Replica Wormholes and Liouville Theory

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Plan

Part I: overview of replica wormholes (islands) and what they mean

Part II: replica wormholes for general n, Liouville theory and Polyakov's conjecture

Aspects of the information paradox solved

In 2019 the game changed due to the west and east coast Replica Wormhole papers

Replica wormholes and the black hole interior

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Replica Wormholes and the Entropy of Hawking Radiation

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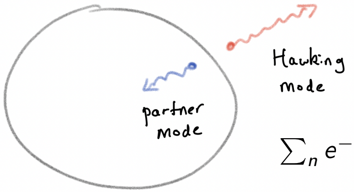
1. We now know what Hawking "missed"
2. Effective theory (i.e. semi-classical gravity) finds a way to be consistent with unitarity: no information is lost when a BH evaporates
3. Effective theory finds a clever way to hide the microscopic physics (string theory fuzzballs??) by introducing a new concept the "island" (an observer-dependent saddle, the replica wormhole)

Hawking's paradox

The Hawking process gives rise to an entanglement paradox



$$\sum_n e^{-n\omega/2T}|n\rangle|\widetilde{n}\rangle$$

- *Each emitted mode has an entangled partner behind the horizon*

There is a flux of entanglement ... measured by the thermodynamic entropy of the outgoing modes

$$\boxed{\frac{dS}{dt} = -\int \frac{d\omega}{2\pi} \sum_n p_n(\omega) \log p_n(\omega) \qquad p_n(\omega) = \mathcal{Z}^{-1} e^{-n\omega/T}}$$

- *When the BH evaporates away where are the partners?*

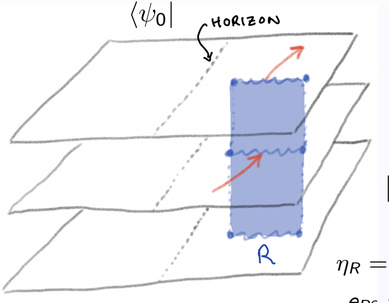
QFT entropy calculations

- Computations of QFT entropies are standard $\rho_R = \mathrm{Tr}_{R^c} |\psi\rangle\langle\psi|$

$$e^{(1-n)S^{(n)}(R)} \equiv \mathrm{Tr}\, \rho_R^n = \langle\psi|^{\otimes n} e_{R^c} \otimes \eta_R |\psi\rangle^{\otimes n}$$

- Partition function on a replicated spacetime glued together in the right way

$\langle\psi_0|$

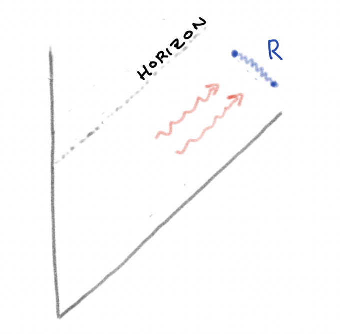


$|\psi_0\rangle$

$|\psi\rangle = U|\psi_0\rangle$

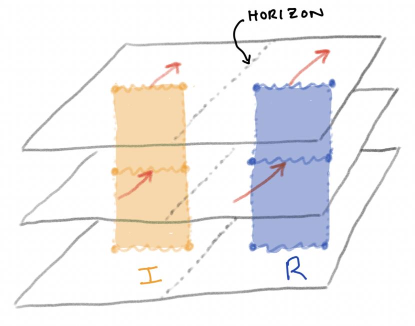
$\eta_R =$ cyclic perm on R

$e_{R^c} =$ identity on R^c

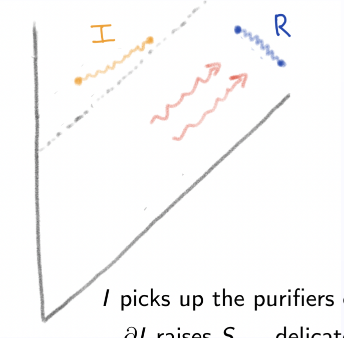


The replica wormhole (island)

- In semi-classical gravity geometry does not fluctuate but that doesn't stop the boundary conditions fluctuating
- Gravity can conjure up new saddles where replicas are connected together in an additional region I (the island)



$\eta_R \to \eta_R \otimes \eta_I$



I picks up the purifiers of R lowering S

∂I raises S ... delicate competition

Saddle with the smallest S dominates

A variational problem

- In the $n \to 1$ limit, boils down to a remarkable variational problem

$$S(\rho_R^{micro}) = \min_I S_I(R) \qquad S_I(R) \equiv \underset{\partial I}{ext}\left\{ \frac{A(\partial I)}{4G} + S(\rho_{R\cup I}^{sc}) \right\}$$

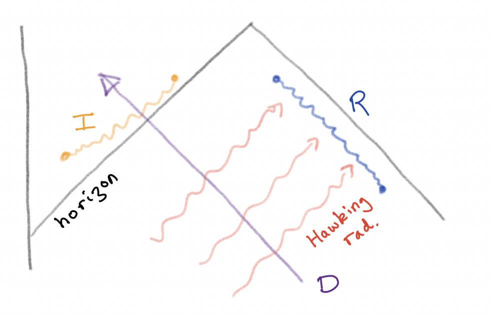
$\partial I =$ Quantum Extremal Surface(s)

- What does it mean?
  - *When there is an island $S(\rho_R^{micro}) \neq S(\rho_R^{sc})$ implies that the microscopic state of the radiation ρ_R^{micro} is NOT equal to the semi-classical state ρ_R^{sc}*
  - *The true state of the Hawking radiation ρ_R^{micro} is subtly correlated at the microscopic level and unitary can—and is—maintained (Page curve). We can compute the correlations via the mutual information*

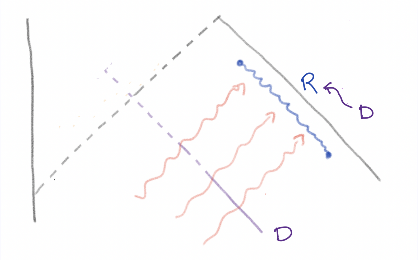
$$I(R_1, R_2) = S(R_1) + S(R_2) - S(R_1 \cup R_2)$$

Correlations appear when the island for $R_1 \cup R_2$ is not the union of the islands for R_1 and R_2.

• If you have access to a subset R and an island I saddle dominates then the modes inside the BH on I can really (i.e. at the microscopic level) be "decoded" from your R. This includes things that fall into the BH and lie in I.



semi-classical



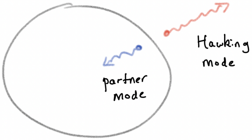
microscopic?

So the inside of the BH in I (at the semi-classical level) is not really inside the BH (at the microscopic level).

$$\boxed{S(\rho_R^{micro}) \overset{ext}{\sim} \frac{\mathrm{A}(\partial I)}{4G} + S(\rho_{R\cup I}^{sc})}$$

1. The island/QES formula (generalized entropy) is a UV safe version of the Bekenstein-Hawking entropy formula.
2. QFT in quantum gravity is less divergent than in a fixed geometry (cf. developments in operator algebras of QFT in QG).

What is happening at the micro level?



$$|\psi\rangle^{sc} = \prod_{modes} \sum_{n} e^{-n\omega/2T} |n\rangle|\widetilde{n}\rangle$$

implies $\rho_R^{sc} = \prod_{modes} \sum_n e^{-n\omega/T} |n\rangle\langle n|$, i.e. a thermal state.

- At the microscopic level the interior states $|\widetilde{n}\rangle$ are not quite orthogonal (the BH is not big enough to contain them as independent states)

$$\langle \widetilde{n}|\widetilde{m}\rangle = \delta_{nm} + Z_{nm} \qquad Z_{nm} \sim \mathscr{O}(e^{-1/G})$$

hence ρ_R^{micro} is not quite thermal.

- The "fluctuations" Z_{nm} are quasi-random.
- When R is big enough the small fluctuations can dominate and cause the island saddle.
- The semi-classical state is an average over the quasi-randomness $\overline{\langle \widetilde{n}|\widetilde{m}\rangle} = \delta_{nm}$.

Part II: an island formula for $S^{(n)}(R)$?

Question: can we understand replica wormholes (island saddles) without taking the $n \to 1$ limit?

• Dong's generalization of the RT formula in holography suggests

$$\widetilde{S}^{(n)}(\rho_R^{micro}) \sim \frac{A(\partial I)}{4G} + \widetilde{S}^{(n)}(\rho_{R\cup I}^{sc}) \qquad \boxed{\widetilde{S}^{(n)}(\rho) = n^2 \partial_n \Big(\frac{n-1}{n} S^{(n)}(\rho) \Big)}$$

involving the **modular entropy** and NOT the Rényi entropy.

• It is a hard problem because the QES ∂I has non-trivial back-reaction when $n \neq 1$.
• Work in the context of Jachiw Teitleboim gravity ... the s-wave sector of a near-extremal charged BH in $3 + 1$ (more than just a toy model!)

Even so there are many steps ... here I will focus on a few features involving Liouville Theory

BH in JT gravity

1. Near horizon geometry of near-extremal BH is $\mathrm{AdS}_2 \times S^2$
2. JT gravity has the 2d metric and a scalar field ϕ (dilaton) which is the radius of the S^2
3. ϕ is a constraint field that enforces the metric to be AdS_2 the Poincaré disc (Euclidean) $|w| \leq 1$

$$ds^2 = e^{2\rho}\,|dw|^2 \qquad e^{2\rho} = \frac{1}{(1-|w|^2)^2}$$

4. CFT matter couples to metric and sources the dilaton $e^{2\rho}\partial_w\big(e^{-2\rho}\partial_w\phi\big) = 8\pi G T_{ww}$, etc
5. JT gravity reduces to the dynamics of the boundary map $w(\tau) = e^{i\theta(\tau)}$ with Schwarzian action

$$\boxed{S_{\mathrm{JT}} \sim \frac{1}{G}\int d\tau\,\{w,\tau\}}$$

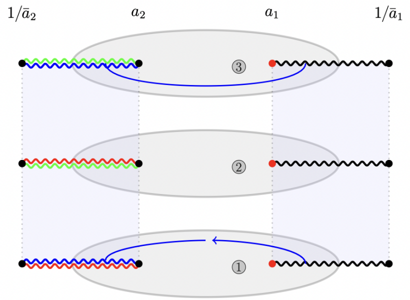
(Many connections with integrability, matrix models, quantum groups, etc ...)

Replica wormhole

- An n-fold branched cover of the Poincaré disc $|w| \leq 1$
- Conformal factor ρ satisfies the Liouville equation with sources at branch points

$$-\partial_w \partial_{\bar{w}} \rho + e^{2\rho} = 2\pi \left(1 - \frac{1}{n}\right) \sum_j \delta^{(2)}(w - a_j)$$

- For N QES $\{a_j\}$ we have a genus $(n-1)(N-1)$ surface (incl mirror points $w \to 1/\bar{w}$)



In order to make progress we need an explicit description of the cover

Fuchsian uniformization

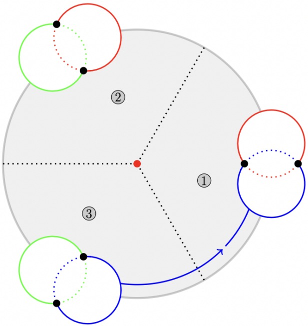
Liouville stress tensor

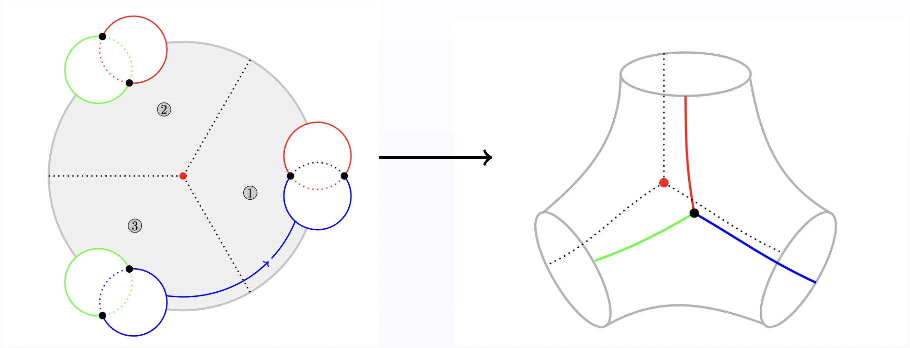
$$\mathcal{T}(w) = -(\partial_w \rho)^2 + \partial_w^2 \rho \overset{w \to a_j}{=} \frac{\varepsilon}{(w-a_j)^2} + \frac{c_j}{w-a_j} + \cdots$$

with $\varepsilon = \frac{n^2-1}{4n^2}$. Finding the **accessory parameters** $c_j = c_j(a_k)$ is a classic unsolved problem

- Next consider auxiliary Fuchsian equation $\boxed{(\partial_w^2 + \mathcal{T}(w))\psi = 0}$
- Coordinate in the cover $W(w) = \psi_1(w)/\psi_2(w)$ and

$$\boxed{\mathcal{T}(w) = \frac{1}{2}\{W(w), w\}}$$

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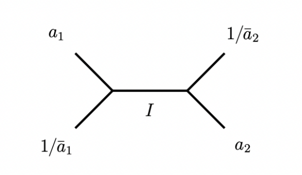
Accessory parameters c_j

• We need the c_j to compute the gravitational action (the area term) but this is unsolved ... ?

Polyakov's conjecture

The accessory parameters are related to the classical limit of a particular conformal block $\mathcal{F}$ of quantum Liouville theory with dimensions $h = \varepsilon c/6$.



Classical limit $f(a_j) = \lim_{c\to\infty} -\frac{6}{c}\log\mathcal{F}(a_j)$ then

$$\boxed{c_j = -\partial_{a_j} f(a_j)}$$

• So c_j can be computed efficiently as a series in $q = e^{-\pi K(1-x)/K(x)}$ which is small when $x \to 0$.

Is that sufficient? ...

Finally ...

1. We can compute the gravitational action in certain limits (late time) corresponding to the $x \to 0$ limit of the conformal block.
2. We can confirm the Dong-inspired guess for a **modular entropy** island type formula $\partial I = \{a_j\}$

$$\widetilde{S}^{(n)}(\rho_R^{micro}) \sim \sum_j \frac{\phi(a_j)}{4G} + \sum_{jk} \xi_{jk} + \widetilde{S}^{(n)}(\rho_{R\cup I}^{sc})$$

   up to "interaction" terms ξ_{jk} between pairs of QES which are suppressed in the late time limit (and other subleading pieces)
3. This analysis was done for the eternal BH and we are now trying to extend to an evaporating BH