Integrable Monopoles

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Based on:

- C.K., & K.Zarembo, ArXiv:2305.03649

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Motivation

- ”The existence of magnetic monopoles seems to be one of the safest bets that one can make about physics not yet seen” (Joe Polchinski, at the Dirac Centennial Symposium)
- A 4D QFT containing a Monopole and a Higgs particle
- A novel example of an integrable dCFT based on N=4 SYM
- Novel insights on S-duality (‘t Hooft line dual to Wilson line)
- Possibly the simplest susy dCFT to attack with the boundary conformal bootstrap program.

Plan of the talk

I. Introducing the monopole and the Higgs

II. Prediction from S-duality

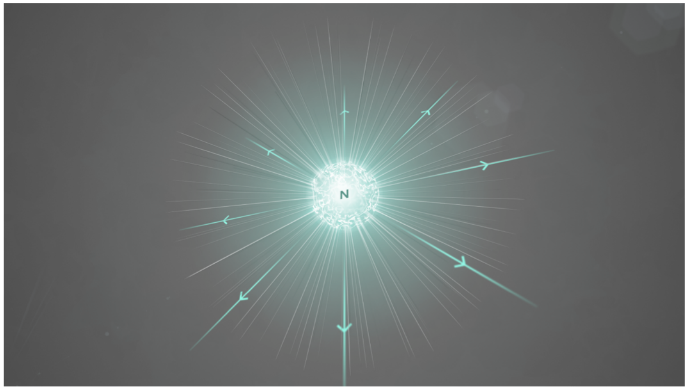
III. Confirmation of prediction via exactly solvable QM

IV. Novel exact results from Integrability

V. Conclusion and open problems

Introducing the monopole

$$\vec{B} = \frac{B\vec{r}}{2r^3}$$



Dirac '31

$$\frac{q_e q_m}{2\pi\epsilon_0 \hbar c^2} \in \mathbb{Z}$$

Dirac quantization condition: $B \in \mathbb{Z}$

't Hooft loop: world line of a monopole (static at the origin)

Disorder operator:
Prescribes certain singular behaviour of the gauge field

$$A_\phi = \frac{B}{2r}\frac{1-\cos\theta}{\sin\theta}, \qquad A_r = A_\theta = 0$$

A monopole in $\mathcal{N} = 4$ SYM

$$\mathbf{A}^{cl}_{\phi} = \mathbf{B}\,\frac{1-\cos\theta}{2r\sin\theta}, \qquad \mathbf{A}^{cl}_{r} = \mathbf{A}^{cl}_{\theta} = \mathbf{0},$$

Kapustin '05

$$\Phi^{cl}_{I} = \mathbf{B}\,\frac{n_I}{2r}, \qquad I = 1, 2, \ldots, 6, \qquad n_I \in \mathbb{Z} \qquad \sum_I n_I^2 = 1$$

Simplest case: $\mathbf{B} = \mathrm{Diag}(1, 0, \ldots)$

$$n_I = (1, 0, 0, 0, 0, 0)$$

Supersymmetry conserved: 1/2 BPS configuration

$$PSU(2,2|4) \longrightarrow OSp(4^*|4)$$

Set-up constitutes a 1D dCFT (co-dimension = 3)

Dualities

AdS/CFT: System dual to D1-D3 brane system — Diaconescu '97

Corresponding string boundary conditions integrable — Dekel & Oz '11

Defect field theory should likewise be integrable

S-duality of $\mathcal{N} = 4$ SYM, gauge group $U(N)$ — Montonen & Olive '77

$$\lambda \longleftrightarrow \frac{16\pi^2 N^2}{\lambda}$$

Wilson loop $W(C) \longleftrightarrow$ 't Hooft loop $T(C)$

Chiral primary, $\mathcal{O} \longleftrightarrow$ Chiral primary $\mathcal{O}$

Exact results for WL give prediction for TL correlators for protected operators

Integrability gives access to correlators for non-protected $\mathcal{O}$'s

't Hooft loop correlators

Consider: $\langle \mathcal{O}_a \rangle_T \equiv \frac{\langle T(C)\mathcal{O}_a(x)\rangle}{\langle T(C)\rangle} = \frac{C_a}{(2|x_\perp|)^{\Delta_a}}$

Consider protected case: $\mathcal{O}_L = \frac{1}{\sqrt{L}}\left(\frac{4\pi^2}{\lambda}\right)^{\frac{L}{2}} \operatorname{tr} Z^L$

$\langle \mathcal{O}_L \rangle_W \equiv \frac{\langle W(C)\mathcal{O}_L(x)\rangle}{\langle W(C)\rangle}$ known exactly from localization.

$$\langle \mathcal{O}_L(x)\rangle_W = \frac{1}{\sqrt{L}}\left(\frac{\sqrt{\lambda}}{2NR}\right)^L \mathrm{e}^{\frac{\lambda}{8N}} \sum_{k=1}^{L} L_{N-k}^{L}\left(-\frac{\lambda}{4N}\right)$$

Pestun '07

Okuyama & Semenoff '07

Gives testable prediction for $\langle \mathcal{O}_L(x)\rangle_T$

For non-protected operators:

$\langle \mathcal{O}_L(x)\rangle_T$ accessible by integrability

Short break

A testable S-duality prediction

Weak coupling prediction from S-duality for $Z = \Phi_1 + i\Phi_2$

$$\langle \mathcal{O}_L \rangle_T \sim \frac{1}{(2r)^L} \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{\frac{L}{2}} \left(1 + \frac{g_{YM}^2 (N-1)}{4\pi^2} L + \dots \right)$$

Leading order: Insert $Z = \Phi_1 + i\Phi_2 = \frac{1}{2r}$

Next to leading order: Quantize around the monopole background in $\mathcal{N} = 4$ SYM

I. For Φ_2: Consider scalar particle in monopole potential Dirac '31

II. For Φ_1: Scalar coupled to spin-1 particle in monopole potential

Both are beautiful, exactly solvable quantum mechanical systems

Scalar in monopole potential

Dirac 31, Tamm '31
Fierz '44

$$\hat{H} = -(\partial^k + iA^k_{cl})(\partial_k + iA^{cl}_k) + \frac{B}{4r^2}, \qquad \hat{H}\Phi = E\Phi$$

Define $L_\pm, L_z$ with standard SU(2) commutation relations and $[\vec{L}, \hat{H}]=0$

$$L_\pm = L_x \pm iL_y = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi} + \frac{B}{2}\frac{\sin\theta}{1+\cos\theta}\right]$$

$$L_z = -i\frac{\partial}{\partial\phi} + \frac{B}{2}$$

Eigenfunctions in terms of monopole spherical harmonics

$$L^2 Y_{lm} = l(l+1)Y_{lm}, \qquad L_z Y_{lm} = mY_{lm}$$

$$Y_{lm}(\theta,\phi) = e^{i(m-\frac{B}{2})\phi}\, U_{lm}(\theta)$$

← Jacobi polynomial

SU(2) representation theory $\Longrightarrow m \in \frac{\mathbb{Z}}{2}$

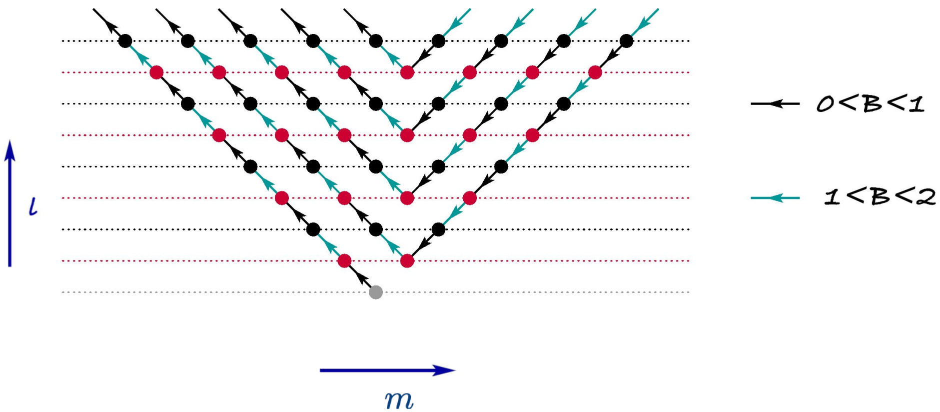
Single valuedness of wavefunction $\Longrightarrow B \in \mathbb{Z}$ (Dirac quantization condition)

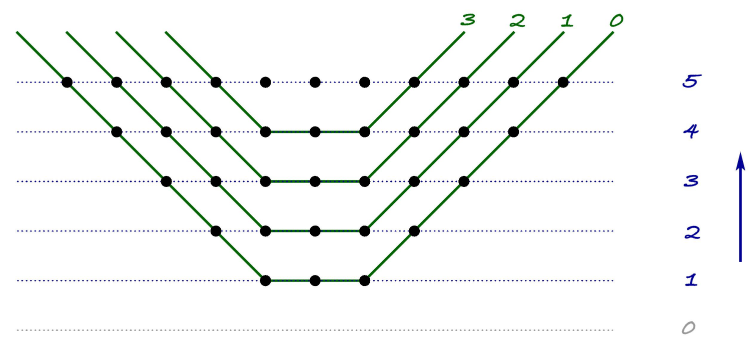
Spectral flow

Wilczek '82

$$B = 0: \; l = 0, 1, 2, \ldots, \; m = -l, \ldots, l$$

$$B = 1: \; l = \frac{1}{2}, \frac{3}{2}, \ldots, \; m = -l, \ldots, l$$





In general: $l = \frac{B}{2}, \frac{B}{2} + 1, \ldots$

Radial wave function

$$\rho(r) = (\sqrt{E}r)^{-1/2} J_\nu(\sqrt{E}r), \qquad \nu = l + \frac{1}{2},$$ No bound states

Scalar and vector in monopole potential

$$\hat{H}\begin{pmatrix}\Phi_1\\ \vec{A}\end{pmatrix} = \frac{1}{r^2}\begin{pmatrix} r^2 p_r^2 + \mathbf{L}^2 & -iB\,\hat{\mathbf{r}}^{\mathbf{T}} \\ iB\hat{\mathbf{r}} & r^2 p_r^2 + \mathbf{L}^2 - iB\,\hat{\mathbf{r}}\times \end{pmatrix}\begin{pmatrix}\Phi_1\\ \vec{A}\end{pmatrix} = E\begin{pmatrix}\Phi_1\\ \vec{A}\end{pmatrix}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

For B=1: $\ell = \frac{1}{2}, \frac{3}{2}, \ldots$
$s = 1$

For $\ell \geq \dfrac{3}{2}$: $J = \ell - 1, \ell, \ell + 1,$

For $\ell = \dfrac{1}{2}$: $J = \ell, \ell + 1.$

$\vec{A}$-eigenfunctions via vector monopole spherical harmonics

Olsen, Osland & Wu '90

$$\begin{aligned}
(\mathbf{L}+\mathbf{S})^2\,\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi) &= J(J+1)\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi),\\
(\mathbf{L_z}+\mathbf{S_z})\,\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi) &= M\,\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi),\\
\mathbf{L^2}\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi) &= \ell(\ell+1)\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi)\\
\mathbf{S^2}\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi) &= 2\,\mathbf{Y}^{(q)}_{J\ell M}(\theta,\phi)
\end{aligned}$$

$$\ell = J-1, J, J+1$$

Scalar and vector in monopole potential

Mode expansion

$$(\Phi_1)_{JM}(x) = C\, F_J(r) Y_{JM}(\theta,\phi)$$

$$\vec{A}_{JM}(x) = F_J(r)\left[C_- \mathbf{Y_{JJ-1M}}(\theta,\phi) + C_0 \mathbf{Y_{JJM}}(\theta,\phi) + C_+ \mathbf{Y_{JJ+1M}}(\theta,\phi)\right]$$

$F_J(r) = (\sqrt{E}r)^{-1/2} J_\nu(\sqrt{E}r), \quad J_\nu$ Bessel function, $\quad \nu = \nu(J)$

Constants and $\nu(J)$ determined by the non-trivial solutions of

$$\begin{bmatrix} a & i\,\mathcal{C}_{J-1} & i\,\mathcal{C}_J & i\,\mathcal{C}_{J+1} \\ -i\,\mathcal{C}_{J-1} & a + 2J - \mathcal{A}_{J-1J-1} & -\mathcal{A}_{JJ-1} & 0 \\ -i\mathcal{C}_J & -\mathcal{A}_{J-1J} & a - \mathcal{A}_{JJ} & -\mathcal{A}_{J+1J} \\ -iC_{J+1} & 0 & -\mathcal{A}_{JJ+1} & a - 2(J+1) - \mathcal{A}_{J+1J+1} \end{bmatrix} \begin{bmatrix} C \\ C_- \\ C_0 \\ C_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$a = \nu - \left(J + \frac{1}{2}\right)^2, \qquad \mathcal{A}$'s and $\mathcal{C}$'s Clebsch-Gordon's

Spectrum

Solution for ν:

$$\nu = \left\{ J - \frac{1}{2}, J + \frac{1}{2}, J + \frac{1}{2}, J + \frac{3}{2} \right\}, \qquad J \geq 3/2.$$

Interesting contrast to a gauge field with no scalar coupling

Olsen, Osland & Wu, '90

$$\nu = \left\{ \left[\frac{1}{4} + \left(\sqrt{J^2 + J} - 1 \right)^2 \right]^{1/2}, J + \frac{1}{2}, \left[\frac{1}{4} + \left(\sqrt{J^2 + J} + 1 \right)^2 \right]^{1/2} \right\}$$

Coupling to scalar is dictated by susy of $\mathcal{N} = 4$ SYM

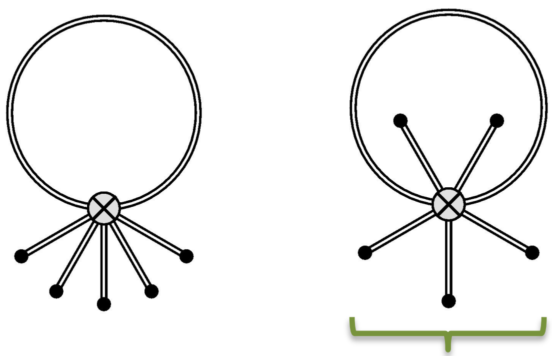
The simple spectrum is a manifestation of underlying integrability

Propagators can be found by spectral decomposition

Become propagators in auxiliary $AdS_2 \times S^2$ space
with the 't Hooft line as boundary

Confirmation of prediction

Only only one contraction needed



Non-planar, vanishing

$$\langle \mathcal{O}_L(\mathbf{x}) \rangle_T^{1-loop} = \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{L/2} \frac{1}{(2r)^L} \left\{ \frac{g_{YM}^2 (N-1) L}{4\pi^2} \sum_{J=\frac{1}{2},\frac{3}{2},\ldots} \frac{1}{4} \left(\frac{1}{J^2} - \frac{1}{(J+1)^2} \right) \right\}$$

$$= \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{L/2} \frac{1}{(2r)^L} \left(\frac{g_{YM}^2 (N-1) L}{4\pi^2} \right),$$

Non-protected operators

Generic operators built from scalars

$$\mathcal{O} = \Psi^{I_1 \ldots I_L} \, \mathrm{tr} \, \Phi_{I_1} \ldots \Phi_{I_L} \, ,$$

Good conformal operators are eigenstates of $\hat{H}$ Minahan & Zarembo '02

$$\hat{H} = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{L} \left(2 - 2P_{\ell\ell+1} + K_{\ell\ell+1}\right) ,$$

Eigenstates characterized by three sets of rapidities

$$|u_{1i}, u_{2j}, u_{3k}\rangle$$

Fullfil a set of algebraic Bethe equations

$$\left(\frac{u_{aj} - \frac{iq_a}{2}}{u_{aj} + \frac{iq_a}{2}}\right)^L \prod_{bk} \frac{u_{aj} - u_{bk} + \frac{iM_{ab}}{2}}{u_{aj} - u_{bk} - \frac{iM_{ab}}{2}} = -1,$$

Correlator with 't Hooft loop

Can be expressed as overlap with boundary state. At leading order

de Leeuw, C.K.
Zarembo '15

$$\langle \mathrm{Bst}| = \mathrm{Bst}_{I_1 \ldots I_L} \Phi_{I_1} \ldots \Phi_{I_L}, \qquad \mathrm{Bst}_{I_1 \ldots I_L} = n_{I_1} \ldots n_{I_L}$$

Overlap formula at leading order

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi^2}{\lambda r^2} \right)^{\frac{L}{2}} L^{-\frac{1}{2}} \, \frac{\langle \mathrm{Bst} \,|\Psi\rangle}{\langle \Psi \,|\Psi\rangle^{\frac{1}{2}}} \, .$$

Integrable boundary state $|\mathrm{Bst}\rangle$

Piroli, Pozsgay,
Vernier '17

de Leeuw, C.K,
Zarembo '15

$$Q_{2n+1}|\mathrm{Bst}\rangle = 0, \;\; n = 1, 2, \ldots$$

Expect closed overlap formula to exist

Expressible entirely in terms of Bethe roots,
and including the Gaudin determinant

Result

For $n_I = \delta_{I,1}$, general scalar operator

de Leeuw, Gombor, C.K., Linardopoulos, Pozsgay '19

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{\pi}{\sqrt{\lambda}\, r} \right)^L \sqrt{\frac{1}{L} \frac{\prod_j u_{2j}^2 \left(u_{2j}^2 + \frac{1}{4}\right)}{\prod_j u_{1j}^2 \left(u_{1j}^2 + \frac{1}{4}\right) \prod_j u_{3j}^2 \left(u_{3j}^2 + \frac{1}{4}\right)} \frac{\det G^+}{\det G^-}},$$

State needs to have paired roots, G = Gaudin matrix

Further results

Closed overlap formula for SL(2) sector

Closed overlap formula in gluon sector

Should be possible to integrability bootstrap the full result the entire theory, all loops

Komatsu & Wang '20

Bajnok & Gombor '20

Conclusions

- Magnetic monopoles --- keep fascinating
- S-duality works beautifully
- Novel example of integrable dCFT (co-dimension 3)

Future directions

- Integrability-bootstrap to get the all loop result
- Extracting other conformal data (bulk-to-boundary couplings, two-point functions, transport coefficients)
- Monopole operators in other dimensions/theories

Thank you