Finite-coupling Multi-gluon Scattering at Clusters of Origin Limits

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QUANTUM UNIVERSE

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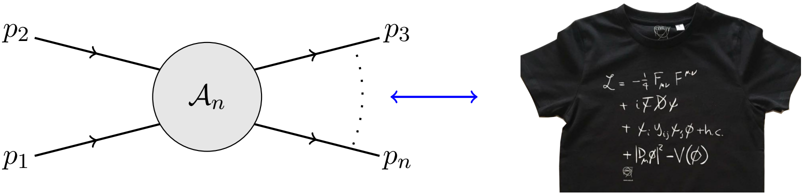
PRL 130, 111602 (2023) with B. Basso, L. Dixon and Y.-T. Liu
PRL 124, 161603 (2020) with B. Basso and L. Dixon
JHEP 08 (2020) 005, JHEP 10 (2021) 007 with N. Henke

Scattering Amplitudes $\mathcal{A}_n$ in Quantum Field Theory



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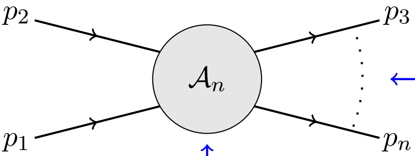
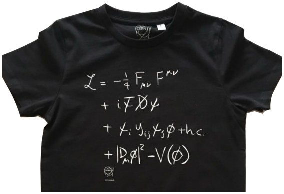
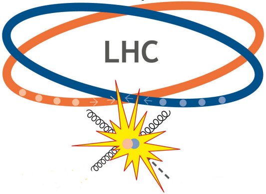




Collider Experiments



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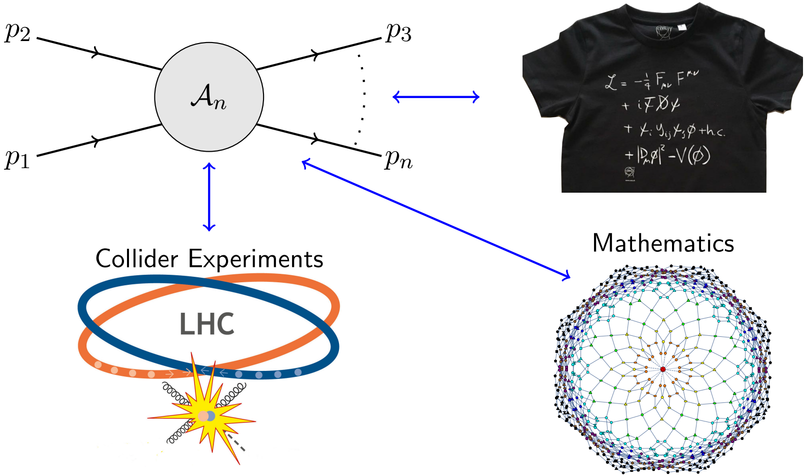

- Theoretical predictions for outcome of elementary particle collisions, central for experiments such as the LHC at nearby CERN, Geneva
- Exhibit remarkably deep mathematical structures

Maximally Supersymmetric Yang-Mills (MSYM) Theory
$SU(N)$ gauge group

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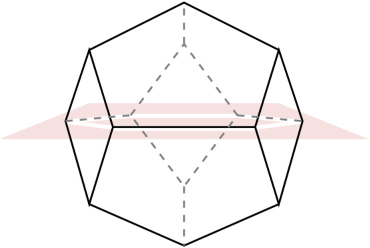

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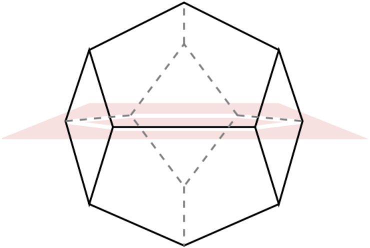

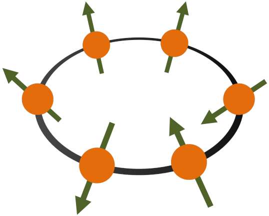

- Here, focus on planar limit $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed.
Integrable structure $\Rightarrow$ Exact physical quantities in $g^2 = \lambda/(4\pi)^2$!
[Minahan,Zarembo'02]... [Beisert,Eden,Staudacher'06]... [Gromov,Kazakov,Leurent,Volin'13]...

Maximally Hellicity Violating (MHV) Gluon Amplitudes

Gluons are massless → helicity $h = \vec{S} \cdot \hat{p} = \pm 1$ good quantum number.

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In planar $\mathcal{N} = 4$ SYM, they are

- remarkably, dual to null polygonal Wilson loops.
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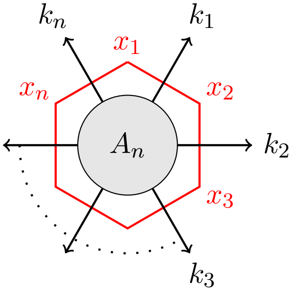

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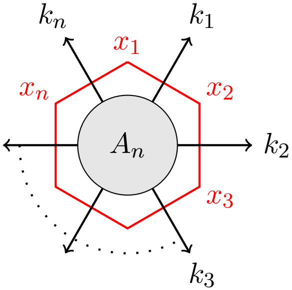

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$$u_{i,j} = \frac{x_{i,j+1}^2 x_{j,i+1}^2}{x_{i,j}^2 x_{j+1,i+1}^2}\,,$$

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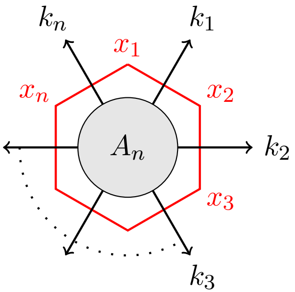

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⇒ First nontrivial amplitude for $n = 6$.

The Amplitude at Strong Coupling

Via gauge/string duality, at leading strong-coupling order $R_n \sim -2g(\text{Area})$ of string ending on null polygon at boundary of AdS space. [Alday,Maldacena]

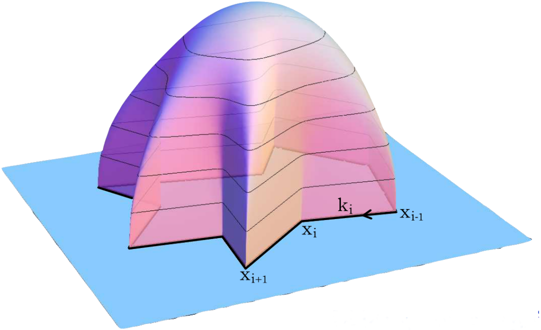


Image Credit: A. Sever

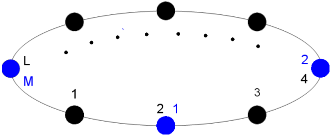
Classically integrable geometric problem ⇒ auxiliary integral equations of Thermodynamic Bethe Ansatz (TBA) type
[Gaiotto,Moore, Neitzke] [Alday, Gaiotto,Maldacena] [Alday,Maldacena,Sever,Vieira]

TBA: An Example

[Zamolodchikov'90]

- M identical relativistic particles of mass m, rapidity β.
- In circle of circumference L, with short-distance pairwise interactions.
- Thermodynamic limit $M, L \to \infty$ with $\frac{M}{L}$ fixed.

Extremize free energy F:



$$e^{-F/T} = \mathrm{Tr}\left[e^{-H/T}\right]$$

$$\Rightarrow \quad \frac{F}{T} = \frac{m}{2\pi} \int \cosh\beta \ln(1 - 1/Y) d\beta$$

given in terms of Y-function $Y = \frac{\rho_1 + \rho}{\rho_1}$ $\leftarrow$total density of states, $\leftarrow$density of occupied states

which obeys nonlinear integral TBA equation:

$$\ln Y = \frac{m}{T}\cosh\beta + \frac{1}{2\pi}\int \phi(\beta - \beta')\ln(1 - 1/Y)d\beta'$$

for some kernel ϕ.

The (Six-particle) Origin of Intriguing Observations

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$$R_6 = -\frac{\Gamma_0 - \Gamma_{\pi/4}}{24}\ln^2(u_1 u_2 u_3) - \frac{\Gamma_{\pi/3} - \Gamma_{\pi/4}}{24}\sum_{i=1}^{3}\ln^2\left(\frac{u_i}{u_{i+1}}\right) + C_0\,,$$

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in terms of *tilted* anomalous dimension & Beisert-Eden-Staudacher kernel,

$$\Gamma_\alpha = 4g^2\left[\frac{1}{1+\mathbb{K}(\alpha)}\right]_{11} = 4g^2\left[1 - \mathbb{K}(\alpha) + \mathbb{K}^2(\alpha) + \dots\right]_{11}\,,$$

$$\mathbb{K}(\alpha) = 2\cos\alpha\begin{bmatrix}\cos\alpha\,\mathbb{K}_{\circ\circ} & \sin\alpha\,\mathbb{K}_{\circ\bullet}\\ \sin\alpha\,\mathbb{K}_{\bullet\circ} & \cos\alpha\,\mathbb{K}_{\bullet\bullet}\end{bmatrix}, \qquad \mathbb{K}_{\circ\circ} = \mathbb{K}_{2n+1,2m+1}\,, \quad \mathbb{K}_{\circ\bullet} = \mathbb{K}_{2n+1,2m} \text{ etc,}$$

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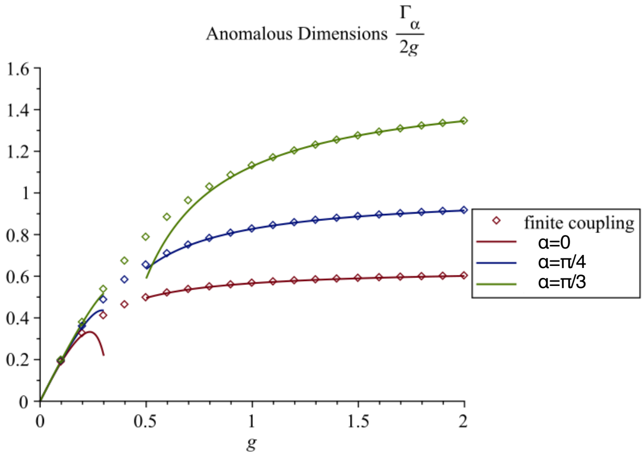
and similarly for C_0. Tilt/deformation removed for $\alpha = \pi/4$.

Comparison: Finite-coupling numerics & weak/strong coupling analytics

[Basso,Dixon,GP'20]



Tilt Appearances & Applications

[Korchemsky]... [Beisert,Eden,Staudacher]

- $\alpha = \pi/4$ recovers usual cusp anomalous dimension,

$$\Gamma_{\pi/4} = 4g^2 - 8\,\zeta_2\, g^4 + 88\,\zeta_4\, g^6 - 4\Big[219\,\zeta_6 + 8\,(\zeta_3)^2\Big]\, g^8 + +\mathcal{O}(g^{10}),$$

- $\alpha = 0$ appeared previously in lightlike limit of "simplest 4-point correlator" of MSYM, [Coronado] [Kostov,Petkova,Serban] [Belitsky,Korchemsky]

$$\Gamma_0 = \frac{2}{\pi^2} \ln\cosh\left(2\pi g\right)$$

and more recently in Coulomb branch amplitudes, [Caron-Huot,Coronado] off-shell Sudakov and higher-point on-shell form factors [Belitsky,Bork,Pikelner,Smirnov] [Sever,Tumanov,Wilhelm]

- $\alpha = \pi/3$ new $\Rightarrow$ application in 3-pt structure constants of operators with large spin, polarization [Bercini,Gonçalves,Vieira]

Physical significance of α? More quantities for other values of α?

This talk

Origins of n-point amplitudes provide first instance of new tilt angle values!

$$R_n = \sum_\alpha \left(\Gamma_\alpha - \Gamma_{\pi/4}\right) \times P_\alpha^{\Sigma_n}\,,$$

where $P_\alpha^{\Sigma_n}$ quadratic-logarithmic polynomials of u_{ij}, and

$$\alpha = \frac{\pi}{2} - \frac{\pi p}{3} - \frac{\pi k}{3(n-4)}\,, \text{ with } k = 1, \ldots, n-5\,,\; p = 0, 1, 2\,.$$

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To arrive at this result, we first found: [Basso,Dixon,Liu,GP]

1. n-gluon generalizations of origin limits → *cluster algebras*
2. Amplitude kinematic dependence, $P_\alpha^{\Sigma_n}$ → *pert. data & bootstrap*
3. Values of α → *thermodynamic Bethe ansatz* (TBA)

Classifying n-gluon origin limits $\mathrm{O}^{(n)}$

$$\mathrm{O}^{(6)}: \quad u_i \equiv u_{i+1,i+4} \to 0\,, \quad i = 1,2,3.$$

⛔ However, in general $n(n-5)/2$ dual conformal cross ratios, but only $3(n-5)$ independent kinematic variables ⇒ Cannot set all $u_{i,j} \to 0$!

💡 ∃ well-defined notion of region of *positive kinematics*, where amplitudes believed to be singularity-free.
[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka] [Arkani-Hamed,Lam,Spradlin]

⇒ Look at *boundary* of this region, as first place for potential origin-type divergent behavior! Completely captured by $Gr(4,n)$ *cluster algebras*.

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- Grouped into overlapping subsets $\{\mathcal{X}_1, \ldots, \mathcal{X}_d\}$ of *rank* d, the *clusters*
- Constructed recursively from initial cluster via *mutations*, encoded d-dimensional matrix B with elements b_{ij}.

Mutation associated to coordinate $\mathcal{X}_k$: [Fock,Goncharov]

$$\mathcal{X}_i \to \mathcal{X}_i' = \begin{cases} 1/\mathcal{X}_i & k = i\,, \\ \mathcal{X}_i \left(1 + \mathcal{X}_k^{-\mathrm{sgn}(b_{ki})}\right)^{-b_{ki}} & k \neq i\,, \end{cases}$$

In new cluster, $B \to B'$ with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{for } i = k \text{ or } j = k \\ b_{ij} + \max\left(0, -b_{ik}\right) b_{kj} + b_{ik} \max\left(0, b_{kj}\right) & \text{otherwise}\,. \end{cases},$$

Exchange graph: Clusters=vertices, mutations=edges

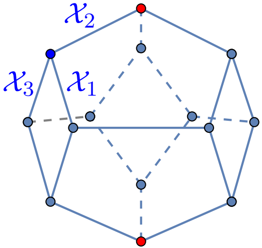
Example: The six-particle positive region

Described by $Gr(4,6) \simeq A_3$ cluster algebra

Initial cluster $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$

Origin limit clusters

Positive region maps to interior of exchange graph/polytope, described by $\infty > \mathcal{X}_i > 0$.



$$u_1 = \frac{\mathcal{X}_2\mathcal{X}_3}{(1+\mathcal{X}_1+\mathcal{X}_1\mathcal{X}_2)(1+\mathcal{X}_2+\mathcal{X}_2\mathcal{X}_3)}\,, \quad u_2 = \frac{\mathcal{X}_1\mathcal{X}_2}{1+\mathcal{X}_1+\mathcal{X}_1\mathcal{X}_2}\,, \quad u_3 = \frac{1}{1+\mathcal{X}_2+\mathcal{X}_2\mathcal{X}_3}\,.$$

In initial cluster, $b_{12} = b_{23} = -b_{21} = -b_{32} = 1 \Rightarrow$

$$\mathcal{X}_1' = \mathcal{X}_1\,(1+\mathcal{X}_2)\,, \quad \mathcal{X}_2' = \frac{1}{\mathcal{X}_2}\,, \quad \mathcal{X}_3' = \frac{\mathcal{X}_2\mathcal{X}_3}{1+\mathcal{X}_2}\,.$$

All $\mathcal{X}_i \to 0$: Boundary vertex. All but one $\mathcal{X}_i \to 0$: Boundary edge etc.

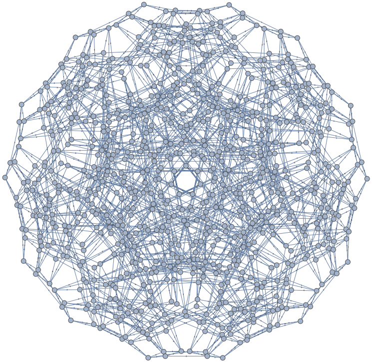
Definition: Higher-point Origins

By analogy with $n = 6$ analysis

Origin point limit

Boundary vertex/cluster $\mathcal{X}_i \to 0$ where $\geq 3(n-5)$ cross ratios vanish

Example: $n = 7$



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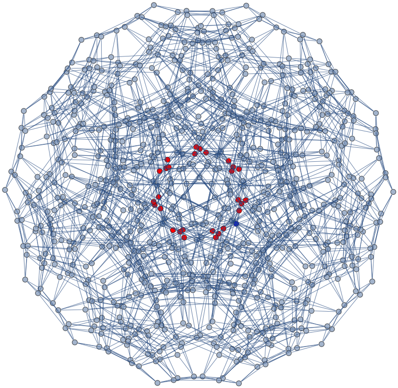
- Find 28 out of 833 clusters:

$\mathrm{O}_i^{(7)}: \quad u_i = 1\,, \quad u_{j\neq i} \to 0.$

- Resolved into L/R:

$u_{i-1}/u_{i+1} \to 0$ faster



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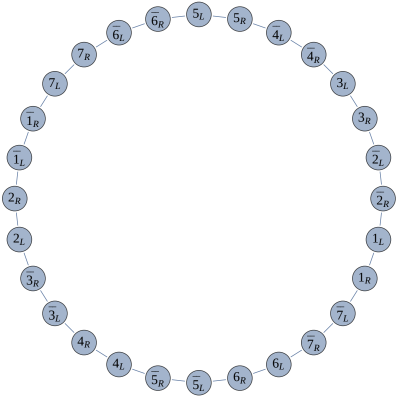
- All of them contiguously connected by mutations



Origin Limit of $n = 7$ Amplitude

Find, based on bootstrapped data through $L = 4$ loops,
[Drummond,GP,Spradlin] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin] [Dixon,Liu]

$$R_7(u_7 + u_1 = 1, u_{i\neq 1,7} \ll 1) = \sum_{i=1}^{3} c_i P_i^{(7)}\,,$$

where

$$P_1^{(7)} = \sum_{i=1}^{6} l_i l_{i+1} + \sum_{i=1}^{5} l_i l_{i+2}\,,$$

$$P_2^{(7)} = -l_1 l_7 + \sum_{i=1}^{7} l_i^2 + \sum_{i=1}^{4} l_i l_{i+3}\,,$$

$$P_3^{(7)} = \sum_{i=1}^{7} l_i l_{i+2} - \sum_{i=1}^{3} l_i l_{i+4}\,,$$

and $l_i \equiv \ln u_i \equiv \ln u_{i+1,i+4}$.

Quadratic-logarithmic behavior not only on origin points, but also on lines between them!

The Higher-point Challenge

$Gr(4,n)$ cluster algebra becomes infinite for $n \geq 8$!

💡 Based on previously observed contiguity, devise algorithm:

1. Start with origin point cluster ($\geq 3(n-5)$ $u_{i,j} \to 0$ as $\mathcal{X}_i \to 0$)
2. Mutate to generate new origin points until condition is no longer met

For $n = 8$, find 1188 clusters.

✓ All contained in 121460 clusters selected by natural proposal to render $Gr(4,n)$ finite by "tropicalization"
[Henke, GP'19] [Arkani-Hamed,Lam,Spradlin'19] [Drummond,Foster,Gurdogan,Kalousios'19B]

Origins for n=8 particles: Exchange Graph

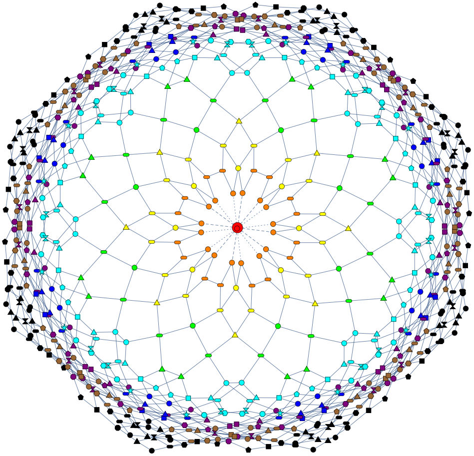


Figure: Different origin limit classes color-coded as 1, 2, 3, 4, 5, 6, 7, 8, 9. May be viewed as half-sphere with two O_1 at north pole and with O_9's at equator. Missing half-sphere is parity image, omitted for simplicity. Same-colored vertices of different shape denote different directions of approach within each origin class.

Origin Class	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	v_1	v_2	v_3	v_4	#
O_1(super)	0	0	0	0	0	0	0	0	0	1	0	1	4
O_2	0	0	0	0	0	0	0	1	0	1	0	1	64
O_3	0	0	0	0	0	0	0	1	0	0	1	1	64
O_4	0	0	0	0	0	0	1	1	0	0	1	0	80
O_5	0	0	0	0	0	1	0	1	0	0	1	0	288
O_6	0	0	0	0	1	0	0	1	0	1	0	0	128
O_7	0	0	0	0	1	0	0	1	0	0	1	0	256
O_8	0	0	0	1	0	0	0	1	0	1	0	0	128
O_9	0	0	0	1	0	0	0	1	0	0	0	1	176

Table: All dihedrally inequivalent origin classes for $n = 8$. Zeros represent infinitesimal values. There are nine infinitesimal cross ratios for all origins except for the *super-origin* O_1 which has ten. All nonzero cross ratios are close to unity. The last column lists the number in each class, taking into account dihedral symmetry, parity, and direction of approach.

$$u_i \equiv u_{i+1,i+4}\,, \qquad v_i \equiv u_{i+1,i+5}$$

Perturbative data and bootstrap

As with $n = 7$, from $n = 8$ perturbative data ($L = 2$ and symbol-level $L = 3$) [Golden,McLeod'21][Li,Zhang'21]

- Amplitude indeed exhibits exponentiated quadratic-logarithmic behavior in all origin points.
- Also in higher-dimensional subspaces of kinematics, dictated by cluster algebras! $A_1 \times A_1$, $2 \times A_2$, $2 \times A_3$ up to dihedral transformations.
- Turn logic around: Assume latter behavior and dihedral symmetry, continuity, dual superconformal symmetry $\Rightarrow$ Fixes all $\log u_i$ polynomials in all different origin limits.

Outline

The TBA for Amplitudes at Strong Coupling

In general kinematics, TBA equations for $3(n-5)$ Y-functions, [Alday,Maldacena,Sever,Vieira]

$$\ln Y_{a,s}(\theta) = I_{a,s}(\theta) + \sum_{b,t} \int \frac{k_a(\theta) d\theta'}{2\pi k_b(\theta')} K_{a,s}^{b,t}(\theta - \theta') \ln\left(1 + Y_{b,t}(\theta')\right),$$

summed over $b = 0, \pm 1$, $t = s, s \pm 1$, for some kernels K, with

$$k_a(\theta) = i^a \sinh\left(2\theta - i\pi a/2\right)$$

and driving terms

$$I_{a,s}(\theta) = a\varphi_s - m_a \tau_s \cosh\theta + (-1)^s i m_a \sigma_s \sinh\theta\,,$$

depending on convenient kinematic variables $\{\sigma_s, \tau_s, \varphi_s\}$, [Basso,Sever,Vieira] $s = 1 \ldots n-5$. $1/\tau_s \sim$ temperatures, $\phi_s \sim$ chemical potentials.

Solving the TBA at Origins

- Different origins correspond to $\tau_s, |\varphi_s| \gg 1$, $|\varphi_s| - \tau_s \gg 1$. Labeled by sequence $\Sigma_n = (h_1, \ldots, h_{n-5})$ with $h_s = \varphi_s / |\varphi_s|$.
- Expect $Y_{a,s} = \begin{cases} \gg 1 & \text{if } a = h_s \\ 0 & \text{otherwise} \end{cases}$
- TBA linearizes: $\ln(1 + Y_{b,t}) \to \delta_{b,h_t} \ln Y_{h_t,t}$

Solve by Fourier transform ($\ln z$ conjugate to θ).

$$\hat{f}(z) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi \cosh(2\theta)} z^{2i\theta/\pi} f(\theta)\,.$$

E.g. $n = 6$, only $Y_{1,1}$ survives and

$$\widehat{\ln Y_{1,1}}(z) = -\frac{\sqrt{z}(\ln u_1 - z \ln u_2 + z^2 \ln u_3)}{2(1 + z^3)}$$

Minimal Area=Free Energy

[Alday,Maldacena,Sever,Vieira] [Bonini,Fioravanti,Piscaglia,Rossi]

$$R_n \simeq -g \sum_{a,s} (-1)^a \int \frac{d\theta}{\pi k_a(\theta)^2} \left[\mathrm{Li}_2(-Y_{a,s}) + \frac{1}{2} \log(1+Y_{a,s}) \log(Y_{a,s}/I_{a,s}) \right]$$

$$= \ldots = -\frac{4g}{\pi} \int_0^\infty \frac{dz}{z} \mathcal{S}_n(z)$$

$$\mathcal{S}_n(z) \equiv \sum_{s=1}^{n-5} \widehat{I}_s(1/z) \widehat{\ln Y_s}(z) = \frac{z(1-z^3) P^{\Sigma_n}(z)}{(1+z)(1+z^2)(1-z^{3(n-4)})}$$

where $\mathcal{P}_n^{\Sigma}(z)$ polynomial of degree $3n-14$ in z and quadratic in $\{\sigma_s, \tau_s, \varphi_s\}$.

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$$= \ldots = -\frac{4g}{\pi} \int_0^\infty \frac{dz}{z} \mathcal{S}_n(z) = -\frac{\sqrt{\lambda}}{\pi^2} \oint \frac{dz}{2\pi i z} \log(-z) \mathcal{S}_n(z)\,,$$

$$\mathcal{S}_n(z) \equiv \sum_{s=1}^{n-5} \widehat{I}_s(1/z) \widehat{\ln Y_s}(z) = \frac{z(1-z^3) P^{\Sigma_n}(z)}{(1+z)(1+z^2)(1-z^{3(n-4)})}$$

where $\mathcal{P}_n^{\Sigma}(z)$ polynomial of degree $3n-14$ in z and quadratic in $\{\sigma_s, \tau_s, \varphi_s\}$.

Integral of rational function. Rewrite as contour integral and deform contour to surround poles of $\mathcal{S}_n(z)$, all on unit circle.

Finite-coupling Origins at any Multiplicity

For $n = 6$, sum over residues matches strong-coupling expansion of known finite-coupling formula with

$$\Gamma_\alpha(g) =: \mathcal{G}(z = -e^{2i\alpha}, g) \simeq \frac{2\alpha\sqrt{\lambda}}{\pi^2 \sin(2\alpha)}\,,$$

i.e. $\mathcal{S}_n(z)$ predicts values of α (+kinematic dependence P^{Σ_n})!

Finite-coupling conjecture $\forall n$: Move $\mathcal{G}(z,g)$ as exact function of g inside integral:

$$R_n = -\frac{1}{2}\oint_{C_n} \frac{dz}{2\pi i z}(z-1/z)\tilde{\mathcal{G}}(z,g)\mathcal{S}_n(z)\,,$$

with $\tilde{\mathcal{G}}(z,g) \equiv \mathcal{G}(z,g) - \Gamma_{\text{cusp}}(g)$.

Conclusions

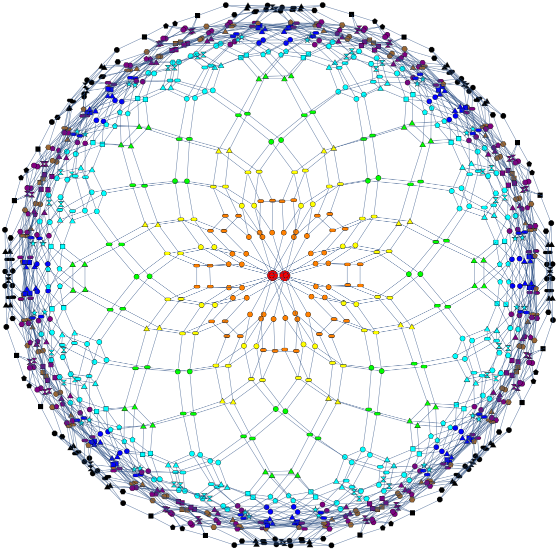
Finite-coupling amplitudes in origin limits for any particle number n!

1. *Cluster algebras* ⇒ Classification of origin limits
2. *Perturbative bootstrap* ⇒ $\log^2(u_{ij})$ kinematic behavior
3. *Thermodynamic Bethe Ansatz* ⇒ coupling-dependent coefficients Γ_α

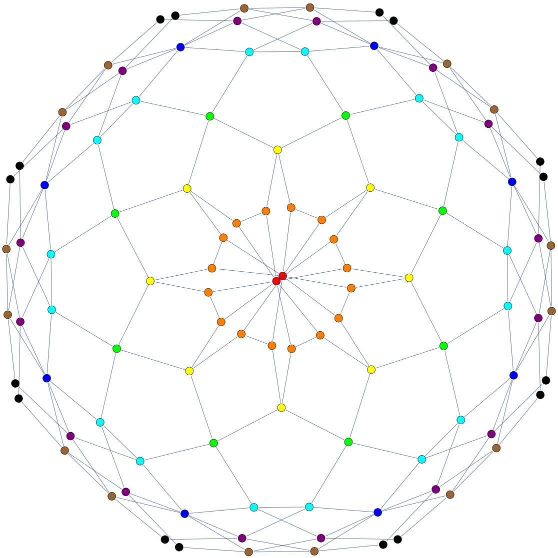
Next Stage

- $\log^0(u_{i,j})$ terms?
- More integrable limits? Exact scattering in general kinematics?
- Maximal transcendentality principle relates $\Gamma_{\pi/4}$ in MSYM and QCD. Other values of α? [Kotikov,Lipatov'02]
- Origin story for higher-point correlators? [Vieira,Gonçalves,Bercini'20]

All eight-particle origin clusters



Octagon origins:Mod out by direction of approach to limit



Kinematic dependence in origin limits

In terms of convenient (OPE) variables $\varphi_s, \tau_s, \sigma_s$, inequivalent limits correspond to $|\varphi_s|, \tau_s \to \infty$ and are thus labeled by $\Sigma_n = (h_1, \ldots, h_{n-5})$ with $h_s = \varphi_s/|\varphi_s|$.

Kinematic dependence then encoded in quadratic polynomial

$$P_\alpha^{\Sigma_n} = -\frac{\cos\alpha\cos(3\alpha)}{12(n-4)\cos(2\alpha)}|Q_n^\Sigma(-e^{2i\alpha})|^2\,,$$

where

$$Q_n^\Sigma(z) = \sum_{s=1}^{n-5}(-1)^{s+1}\frac{1-z^{3(n-4-s)}}{1-z^3}e_{h_s,s}(z)\prod_{i=1}^{s-1}b_i(z)\,,$$

with

$$e_{\pm,s} = [\pm\varphi_s - \tau_s + (-1)^s\sigma_s] - 2\tau_s z + [\pm\varphi_s - \tau_s - (-1)^s\sigma_s]z^2\,,$$

$b_s = \frac{1}{2}(1-h_s h_{s+1})z - \frac{1}{2}(1+h_s h_{s+1})z^3$ for s odd, and similarly with $b_s \to z^3 b_s(1/z)$ for s even.

Weak coupling expansion of Γ_α

	$L=1$	$L=2$	$L=3$	$L=4$
Γ_{oct}	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$
Γ_{cusp}	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6-32\zeta_3^2$
Γ_{hex}	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6-24\zeta_3^2$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6+2\zeta_3^2$	$\frac{67645}{32}\zeta_8+6\zeta_2\zeta_3^2-40\zeta_3\zeta_5$

$$
\begin{aligned}
\frac{\Gamma_\alpha}{4g^2} &= 1-4c^2\zeta_2 g^2+8c^2(3+5c^2)\zeta_4 g^4 \\
&\quad -8c^2\left[(25+42c^2+35c^4)\zeta_6+4s^2\,\zeta_3^2\right]g^6+\dots\,, \\
D(\alpha) &= 4c^2\zeta_2 g^2-4c^2(3+5c^2)\zeta_4 g^4 \\
&\quad +\frac{8}{3}c^2\left[(30+63c^2+35c^4)\zeta_6+12s^2\,\zeta_3^2\right]g^6+\dots\,,
\end{aligned}
$$

$$
\Gamma_{\text{oct}}=\Gamma_0\,,\quad \Gamma_{\text{cusp}}=\Gamma_{\pi/4}\,,\quad \Gamma_{\text{hex}}=\Gamma_{\pi/3}
$$

Strong coupling: Expansion & Comparison with String Theory

Expanded Γ_α to four orders in $1/g$, and C_0 to two. For example,

$$\Gamma_\alpha = \frac{8\alpha g}{\pi \sin(2\alpha)} + \mathcal{O}(g^0)\,, \quad D(\alpha) = 4\pi g\left[\frac{1}{4} - \frac{\alpha^2}{\pi^2}\right] + \mathcal{O}(g^0)\,.$$

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Via gauge/string duality, at leading strong-coupling order $\mathcal{W} \sim e^{-2g(\mathrm{Area})}$ of string ending on $\mathcal{W}$ at boundary of AdS space. [Alday,Maldacena]

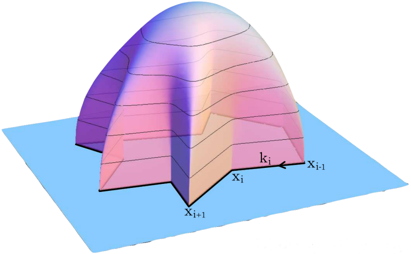

Image Credit: A. Sever

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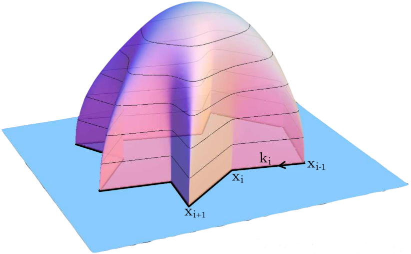
At $u_1 = u_2 = u_3 \to 0$:
[Alday,Gaiotto,Maldacena] [Basso,Sever,Vieira]

$$\frac{\ln \mathcal{E}_6}{\Gamma_{\text{cusp}}} = -\frac{3}{4\pi}\ln^2 u - \frac{\pi^2}{12} - \frac{\pi}{6} + \frac{\pi}{72} + \mathcal{O}\left(u^{-1}\right)$$



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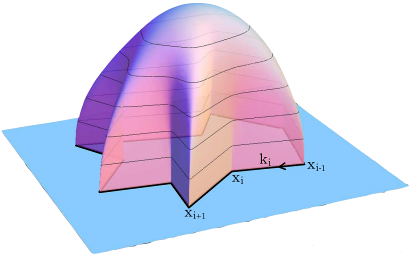
Perfect agreement!



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Perfect agreement!

Can also confirm Γ_{hex}. [Ito,Satoh,Suzuki]

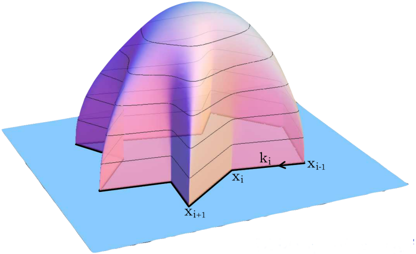


Image Credit: A. Sever

Strong coupling expansion of Γ_α

Letting $a = \alpha/\pi$, find

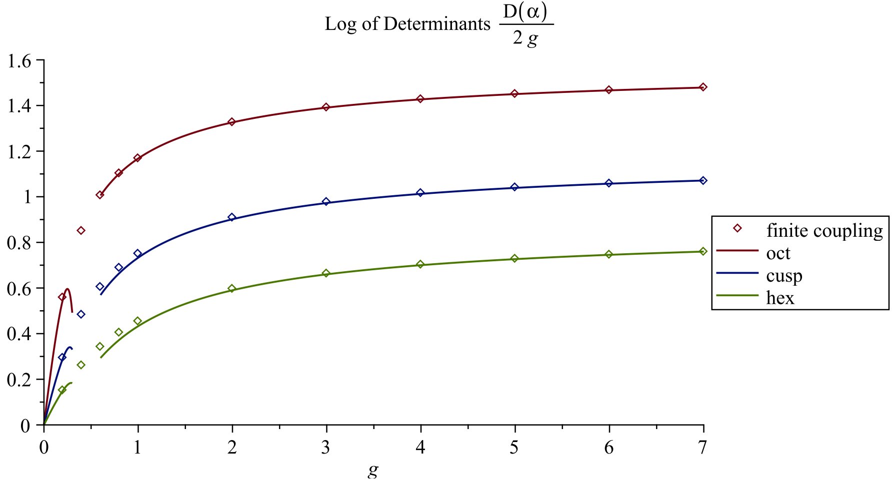
$$\Gamma_\alpha = \frac{8ag}{\sin(2\pi a)}\left[1 - \frac{s_1}{2\sqrt{\lambda}} - \frac{as_2}{4\lambda} - \frac{a(s_1 s_2 + as_3)}{8(\sqrt{\lambda})^3} + \dots\right],$$

where

$$s_{k+1} = \{\psi_k(1) - \psi_k(\tfrac{1}{2} + a)\} + (-1)^k\{\psi_k(1) - \psi_k(\tfrac{1}{2} - a)\},$$

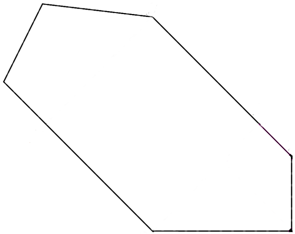
and $\psi_k(z) = \partial_z^{k+1} \ln\Gamma(z)$ the polygamma function.

Comparison: Finite-coupling numerics & weak/strong coupling analytics



$$D_{\rm oct} = D(0)\,, \quad D_{\rm cusp} = D(\pi/4)\,, \quad D_{\rm hex} = D(\pi/3)$$

Integrability in Scattering Amplitudes/Wilson Loops?



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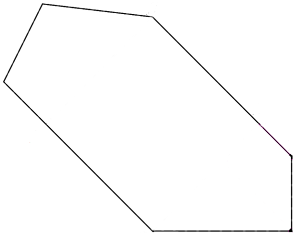
Not yet understood in general kinematics. Good starting point, however, particular collinear limit.



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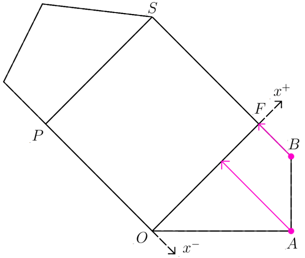
$$u_2 = \frac{1}{e^{2\tau} + 1}\,, \quad u_1 = e^{2\tau+2\sigma} u_2 u_3\,,$$

$$u_3 = \frac{1}{1 + e^{2\sigma} + 2e^{\sigma-\tau}\cosh\varphi + e^{-2\tau}}\,.$$



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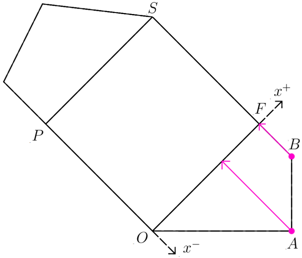
In convenient normalization,

$$\mathcal{W}_6 \equiv \mathcal{E}_6 e^{\frac{1}{2}\Gamma_{\rm cusp}(\sigma^2+\tau^2+\zeta_2)}$$



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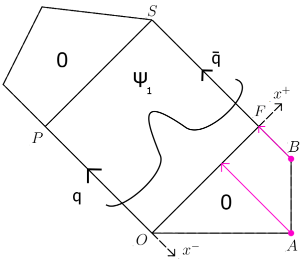
In convenient normalization, conformal symmetry implies

$$\mathcal{W}_6 = \sum_{\psi_i} e^{-E_i\tau + ip_i\sigma + a_i\phi}\, \mathcal{P}(0|\psi_i)\mathcal{P}(\psi_i|0)$$



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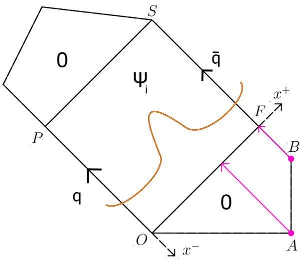

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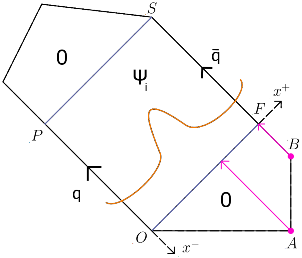

Wilson Loop 'Operator Product Expansion (OPE)'
[Alday,Gaiotto,Maldacena,Sever,Vieira]

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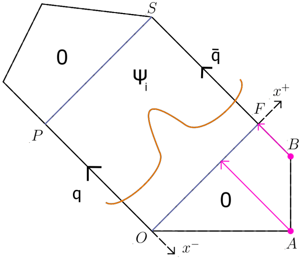

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MSYM: ψ_i mapped to excitations of integrable $SL(2,\mathbb{R})$ spin chain, equivalently of Gubser-Polyakov-Klebanov string $\Rightarrow$ exact $E, \mathcal{P}$
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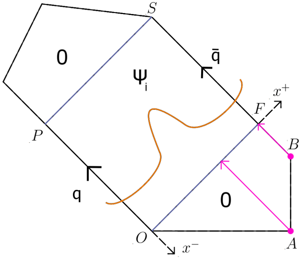

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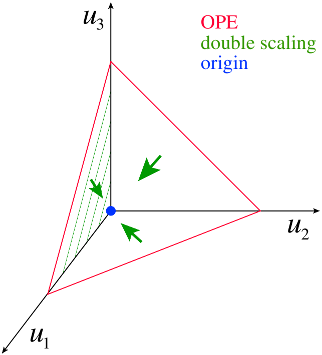
[Belitsky,Bonini,Bork,Caetano,Cordova,Drummond,Fioravanti,Hippel,Lam,Onishchenko, GP,Piscaglia,Rossi. . .

A Path to Originality



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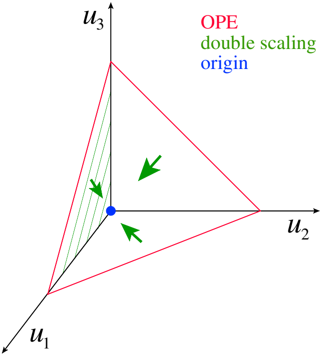

- Origin does not intersect collinear limit

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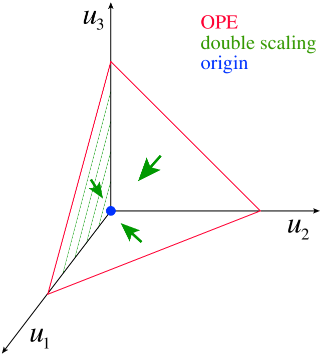

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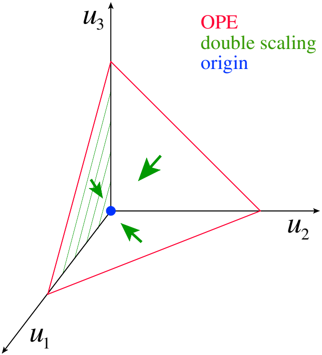

- Origin does not intersect collinear limit
- However part of double scaling limit: Only simpler, gluon flux tube excitations contribute, [Basso,Sever,Vieira] [Drummond,GP]

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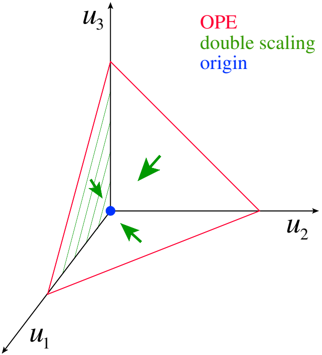

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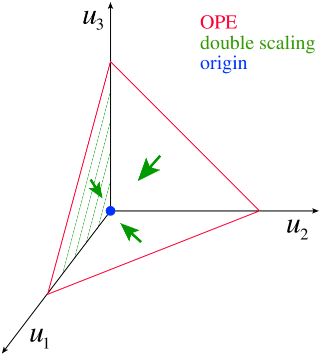

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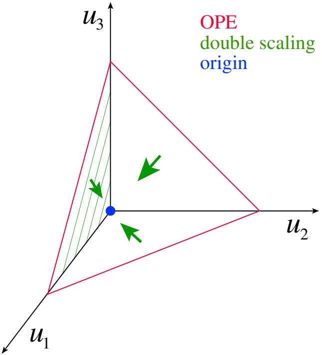

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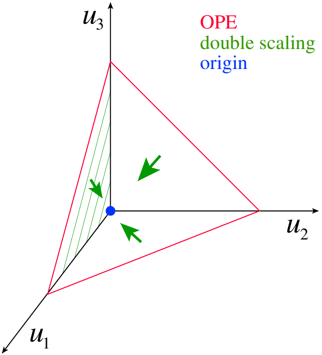

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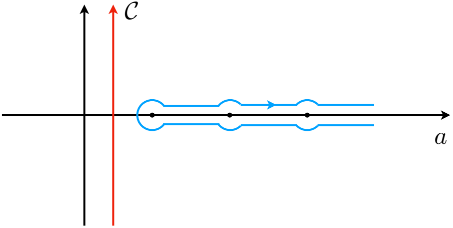
- As we'll see however, not really necessary! ☺

Sommerfeld-Watson Transform

Similar to Regge theory, where it amounts to analytic continuation in spin,

$$\sum_{a\geqslant 1}(-1)^a f(a) \to \int_{+\infty-i\epsilon}^{+\infty+i\epsilon} \frac{i f(a) da}{2\sin(\pi a)}\,,$$

provided $f(z)$ decays faster than $1/z$ as $z\to\infty$.



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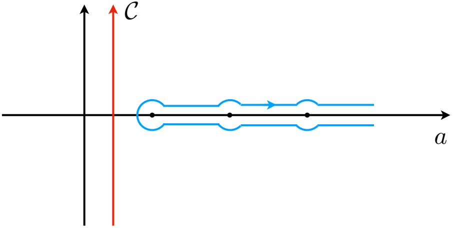

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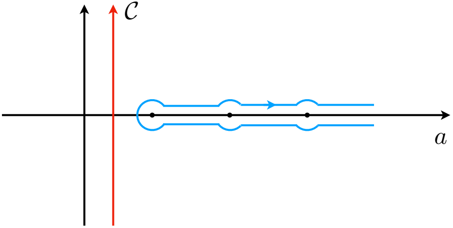

Finally, closing contour around $a = 0$ on the left-hand side yields all nonvanishing terms at origin at finite coupling!

Secretly Gaussian integral

Origin=OPE integrand in modified integration contour. Can recast as infinite-dimensional integral,

$$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i F(\vec{\xi}\,)\, e^{-\vec{\xi}\cdot M \cdot \vec{\xi}}\,,$$

where $M = (1 + \mathbb{K}) \cdot \mathbb{Q}$ and $F(\xi, \phi, \tau, \sigma)$ complicated Fredholm determinant. Remarkably, observe that perturbatively

$$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i \, e^{-\vec{\xi}\cdot (M+\delta M) \cdot \vec{\xi}}\,,$$

becomes Gaussian but with modified kernel⇒ evaluate explicitly!

Higher Loops and Legs: The Amplitude Bootstrap

Evade Feynman diagrams by exploiting analytic structure

QFT Property	Computation

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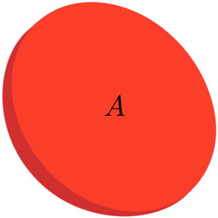

QFT Property	Computation
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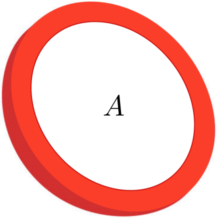

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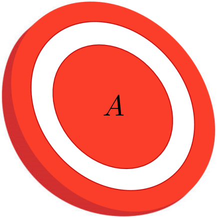

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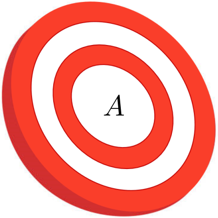

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See also recent $S(A_7) \to A_7$ work by [Dixon,Liu]