$
$$\left\{ (3 + (8N + 3t - 6)\, t)\, P_N^{(1,-2)} \left(\frac{1+t^2}{1-t^2} \right) + \frac{1}{1+t} \left(3t^2 - 8Nt - 3\right) P_N^{(1,-1)} \left(\frac{1+t^2}{1-t^2} \right) \right\}$$

$$B_{SU(2)}(t) = \frac{3(3t - 10t^2 + 3t^3)}{2(1+t)^5}$$

$$B_{SU(3)}(t) = \frac{9t(2 - 11t + 14t^2 - 11t^3 + 2t^4)}{(1+t)^7}$$

$$B_{SU(4)}(t) = \frac{15t(3 - 23t + 50t^2 - 72t^3 + 50t^4 - 23t^5 + 3t^6)}{(1+t)^9}$$

Nice palindromic polynomials

Exact Results for SU(N)

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$$\mathcal{Q}_{SU(N)}(t) = -\frac{1}{4} N(N-1)(1-t)^{N-1}(1+t)^{N+1} \left\{ \left(3 + (8N+3t-6)\,t\right) P_N^{(1,-2)}\left(\frac{1+t^2}{1-t^2}\right) + \frac{1}{1+t}\left(3t^2 - 8Nt - 3\right) P_N^{(1,-1)}\left(\frac{1+t^2}{1-t^2}\right) \right\}$$

Generating series are nice:

$$F_{SU}(t,z) = \sum_{N=0}^{\infty} B_{SU(N)}(t) z^N$$

$$F_{SU}(t,z) = \frac{3tz^2[(t+1)^2(t-3)(3t-1) - z(t-1)^2(t+3)(3t+1)]}{2(1-z)^{\frac{3}{2}}((t+1)^2 - (t-1)^2 z)^{\frac{7}{2}}}$$

SU(N) @ Large N

[DD,Green,Wen,Xie]

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}} \Big[\oint_\gamma \frac{F_{SU}(t,z)}{z^{N+1}} dz\Big] dt$$

Finite N perturbative expansion

$$C_{SU(2)}(\tau,\bar\tau) \sim \frac{9\zeta(3)}{y} - \frac{225\zeta(5)}{2y^2} + \frac{2205\zeta(7)}{2y^3} - \frac{42525\zeta(9)}{4y^4} + O(y^{-5})$$

Match with [Eden,Heslop,Korchemsky,Sokatchev]

$$y = \frac{4\pi^2}{g^2_{YM}} = \pi\tau_2$$

$$y \gg 1 \Leftrightarrow g^2_{YM} \to 0$$

with apologies to Burkhard for inadvertently omitting his name during the talk!

SU(N) @ Large N

[DD,Green,Wen,Xie]

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} \Big[\oint_\gamma \frac{F_{SU}(t,z)}{z^{N+1}} dz \Big] dt$$

@Large-N we can split into P and NP contributions:

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) \overset{N\to\infty}{=} \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} \Big[B^{\text{Pert}}_{SU(N)}(t) + B^{\text{N.P.}}_{SU(N)}(t) \Big] dt$$

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = \mathcal{C}^{P}_{SU(N)}(\tau,\bar\tau) + \mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau)$$

$$\mathcal{C}^{P}_{SU(N)}(\tau,\bar\tau) = \frac{N^2}{4} + \sum_{r=0}^{\infty} N^{\frac{1}{2}-r} f_r(\tau,\bar\tau) \qquad \mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau) = O(N^2 e^{-\sqrt{N}})$$

Large-N Perturbative

[DD,Green,Wen,Xie]

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = \mathcal{C}^{P}_{SU(N)}(\tau,\bar\tau) + \mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau)$$

The large-N expansion changes dramatically only half-integer index Eisensteins appear:

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4} E(\tfrac{3}{2};\tau,\bar\tau) + \frac{45}{2^8 N^{\frac{1}{2}}} E(\tfrac{5}{2};\tau,\bar\tau)$$
$$+ \frac{3}{N^{\frac{3}{2}}}\Big[\frac{1575}{2^{15}} E(\tfrac{7}{2};\tau,\bar\tau) - \frac{13}{2^{13}} E(\tfrac{3}{2};\tau,\bar\tau)\Big] + O(N^{-\frac{5}{2}})$$

Non-holo Eisensteins

$$E(s;\tau,\bar\tau) = \frac{1}{\pi^s}\sum_{(m,n)\neq(0,0)}\frac{\tau_2^s}{|m+n\tau|^{2s}}$$

$$= \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\pi^s\Gamma(s)}\tau_2^{1-s}$$

$$+\sum_{k\neq 0} e^{2\pi i k\tau_1}\frac{4\sqrt{\tau_2}}{\Gamma(s)}|k|^{s-\frac{1}{2}}\sigma_{1-2s}(|k|)K_{s-\frac{1}{2}}(2\pi|k|\tau_2)$$

$$(\Delta_\tau - s(s-1))E(s;\tau,\bar\tau) = 0$$

Large-N Perturbative

-Fixed g_{YM} large-N (modularity is preserved):

[Chester, Green, Pufu, Wang, Wen - DD, Green, Wen]

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4} E(\tfrac{3}{2};\tau,\bar\tau) + \frac{45}{2^8 N^{\frac{1}{2}}} E(\tfrac{5}{2};\tau,\bar\tau)$$

$$+ \frac{3}{N^{\frac{3}{2}}}\left[\frac{1575}{2^{15}} E(\tfrac{7}{2};\tau,\bar\tau) - \frac{13}{2^{13}} E(\tfrac{3}{2};\tau,\bar\tau)\right] + O(N^{-\frac{5}{2}})$$

4-graviton effective action in type IIB low-energy expansion

$$\tau = \chi + i/g_s$$

$$\mathcal{L}_{eff} = (\alpha')^{-4} g_s^{-2} R + f_1(\tau,\bar\tau)(\alpha')^{-1} g_s^{-1/2} R^4 + f_2(\tau,\bar\tau)\alpha' g_s^{1/2} d^4 R^4 + f_3(\tau,\bar\tau)(\alpha')^2 g_s d^6 R^4 + \ldots$$

N^2 $\quad N^{1/2}$ $\quad N^{-1/2}$ $\quad N^{-1}$

[Green, Gutperle - Green, Vanhove- Green, Miller, Vanhove]

Large-N Non-Perturbative

[DD,Green,Wen,Xie]

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = \mathcal{C}^{P}_{SU(N)}(\tau,\bar\tau) + \mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau)$$

With a more careful large-N analysis we also found some new, non-perturbative corrections:

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau) \to \sum_{(m,n)\neq(0,0)} \exp\Big[-4\sqrt{N}\Big(\frac{\tau_2}{\pi|m+n\tau|^2}\Big)^{-\frac{1}{2}}\Big]\Big(\frac{\tau_2}{\pi|m+n\tau|^2}\Big)^{s}$$

for different values of s half-integer.
Modular invariant, non-perturbative contributions

Holographic Interpretation:

Holo. Dictionary: consider $AdS_5 \times S^5$ with scale L

$$g_{YM}^2 = \frac{4\pi}{\tau_2} = 4\pi g_s \qquad \text{and} \qquad \sqrt{g_{YM}^2 N} = \frac{L^2}{\alpha'}$$

$$T_F = \frac{1}{2\pi\alpha'} \qquad \Rightarrow \qquad T_{p,q} = T_F |p + q\tau|$$

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar{\tau}) \to \sum_{\ell=1}^{\infty} \sum_{\gcd(p,q)=1} \exp\Big(-4\pi L^2 \ell \, T_{p,q}\Big)$$

NP terms are given by sum over ℓ coincident (p,q)-strings euclidean world-sheet wrapping a great S^2 inside S^5

[Some key differences for SO and USp]

’t Hooft Limit

To be more precise,
when we compute the NP terms
via saddle point we find:

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar{\tau}) \sim \sum_{(m,n)\neq(0,0)} \exp\Big[-NA\Big(\sqrt{\frac{\pi|m+n\tau|^2}{4N\tau_2}}\Big)\Big]$$

With:

$$A(x) = 4(x\sqrt{x^2+1} + \text{arcsinh}(x))$$

D3-brane action for multiply wound Wilson loops

[Drukker,Fiol]

't Hooft Limit

To be more precise,
when we compute the NP terms
via saddle point we find:

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau) \sim \sum_{(m,n)\neq(0,0)} \exp\Big[-NA\Big(\sqrt{\frac{\pi|m+n\tau|^2}{4N\tau_2}}\Big)\Big]$$

At large-N and fixed $\lambda = Ng^2_{YM} = \dfrac{4\pi N}{\tau_2}$ we focus on zero-mode w.r.t. τ_1

’t Hooft Limit

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau) \sim \sum_{(m,n)\neq(0,0)} \exp\Big[-NA\Big(\sqrt{\frac{\pi|m+n\tau|^2}{4N\tau_2}}\Big)\Big]$$

At large-N and fixed $\lambda = Ng_{YM}^2 = \frac{4\pi N}{\tau_2}$ we focus on zero-mode w.r.t. τ_1

$$1 \ll \lambda \ll N$$

- $(m,n) = (\ell,0)$ → Retrieve F-string world-sheet instantons $e^{-2\ell\sqrt{\lambda}}$ [DD,Green,Wen] found from resurgence at large-λ

Similar effects to [Arutyunov,DD,Savin]
Different from [Basso, Korchemsky]

't Hooft Limit

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau,\bar\tau) \sim \sum_{(m,n)\neq(0,0)} \exp\left[-NA\left(\sqrt{\frac{\pi|m+n\tau|^2}{4N\tau_2}}\right)\right]$$

At large-N and fixed $\lambda = Ng^2_{YM} = \dfrac{4\pi N}{\tau_2}$ we focus on zero-mode w.r.t. τ_1

- 0-mode of infinite sum of dyonic $(m,n) = (m \in \mathbb{Z}, n \neq 0)$

$1 \ll \lambda \ll N$

Retrieve NP effects $e^{-\frac{8\pi \ell N}{\sqrt{\lambda}}} = e^{-2\ell\sqrt{\tilde\lambda}}$ found from resurgence at large-$\tilde\lambda$

[Collier, Perlmutter - Hatsuda, Okuyama]

$$\tilde\lambda = \frac{(4\pi N)^2}{\lambda}$$

A unifying Picture:

Laplace-Difference Equations

A striking non-perturbative result: [DD,Green,Wen]

$$\Delta_\tau \mathcal{C}_{SU(N)}(\tau,\bar\tau) - 4c_{SU(N)}\Big[\mathcal{C}_{SU(N+1)}(\tau,\bar\tau) - 2\,\mathcal{C}_{SU(N)}(\tau,\bar\tau) + \mathcal{C}_{SU(N-1)}(\tau,\bar\tau)\Big] - (N+1)\,\mathcal{C}_{SU(N-1)}(\tau,\bar\tau) + (N-1)\,\mathcal{C}_{SU(N+1)}(\tau,\bar\tau) = 0\,.$$

$$\Delta_\tau \mathcal{C}_{SO(n)}(\tau,\bar\tau) - 2c_{SO(n)}\Big[\mathcal{C}_{SO(n+2)}(\tau,\bar\tau) - 2\,\mathcal{C}_{SO(n)}(\tau,\bar\tau) + \mathcal{C}_{SO(n-2)}(\tau,\bar\tau)\Big] - n\,\mathcal{C}_{SU(n-1)}(\tau,\bar\tau) + (n-1)\,\mathcal{C}_{SU(n)}(\tau,\bar\tau) = 0\,.$$

$$\Delta_\tau \mathcal{C}_{USp(n)}(\tau,\bar\tau) - 2c_{USp(n)}\Big[\mathcal{C}_{USp(n+2)}(\tau,\bar\tau) - 2\,\mathcal{C}_{USp(n)}(\tau,\bar\tau) + \mathcal{C}_{USp(n-2)}(\tau,\bar\tau)\Big] + n\,\mathcal{C}_{SU(n+1)}(2\tau,2\bar\tau) - (n+1)\,\mathcal{C}_{SU(n)}(2\tau,2\bar\tau) = 0\,.$$

Central Charges:

$$c_{SU(N)} = \frac{N^2-1}{4}\,, \qquad c_{SO(n)} = \frac{n(n-1)}{8}\,, \qquad c_{USp(n)} = \frac{n(n+1)}{8}$$

Laplace-Difference Equations

A striking non-perturbative result:

$$\Delta_\tau \mathcal{C}_{SU(N)}(\tau,\bar\tau) - 4c_{SU(N)}\Big[\mathcal{C}_{SU(N+1)}(\tau,\bar\tau) - 2\,\mathcal{C}_{SU(N)}(\tau,\bar\tau) + \mathcal{C}_{SU(N-1)}(\tau,\bar\tau)\Big] \\ - (N+1)\,\mathcal{C}_{SU(N-1)}(\tau,\bar\tau) + (N-1)\,\mathcal{C}_{SU(N+1)}(\tau,\bar\tau) = 0\,.$$

$$\Delta_\tau \mathcal{C}_{SO(n)}(\tau,\bar\tau) - 2c_{SO(n)}\Big[\mathcal{C}_{SO(n+2)}(\tau,\bar\tau) - 2\,\mathcal{C}_{SO(n)}(\tau,\bar\tau) + \mathcal{C}_{SO(n-2)}(\tau,\bar\tau)\Big] \\ - n\,\mathcal{C}_{SU(n-1)}(\tau,\bar\tau) + (n-1)\,\mathcal{C}_{SU(n)}(\tau,\bar\tau) = 0\,.$$

$$\Delta_\tau \mathcal{C}_{USp(n)}(\tau,\bar\tau) - 2c_{USp(n)}\Big[\mathcal{C}_{USp(n+2)}(\tau,\bar\tau) - 2\,\mathcal{C}_{USp(n)}(\tau,\bar\tau) + \mathcal{C}_{USp(n-2)}(\tau,\bar\tau)\Big] \\ + n\,\mathcal{C}_{SU(n+1)}(2\tau,2\bar\tau) - (n+1)\,\mathcal{C}_{SU(n)}(2\tau,2\bar\tau) = 0\,.$$

Everything is determined once we know the initial condition $\mathcal{C}_{SU(2)}(\tau,\bar\tau)$ (and $\mathcal{C}_{SU(1)} = 0$)

Conclusions:

- First non-trivial example of GNO covariant exact non-protected quantity valid for all N, all couplings and all gauge groups.
- Our ansatz passes many (many) consistency checks, perturbative and non-perturbative. [Billo',Frau, Fucito,Lerda,Morales] (proof via matrix model)
- String theory/QFT origin of these lattice-sum representations?
- String theory/QFT origin of the Laplace-difference equation?
- Semi-classical origin of non-perturbative corrections?
- Other integrated correlator $\partial_m^4 Z_{G_N}(m;\tau,\bar\tau)|_{m=0}$? [w.i.p. with Alday, Chester, Green and Wen]
- Exact formula for integrated correlator and super conformal bootstrap?

Thanks!