$T\bar{T}$ deformations and the pp wave correspondence

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IGST 2023

Zurich, Switzerland
19 June 2023

based on: HN and Cobi Sonnenschein, 2207.02257

Summary:

- 1. **Historical background and introduction**
- 2. $T\bar{T}$ **deformations in 2d, 4d and 1d**

- 3. **Penrose limits of single-trace** $T\bar{T}$ **deformations of** $AdS_3 \times S^3 \times T^4$ **and** $AdS_5 \times S^5$ **vs.** $\mathcal{N} = 4$ **SYM**
- 4. $T\bar{T}$ **deformations of string worldsheet on** $AdS_5 \times S^5$ **pp wave vs.** $\mathcal{N} = 4$ **SYM spin chain deformation**
- 5. **Conclusions.**

1a. Historical background

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•∃ pp wave solutions: In M theory,
(see also Figueroa-O'Farrill, Papadopoulos, hep-th/0106308)

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- + H(x^+, x^i)(dx^+)^2 + \sum_{i=1}^{9} dx_i^2 \\
F_4 &= dx^- \wedge d\varphi\ , \quad \Delta H = \frac{1}{2}|\varphi|^2
\end{aligned}
$$

•In particular, important class: $H(x^+, x^i) = \sum_{i,j} A_{ij}x^ix^j$.
•Kowalski-Glikman solution (PLB 1984)

$$
\begin{aligned}
A_{ij} &= \begin{cases} -\frac{\mu^2}{9}\delta_{ij}\ , & i = 1,2,3 \\ -\frac{\mu^2}{3}\delta_{ij}\ , & i = 4,...,9 \end{cases} \\
\varphi &= \mu\, dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}
$$

has maximal susy! Only other such solutions (theorem) are $Mink_{11}, AdS_4 \times S^7$ and $AdS_7 \times S^4$.

•Blau, Figueroa-O'Farrill,
Hull, Papadopoulos, hep-th/0110242
new type II B solution:

$$\begin{aligned} A_{ij} &= -\mu^2\delta_{ij}\,, \varphi = \mu(\omega + *\omega) \\ \omega &= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 . \end{aligned}$$

Then

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2 \sum_{i=1}^{8} (x^i)^2 (dx^+)^2 + \sum_{i=1}^{8} (dx^i)^2 \\ F_5 &= \pm\frac{\mu}{2} dx^+ \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \end{aligned}$$

is also a max. susy solution of type IIB! Only other is $Mink_{10}$ and $AdS_5 \times S^5$! (also a theorem)

•Moreover, pp wave solutions receive no quantum string corrections Horowitz and Steif (PRL 1990) $\Rightarrow$ exact string solutions! Like $AdS_5 \times S^5$ and Minkowski.

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•String in type IIB pp wave (D. Berenstein, J. Maldacena, HN, 2002): massive scalars,

$$S = \frac{1}{2\pi\alpha'}\int dt \int_0^{2\pi\alpha' p^+} d\sigma \left\{ \sum_I \left[\frac{(\dot{X}^I)^2}{2} - \frac{(X'^I)^2}{2} - \mu^2 \frac{(X^I)^2}{2} \right] + \text{fermi} \right\}$$

•Go to H and discretize $\sigma \Rightarrow$

$$H = \frac{1}{2\pi\alpha'}\sum_i \sum_I \left[\frac{(\dot{X}_i^I)^2}{2} + \frac{(X_i^I - X_{i+1}^I)^2}{2a^2} + \mu^2 \frac{(X_i^I)^2}{2} \right] + \text{fermi}.$$

.

•Can we get it from $\mathcal{N}=4$ SYM on the boundary $S^3\times\mathbb{R}_t$ (of global $AdS_5\times S^5$), KK reduced on S^3 to QM Hamiltonian on $\mathbb{R}_t$? Use CFT operator-state correspondence for $\mathbb{R}^4$ vs. $S^3\times\mathbb{R}_t$ and define a Hamiltonian acting on states.

•Gopakumar and Gross (NPB 1995): states obtained in large N, by considering only planar "words" are acted upon by Cuntz oscillators, $a_\alpha a^\dagger_\beta=\delta_{\alpha\beta}$, $\sum_\alpha a^\dagger_\alpha a_\alpha = 1\!\mathrm{l}-|0\rangle\langle 0|$.

•But now also, for "dilute gas approximation" states, replace by "Cuntz oscillators at each site", $a_j a_j^\dagger = 1\!\mathrm{l}, a_j^\dagger a_j = 1\!\mathrm{l}-|0\rangle\langle 0|$.

.

•Then, the Hamiltonian is (modulo $1/J$ corrections)

$$H = \sum_{I,i} \left\{ b_i^{I\dagger} b_i^I + \frac{g_s N}{2\pi} \left[\left(b_i^I + b_i^{I\dagger} \right) - \left(b_{i+1}^I + b_{i+1}^{I\dagger} \right) \right]^2 \right\}$$

and reduces to string pp wave Ham. for $X_i^I = (b_i^I + b_i^{I\dagger})/\sqrt{2}$.

•Moreover, diagonalization: standard in condensed matter. Go to discrete momentum space, $b_j = \frac{1}{\sqrt{J}} \sum_{n=1}^{J} e^{\frac{2\pi i j n}{J}} a_n$, choose backward $\pm$ forward waves, $a_{\pm n}^I = (c_{n,1}^I \pm c_{n,2}^I)/\sqrt{2}$ and do a Bogoliubov transformation (mixing c's and $c^\dagger$'s into $\tilde{c}$'s), to find free oscillators ($H = \sum_n \omega_n (\tilde{c}_{n,1}^\dagger \tilde{c}_{n,1} + \tilde{c}_{n,2}^\dagger \tilde{c}_{n,2})$) with

$$\omega_n = \sqrt{1 + \frac{4 g_s N}{\pi} \sin^2 \frac{\pi n}{J}}$$

.

•Thus, "BMN states"

$$a_n^I a_{-n}^J|0\rangle = \frac{1}{N^{J/2+1}}\sum_{l=1}^{J}\sum_{k=1}^{J} e^{\frac{2\pi i l n}{J}} e^{-\frac{2\pi i k n}{J}} \mathrm{Tr}\left[Z^l \Phi^I Z^{k-l} \Phi^J Z^{J-k}\right]$$

•Finally: $\omega_n^{1-loop\ approx.} \sim 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{\pi n}{J} = 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}$ $(4\pi g_s N = g_{YM}^2 N = \lambda)$ is $=$ 1+eigenvalue of $H_{XXX1/2}$ (Heisenberg Ham.) $\rightarrow$ integrability, Bethe ansatz (standard one).

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1b. Introduction

- Use pp wave (Penrose limit) to better understand AdS/CFT dualities.

- Understand what deformations we can have for the pp wave method.

- $\exists$ interesting $T\bar{T}$ deformaion: preserves integrability

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•$T\bar{T}$ deformation in holography:
motion in the bulk, vs. deforming the gravity background

•Latter is well defined in $AdS_3 \times S^3 \times T^4$, but for "single-trace deformations", given by TsT transformation in bulk.

•Extend to $AdS_5 \times S^5$: $(TsT)^2$. Take Penrose limit $\rightarrow$ first on $T\bar{T}(AdS_3 \times S^3 \times T^4)$, then on $T\bar{T}(AdS_5 \times S^5)$.

•Interpret in $\mathcal{N} = 4$ SYM: spin chain of dipole theory, likely noncommutative.

.

•$T\bar{T}$ deformation of discretized string worldsheet in $AdS_5 \times S^5 \to$ difficult, not clear.

•Discretize, then deform: QM spin chain $\to$ OK. (use Gross et al., 2019)

•In $\mathcal{N} = 4$ SYM: deform the *large charge sector*, to an equivalent one.

.

2. $T\bar{T}$ deformations in 2d, 4d and 1d

•Deformation defined in 2d, by $T(z)\bar{T}(\bar{z})$ (Zamolodchikov). Equivalently, by $(\det T_{\mu\nu})$ $\to$ understood in terms of renormalized quantities, by normal ordering $\frac{1}{8}(T_{\alpha\beta}T^{\alpha\beta} - (T^{\alpha}{}_{\alpha})^2)$, in point splitting regularization.

•But, equivalently, (Cavaglia, Negro, Szeczenyi, Tateo 2016 and Bonelli, Doroud, Zhou, 2018) realized that can deform classical $\mathcal{L}$ by $\det T_{\mu\nu}$ at each point in deformation,

$$\partial_t S = \frac{1}{2}\int d^2x\sqrt{g}\,[(\epsilon^{\mu\nu}\epsilon^{\rho\sigma})\,T_{\mu\rho}T_{\nu\sigma}] = \int d^2x\sqrt{g}\,\det T_{\mu\nu}$$

.

•Generalization to higher dimensions: several possibilities.

1.$\epsilon^{\mu\nu}\epsilon^{\rho\sigma} = g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}$ can be generalized

2. power of $\det T_{\mu\nu}$: $(-\det T_{\mu\nu})^{\frac{1}{\alpha}}$.

3. (Marika Taylor 2018) with $T^{\mu\nu}T_{\mu\nu} - \frac{1}{D-1}T^{\mu}{}_{\mu}T^{\nu}{}_{\nu}$.

4. ...

•What is the holographic dual?

•McGough, Mezei, Verlinde 2016: RG flow in $r = 1/z$ (radial coordinate) $\Rightarrow$ we can define theory by Dirichlet boundary condition at $z = \epsilon$ instead of at $z = 0$: $T\bar{T}$ deformation.

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•But not satisfying $\rightarrow$ what is the normal holographic dual (defined at $z = 0$) of $T\bar{T}$ deformation?

•For $AdS_3 \times S^3 \times T^4$ with NS flux $\leftrightarrow \mathcal{M}^p/\mathfrak{S}_p$. Instead of double trace $T(z)\bar{T}(\bar{z}) = (\sum_{i=1}^{p} T_i(z))(\sum_{j=1}^{p} \bar{T}_j(\bar{z}))$, single trace def.: construct string worldsheet vertex operators for "single trace",

$$"T(z)\bar{T}(\bar{z})" = \sum_{i=1}^{p} T_i(z)\bar{T}_i(\bar{z})$$

$\rightarrow$ similar properties to double trace deformation Giveon, Itzhaki, Kutasov, 2017.

•Corresponds to TsT (T-duality/shift/T-duality) transformation (via string worldsheet vertex operators) on CFT_2 directions x and t.

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$\bullet AdS_3 \times S^3 \times T^4$ with NS flux,

$$R^{-2}ds^2 = e^{2\rho}(-dt^2 + dx^2) + d\rho^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + ds^2(T^4)$$

$$H = -2e^{2\rho} dt \wedge dx \wedge d\rho + \frac{1}{4}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \,,$$

via TsT with shift $x \to x + \gamma t$, $-2\gamma = l_s^2/\tilde{R}^2$, leads to $(R^2 \sim l_s^2 k)$

$$\frac{ds^2}{l_s^2} = \frac{k(-dt^2 + dx^2)}{\frac{l_s^2}{\tilde{R}^2} + e^{-2\phi}} + kd\phi^2 + kds^2_{S^3} + ds^2_{T^4}$$

$$e^{2\Phi} = \frac{vk}{p}\frac{e^{-2\phi}}{\frac{l_s^2}{\tilde{R}^2} + e^{-2\phi}} \equiv e^{2\Phi_0}\frac{e^{-2\phi}}{\frac{l_s^2}{\tilde{R}^2} + e^{-2\phi}} \,,$$

$$H = -\frac{2e^{2\phi}}{\left(1 + \frac{l_s^2}{\tilde{R}^2}e^{2\phi}\right)^2} dt \wedge dx \wedge d\phi + \frac{1}{4}\sigma_1 \wedge \sigma_2 \wedge \sigma_3$$

.

•$AdS_5 \times S^5$: do 2TsT's, on (01) and (23) (conjectured to be dual to noncommmutative theory, Maldacena, Russo 1999; Hashimoto, Itzhaki 1999) on Euclidean theory, then back:

$$
\begin{aligned}
ds^2 &= \frac{e^{2\rho}(-dt^2+d\vec{x}^2)}{1+\gamma^2 e^{4\rho}} + d\rho^2 + ds^2_{S^5} \\
B_{01} &= B_{23} = \frac{\gamma e^{2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}} \Rightarrow H_{23\rho} = H_{01\rho} = \partial_\rho B_{01} = -\frac{4\gamma}{(e^{-2\rho}+\gamma^2 e^{2\rho})^2} \\
\Phi &= \Phi_0 - \log e^{2\rho} - \log(e^{-2\rho}+\gamma^2 e^{2\rho}) \Rightarrow e^{2\Phi} = e^{2\Phi_0}\left(\frac{e^{-2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}}\right)^2.
\end{aligned}
$$

.

•One dimension (QM) (Gross, Kruthoff, Rolph, Shaghoulian, 2019). In 2d,

$$\frac{\partial S_E(\lambda)}{\partial\lambda} = \int d^2x\sqrt{\gamma}\,8T\bar{T}\ ,$$

but, assuming the McGough, Mezei, Verlinde holography, on the flow $\Rightarrow T^{\mu}{}_{\mu} = -16\lambda T\bar{T}$ and then one finds

$$\frac{\partial S_E(\lambda)}{\partial\lambda} = \int d^2x\sqrt{\gamma}\frac{(T^{\tau}{}_{\tau})^2 + T_{\tau\phi}T^{\tau\phi}}{1/2 - 2\lambda T^{\tau}{}_{\tau}}$$

and the energy deformation

$$E(\lambda) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda E_0 + 16\lambda^2 J^2}\right)\ .$$

.

•Dimensional reduction: $T_{\tau\phi} = T^{\tau\phi} = 0$ and $T^{\tau}{}_{\tau} = T$, then as if $J = 0$ above,

$$E(\lambda) = \frac{1 - \sqrt{1 - 8\lambda E_0}}{4\lambda} \,,$$

leading to a deformed Hamiltonian that is a function of the undeformed one,

$$H(\lambda) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda H_0}\right) = f_\lambda(H_0).$$

.

3. Penrose limits of single-trace $T\bar{T}$ deformations of $AdS_3 \times S^3 \times T^4$ and $AdS_5 \times S^5$ vs. $\mathcal{N} = 4$ SYM

•After a transformation of coordinates, we can put the metric in the form in the Penrose theorem,

$$R^{-2}ds^2 = 2dVdU + \alpha dV^2 + \sum_i \beta_i dV dY^i + \sum_{i,j} C_{ij} dY^i dY^j$$

after which we do Penrose's rescaling

$$U = u\ , \ \ V = \frac{v}{R^2}\ , \ \ \ Y^i = \frac{y^i}{R}\ , \ \ \ R \to \infty$$

•Results in pp wave in Rosen coordinates; need to transform to usual Brinkmann coords.

.

• $AdS_3 \times S^3 \times T^4$: motion in (t, ρ, ψ) gives pp wave metric

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- + H(x^+)(dx^+)^2 + d\tilde{\varphi}^2 + d\tilde{x}^2 + d\vec{\tilde{y}}_2^2 + ds^2(T^4) \ , \\
H(x^+) &= A_{\tilde{\varphi}\tilde{\varphi}}\tilde{\varphi}^2 + A_{\tilde{x}\tilde{x}}\tilde{x}_2^2 + A_{\tilde{y}\tilde{y}}\vec{\tilde{y}}_2^2 .
\end{aligned}
$$

where

$$
\begin{aligned}
A_{\tilde{y}\tilde{y}} &= -16\mu^2 \ , \quad A_{\tilde{\varphi}\tilde{\varphi}} = -8\mu^2 \frac{1-4\gamma e^{2\rho}}{(1+2\gamma e^{2\rho})^2} \\
A_{\tilde{x}\tilde{x}} &= -\frac{4}{(1+2\gamma e^{2\rho})^2}\left[(1+2\gamma e^{2\rho})(\gamma E^2+\mu^2) - 6\gamma\mu^2 e^{2\rho}\right] \ , \\
e^{\rho(x^+)} &= \frac{E}{\sqrt{4\mu^2-2\gamma}} \sin\left(x^+\sqrt{4\mu^2-2\gamma}\right) \ , \ \text{if} \ 4\mu^2-2\gamma > 0 \ ,
\end{aligned}
$$

and also a B field: only $\rho(x^+)$ dependence.

.

•In light-cone gauge $x^+ = \tau$ and conformal gauge, the string action is

$$
\begin{aligned}
S_{\text{string}} \;=\; & -\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+} d\sigma \int d\tau \Bigg[\eta^{ab}\sum_{i\neq\pm}\partial_a X^i \partial_b X^i - 8\mu^2\tilde{\varphi}^2\frac{1-4\gamma e^{2\rho(\tau)}}{1+2\gamma e^{2\rho(\tau)}} - 16\mu^2\tilde{\vec{y}}_2^{\,2} \\
& -4\tilde{x}^2\frac{(1+2\gamma e^{2\rho(\tau)})(\mu^2+\gamma E^2) - 6\gamma\mu^2 e^{2\rho}}{(1+2\gamma e^{2\rho(\tau)})^2} \\
& -E\partial_1 x' + 4\mu\tau \sin^2(4\mu\tau)(\partial_0 y_1 \partial_1 y_2 - \partial_1 y_1 \partial_0 y_2)\Bigg] \,.
\end{aligned}
$$

.

$\bullet AdS_5 \times S^5$: motion in (t, ρ, ψ), so pp wave metric

$$ds^2 = 2dx^+ dx^- + \left[A_{\varphi\varphi}\varphi^2 + A_{\tilde{x}\tilde{x}}\vec{\tilde{x}}_3^2 + A_{\tilde{y}\tilde{y}}\vec{\tilde{y}}_4^2\right](dx^+)^2 + d\tilde{\varphi}^2 + (d\vec{\tilde{x}}_3)^2 + (d\vec{\tilde{y}}_4)^2\ ,$$

$$\begin{aligned} A_{\tilde{y}\tilde{y}} &= -\mu^2\ ,\ \ A_{\varphi\varphi} = -2\mu^2\frac{1-8\gamma^2 e^{4\rho}-\gamma^4 e^{8\rho}}{(1+\gamma^2 e^{4\rho})^2} \\ A_{\tilde{x}\tilde{x}} &= -\frac{E^2(1+\gamma^2 e^{4\rho})-\mu^2 e^{2\rho}}{(1+\gamma^2 e^{4\rho})^2} 2\gamma^2 e^{2\rho}(3-\gamma^2 e^{4\rho}) - (-2E^2\gamma^2 e^{2\rho}+\mu^2)\frac{1-\gamma^2 e^{4\rho}}{1+\gamma^2 e^{4\rho}}\ , \end{aligned}$$

and a B field: also $\rho(x^+)$ dependence

.

•The string action in light-cone gauge $x^+ = \tau$ and conformal gauge is

$$
\begin{aligned}
S_{\text{string}} \;=\; & -\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+} d\sigma \int d\tau \left\{ \eta^{ab}\sum_{i\neq\pm}\partial_a X^i \partial_b X^i - 2\mu^2\tilde{\varphi}^2 \frac{1-8\gamma^2 e^{4\rho(\tau)}-\gamma^4 e^{8\rho(\tau)}}{(1+\gamma^2 e^{4\rho(\tau)})^2} \right. \\
& -\mu^2 \vec{\tilde{y}}_4^2 - \vec{\tilde{x}}_3^2 \left[(-2E^2\gamma^2 e^{2\rho(\tau)} + \mu^2)\frac{1-\gamma^2 e^{4\rho(\tau)}}{1+\gamma^2 e^{4\rho(\tau)}} \right. \\
& \left.\left. + \frac{E^2(1+\gamma^2 e^{4\rho(\tau)}) - \mu^2 e^{2\rho(\tau)}}{(1+\gamma^2 e^{4\rho(\tau)})^2} 2\gamma^2 e^{2\rho(\tau)}(3-\gamma^2 e^{4\rho(\tau)}) \right] \right\} + \gamma e^{2\rho(\tau)}\partial_\sigma x_1' \,.
\end{aligned}
$$

.

$\mathcal{N}=4$ SYM interpretation:

•As usual, $X^i = X_0^i \,\text{exp}[-i\omega\tau + ik_i\sigma]$, $k_{i,n} = \frac{n_i}{\alpha' p^+}$.
•Only simple modes are $\tilde{y}_i$, with the usual

$$\frac{\omega_y}{\mu} = \sqrt{1 + \frac{n_i^2}{(\mu\alpha' p^+)^2}}.$$

.

Symmetries

•$AdS_5 \times S^5$: In pp limit, $PSU(2,2|4) \supset SO(4,2) \times SO(6)$ breaks to $[SO(4)_1 \times SO(2)_1] \times [SO(4)_2 \times SO(2)_2]$. $SO(2)_1$: X^+ translations, $SO(2)_2$: X^- translations.
•$T\bar{T}$ def.: $X^+ = \tau$ not a symmetry $\Rightarrow \nexists SO(2)_1; SO(4)_1 \to SO(3) \to SO(2)'_1$. So $\left[SO(2)'_1\right] \times [SO(6)] \to \left[SO(2)'_1\right] \times [SO(4) \times SO(2)_2]$
•But $\exists$ also generators e_i and e_i^*, and $e = -p_-$. Undeformed or deformed, same:

$$\begin{aligned}
\xi_{e_i} &= -\cos(\mu x^+)\partial_i - \mu \sin(\mu x^+)\tilde{y}^i\partial_- \\
\xi_{e_i^*} &= -\mu \sin(\mu x^+)\partial_i + \mu^2 \cos(\mu x^+)\tilde{y}^i\partial_- \,.
\end{aligned}$$

•pp algebra: $a_i \sim e_i + e_i^*$, $\quad M_{ij} = x_i\partial_j - x_j\partial_i = i(a_i^\dagger a_j - a_j^\dagger a_i)$

$$h = -p_+ = -\mu \sum_i a_i^\dagger a_i \,, \qquad M_{23} = x_2\partial_3 - x_3\partial_2.$$

.

•$SO(4)_2$ maintained $\leftrightarrow$ R-symmetry of 4 of the fermions (with $J > 0$) $\Rightarrow$ susy still $\mathcal{N} = 4$.
•Spin chain: $Z = X^1 + iX^2$ charged under $J = i\partial_\psi$, Φ^i , $i = 1, 2, 3, 4$. Insertions into $\mathrm{Tr}\,[Z^J]$ of string modes.
•Undeformed case: $D_i Z = \partial_i Z + [A_i, Z]$ and Φ^i.
•Deformed case: still 2 index refers to transverse scalar symmetry, so insertions of Φ^i unchanged. Also, interactions are still $[\Phi^i, \Phi^j]^2$.
•But $D_i Z$ insertions changed: (01) and (23) singled out $\Rightarrow$ dipole theory, likely noncommutative.

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4. $T\bar{T}$ deformations of string worldsheet on $AdS_5 \times S^5$ pp wave, vs. $\mathcal{N} = 4$ SYM spin chain deformation

•One possibility: $T\bar{T}$ deform string worldsheet, then discretize. Still corresponds to a spin chain? Unclear (likely no):

$$
\begin{aligned}
L &= \int dx \mathcal{L} \to \sum_{I,i} L_i^I \\
L_i^I &= -\frac{\sqrt{1 + 2\lambda(-(\dot{X}_i^I)^2 + (X_i^I - X_{i+1}^I)^2/a^2)(1 - \lambda\mu_I^2(X_i^I)^2/2)} - (1 - \lambda\mu_I^2(X_i^I)^2)}{2\lambda(1 - \lambda\mu_I^2(X_i^I)^2/2)}.
\end{aligned}
$$

.

•We could try the same, with

$$X_i^I = \frac{a_i^I e^{-i\mu_I t} + (a_i^I)^\dagger e^{i\mu t}}{\sqrt{2}} \, ,$$

•Following the same steps doesn't give a diagonal Hamiltonian, since $A_j^I \equiv \left[(a_j^I + a_j^{\dagger I}) - (a_{j+1}^I + a_{j+1}^{\dagger I}) \right]$ doesn't lead to $\sum_j (A_j^I)^2$ as before. Now there is time dependence.

.

•Also in (Pozsgay, Jiang, Takacs, 2019 and Manchetto, Sfondrini, Yang 2019), argued that for spin chains, $T\bar{T}$ corresponds to Bargheer, Beisert, Loebert 2018,2019 deformation: Bethe-Yang equations

$$e^{ip_jR}\prod_{k\neq j}^{N} S(p_k,p_j) = 1$$

deformed to

$$e^{ip_jR+i\alpha(X_jY-Y_jX)}\prod_{k\neq j}^{N} S(p_j,p_k) = 1$$

$$\Rightarrow \quad S(p_j,p_k) \to e^{i\alpha(X_jY_k-X_kY_j)}S(p_j,p_k)$$

with $X_j = p_j$, $Y_j = H(p_j)$.

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•Note also that Baggio, Sfondrini 2018 consider the above deformation for $AdS_3 \times S^3 \times T^4$ pp wave in T^4 directions and it matches the free massless boson on worldsheet $T\bar{T}$ deformation, but Sfondrini, van Tongeren 2019 consider it for $AdS_5 \times S^5$ and find some other deformation.

.

•But rather, we use the Gross et al. prescription for $T\bar{T}$ deformation of QM system (note: dim.red. of $T\bar{T}$-holography, not $T(\bar{T})$ in 1d). Then,

$$H(\lambda)=\mu\frac{1}{4\lambda\mu}\left[1-\sqrt{1-8\lambda\mu\sum_{i,I}\left(\frac{a_i^I(a_i^I)^\dagger+(a_i^I)^\dagger a_i^I}{2}+\frac{1}{a^2}\left(\frac{a_i^I+(a_i^I)^\dagger}{\sqrt{2}}-\frac{a_{i+1}^I+(a_{i+1}^I)^\dagger}{\sqrt{2}}\right)^2\right)}\right]$$

and so the eigenenergies are

$$E(\lambda,g^2N,n/J)=\mu\frac{1}{4\lambda\mu}\left(1-\sqrt{1-8\lambda\mu\sqrt{1+\frac{g^2N}{\pi^2}\sin^2\frac{\pi n}{J^2}}}\right).$$

•Deformation preserves integrability! (H_0 has conserved quantities $\Rightarrow f(H_0)$ also has).

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Deformation of $\mathcal{N}=4$ SYM

•Symmetries now continue to be the same: $[SO(2)_1 \times SO(4)_1] \times [SO(2)_2 \times SO(4)_2]$, also $\mathcal{N}=4$ susy (unique!) $\Rightarrow$ Deformed sector within $\mathcal{N}=4$ SYM.
•pp wave has symmetry operators

$$
\begin{aligned}
&h = \xi_{e^+} = -\partial_+ \, , \;\; \xi_{e^-} = -\partial_- \, , \\
&\xi_{e_i} = -\cos(\mu x^+)\partial_i - \mu \sin(\mu x^+)\tilde{y}^i \partial_- \, , \;\; i = 1, ..., 8 \\
&\xi_{e_i^*} = -\mu \sin(\mu x^+)\partial_i + \mu^2 \cos(\mu x^+)\tilde{y}^i \partial_- \, , \\
&\xi_{M_{ij}} = x_i \partial_j - x_j \partial_i \, , \;\; i,j = 1, ..., 4 or 5, ..., 8.
\end{aligned}
$$

.

•Algebra

$$\begin{aligned} [e_i, e_j^*] &= (\mu e)\delta_{ij} \\ [h, e_i] &= \mu e_i^* \ , \quad [h, e_i^*] = -\mu e_i \ , \\ [M_{ij}, e_k] &= -\delta_{ik} e_j + \delta_{jk} e_i \ , \quad [M_{ij}, e_k^*] = -\delta_{ik} e_j^* + \delta_{jk} e_i^* . \end{aligned}$$

•Define

$$a_i = \frac{e_i + i e_i^*}{\sqrt{2\mu}} \ , \qquad a_i^\dagger = \frac{e_i^* - i e_i}{\sqrt{2\mu}}$$

$$\begin{aligned} M_{ij} &= i(a_i^\dagger a_j - a_j^\dagger a_i) \ , \\ ih &= \frac{\mu}{e} \sum_i a_i^\dagger a_i . \end{aligned}$$

.

- Then

$$(\pm i)H = \mu \sum_i \tilde{a}_i^\dagger \tilde{a}_i = \sum_{n\geq 1} c_n \frac{1}{\lambda}\left[\lambda\mu_0 \sum_i \tilde{a}_{0,i}^\dagger \tilde{a}_{0,i}\right]^n$$

is obtained from

$$\tilde{a}_i \;=\; \sum_{n\geq 0} \tilde{c}_n \left[\lambda\mu \sum_j \tilde{a}_{0,j}^\dagger \tilde{a}_{0,j}\right]^n \tilde{a}_{0,i} \;,$$

$$\frac{1}{4\lambda}\left(1-\sqrt{1-8\lambda x}\right) \equiv \sum_{n\geq 1} c_n \lambda^{n-1} x^n \;\Rightarrow\; \sqrt{\frac{1}{4\lambda}\left(1-\sqrt{1-8\lambda x}\right)} \equiv \sqrt{x}\sum_{n\geq 0}\tilde{c}_n \lambda^n x^n.$$

.

•Thus $(e_0, \tilde{a}_{0,i}, \tilde{a}^\dagger_{0,i})$ and $(e, \tilde{a}_i, \tilde{a}^\dagger_i)$ satisfy the same algebra if also $e = e_0$ and

$$M_{ij} = \sum_{n \geq 0} \left[\lambda \mu_0 \sum_j \tilde{a}^\dagger_{0,j} \tilde{a}_{0,j} \right]^n M_{0,ij}.$$

•Thus we have *equivalence* of the 2 sets.

•Undeformed generators on undeformed BMN operators give

$$(\tilde{a}_{0,i})^\alpha{}_\beta = \frac{\delta}{\delta(\Phi_i)^\alpha{}_\beta}, \quad (\tilde{a}^\dagger_{0,i})^\alpha{}_\beta = (\Phi_i)^\alpha{}_\beta \mathbf{In} ,$$

where Φ^i, $i = 1, ..., 4$ are 4 scalars inserted inside Tr $[Z^J]$ and $\mathbf{In}$ refers to insertion inside the trace.

.

•Operators of deformed sector: vacua: same, $e = e_0 \rightarrow$ undeformed $p^+ \rightarrow J$. Same vacuum Tr $[Z^J]$, but insert

$$a^{\dagger i}_m|0\rangle \sim \sum_l \text{Tr}\left[Z^l \left(\sum_{n\geq 0} \tilde{c}_n \left[\lambda\mu_0 \sum_j \Phi^j \frac{\delta}{\delta\Phi_j}\right]^n\right) \Phi_i Z^{J-l}\right] e^{\frac{2\pi m l}{J}}.$$

•→ *Deformed* BMN sector, equivalent to original one $\rightarrow H(\lambda)$ has both same eigenstates (undeformed BMN sector) and new eigenstates (deformed BMN sector), since they are equivalent. $H(\lambda)$ in *undeformed* eigenstates gives $E(\lambda)$, $H(\lambda)$ in deformed eigenstates gives E_0.

•Obs: $H_\lambda = f(\lambda, H_0)$ gives in general $L_\lambda \neq g(\lambda, L_0)$.

.

5. Conclusions

• Extended holographic dual of "single-trace $T\bar{T}$ deformations" from $AdS_3 \times S^3 \times T^4$ to $AdS_5 \times S^5$ and took Penrose limit: spin chain of a dipole theory (noncommutative?)

• $T\bar{T}$ of string worldsheet on $AdS_5 \times S^5$ pp wave discretized $\rightarrow$ difficult to solve.

• $T\bar{T}$ of spin chain (discretized string worldsheet on pp wave), via Gross et al. gives a deformed $\mathcal{N} = 4$ SYM large charge sector equivalent to the original one.

IGST 2024

June 17 - 21

ICTP-SAIFR, Sao Paulo

Local Organizers:

Thiago Fleury, Horatiu Nastase,
Enrico Olivucci, Pedro Vieira

There will probably be a conference on Bootstrap/Integrability interface the week before IGST. To be confirmed.