Integrable Open Quantum Systems

w. C. Paletta
B. Pozsgay
E. Vernier

Based on: 2101.08279, 2301.01612, 2305.01922 + ...

Integrable models:

- have infinite number of

$$\boxed{\text{commuting, conserved } Q}$$

Often consequence of large symmetry algebra

- special properties:
  - usually solvable (Bethe Ansatz)
  - equilibrate to gen. Gibbs ensemble
  - ⋮

Yang-Baxter integrable:

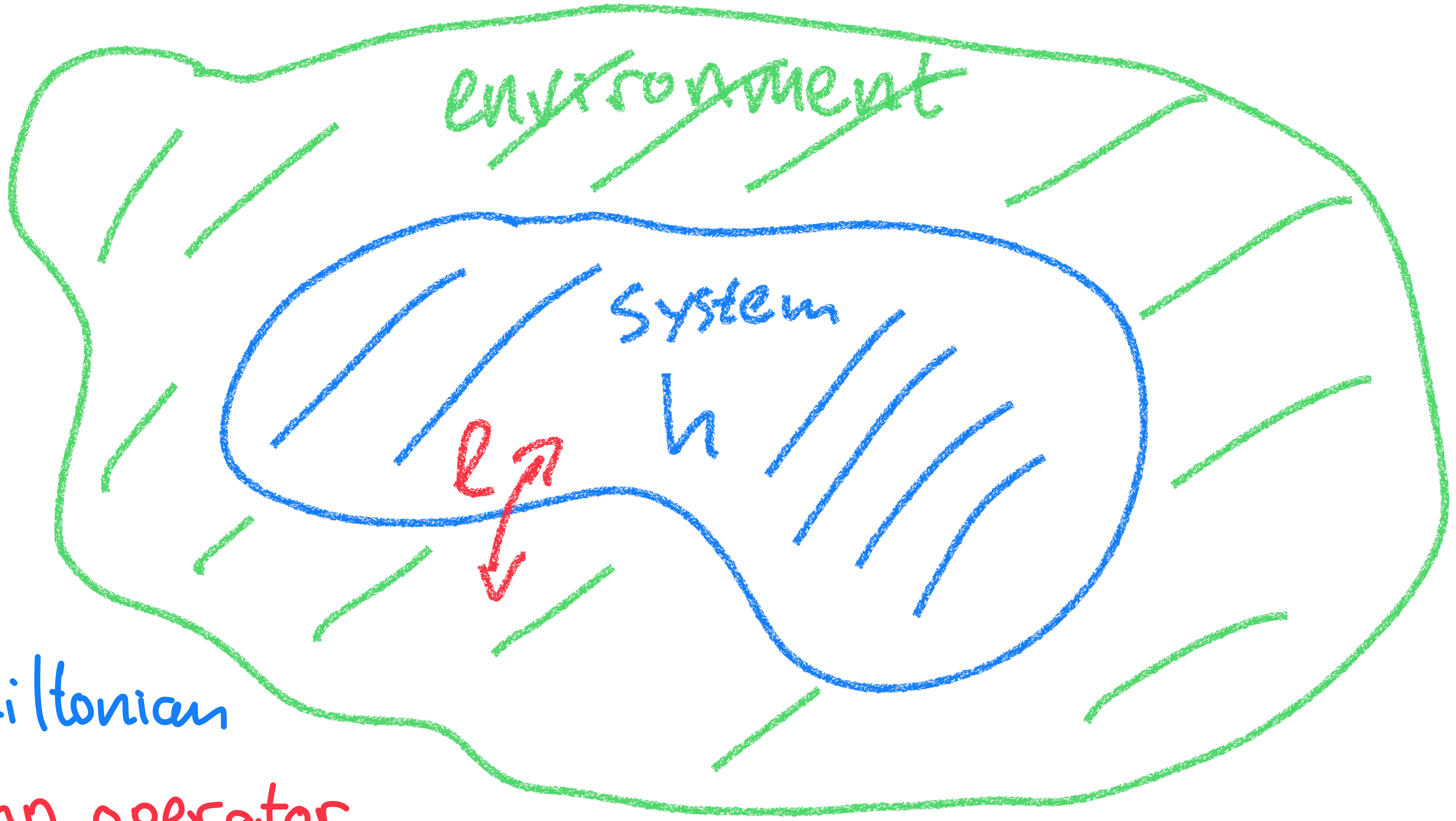
- solution of the YBE

$$R_{12} \, R_{13} \, R_{23} = R_{23} \, R_{13} \, R_{12}$$

General solution $R(u,v)$

- appear in many areas
  - mathematics
  - cond-mat
  - string theory
  - QFT
  - ...
- Closed system
- interest from cond-mat, quantum computing, quantum, circuits

  for interactions with environment



h Hamiltonian

ℓ jump operator

Open quantum systems

- Environment generically breaks integrability

- Open systems are hard to solve

Q:

- are there cases where system remains integrable?
- how many?
- properties (fine-tuning?)

Markovian approximation:

$\Rightarrow$ density matrix $\rho = |\rangle\langle|$

evolves via Lindblad eqn:

$$\dot{\rho} = \mathcal{L}\rho \equiv i[\rho, h] + \sum_a \underbrace{\left[\ell_a \rho \ell_a^\dagger - \tfrac{1}{2}\{\ell_a^\dagger \ell_a, \rho\}\right]}_{\text{dissipator}}$$

hamiltonian

Idea: $\mathcal{L} \sim Q$ conserved charge

- They exist [Medvedyeva Essler Prosen]
  [Ziolkowska, Essler]
  - Hubbard
  - SU(4)

- No real structured approach to finding them

Other approaches

- Free fermions [Prosen, Vernier, Znidaric...]
- Triangular Lindblad operators [Buca, Booker, Medenjak, Jaksch...]
- Boundary driving [Prosen, Ilievski, Popov ...]
- Integrable subspaces [Essler, Piroli ...]

Boost approach

Classifying integrable models

Bottom up approach : [MdL et al '17-'23]

$$\mathcal{H} \xrightarrow{\text{boost}} \{Q_i(\mathcal{H})\} \xrightarrow[\text{Sutherland}]{\{Q_i, Q_j\}=0} R\text{-matrix}$$

Hamiltonian

conserved charges

solution of Yang-Baxter equation

Method is fast, efficient and complete

Complete classification of: - 4×4 [tomorrow]

- 15 vertex (9×9)

- $SU(2) \times SU(2)$ (16×16)

Found many new and interesting models.

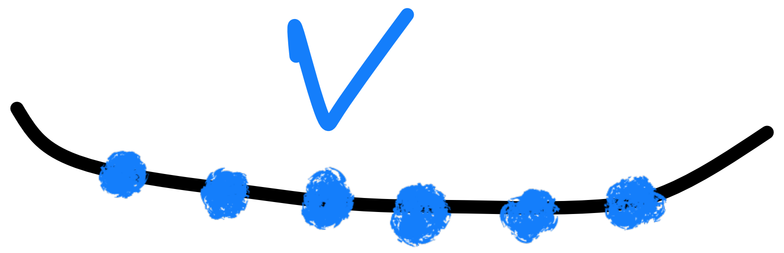
$\Rightarrow$ Apply to Lindblad operator

Apply our method to Lindbladians?

$\rho \sim |\rangle\langle| \iff |\rangle|\rangle\rangle$

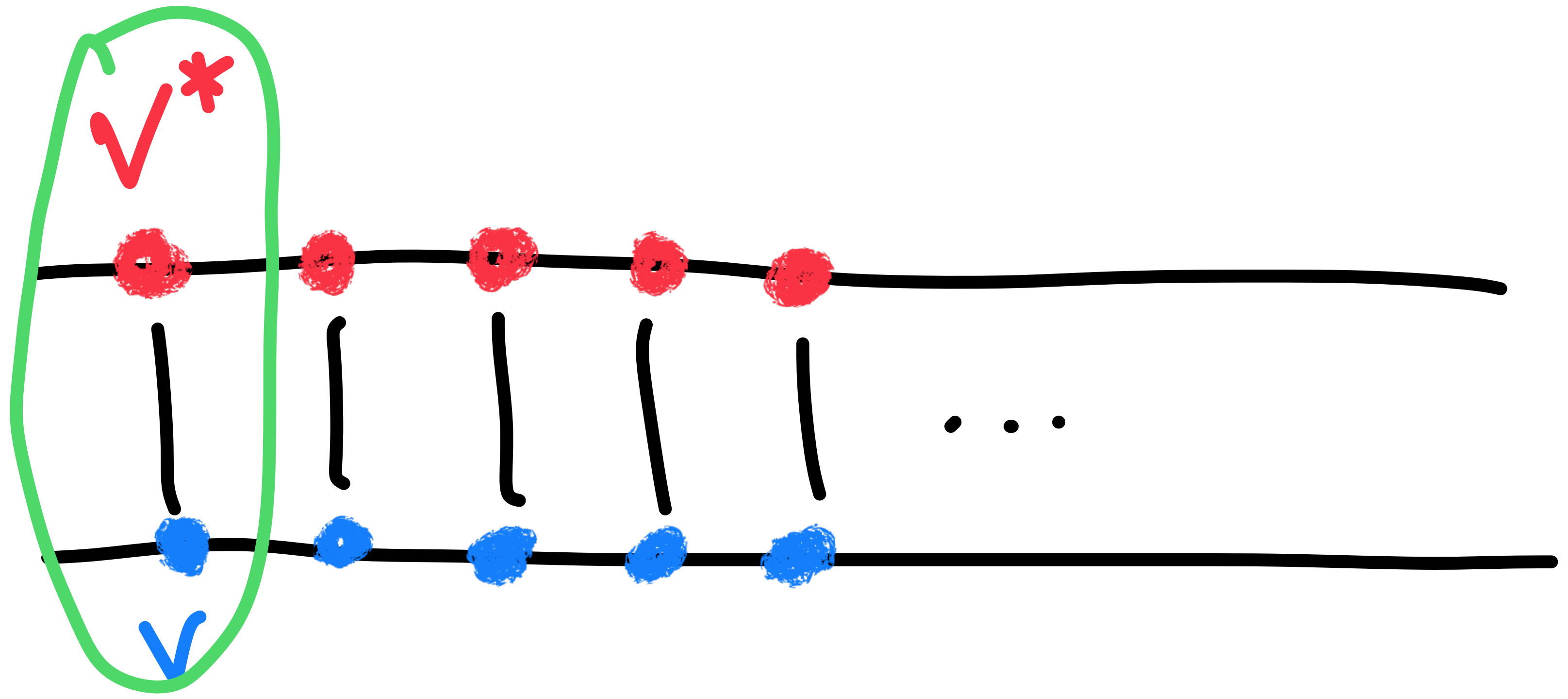
matrix — vector

Local Hilbert space V

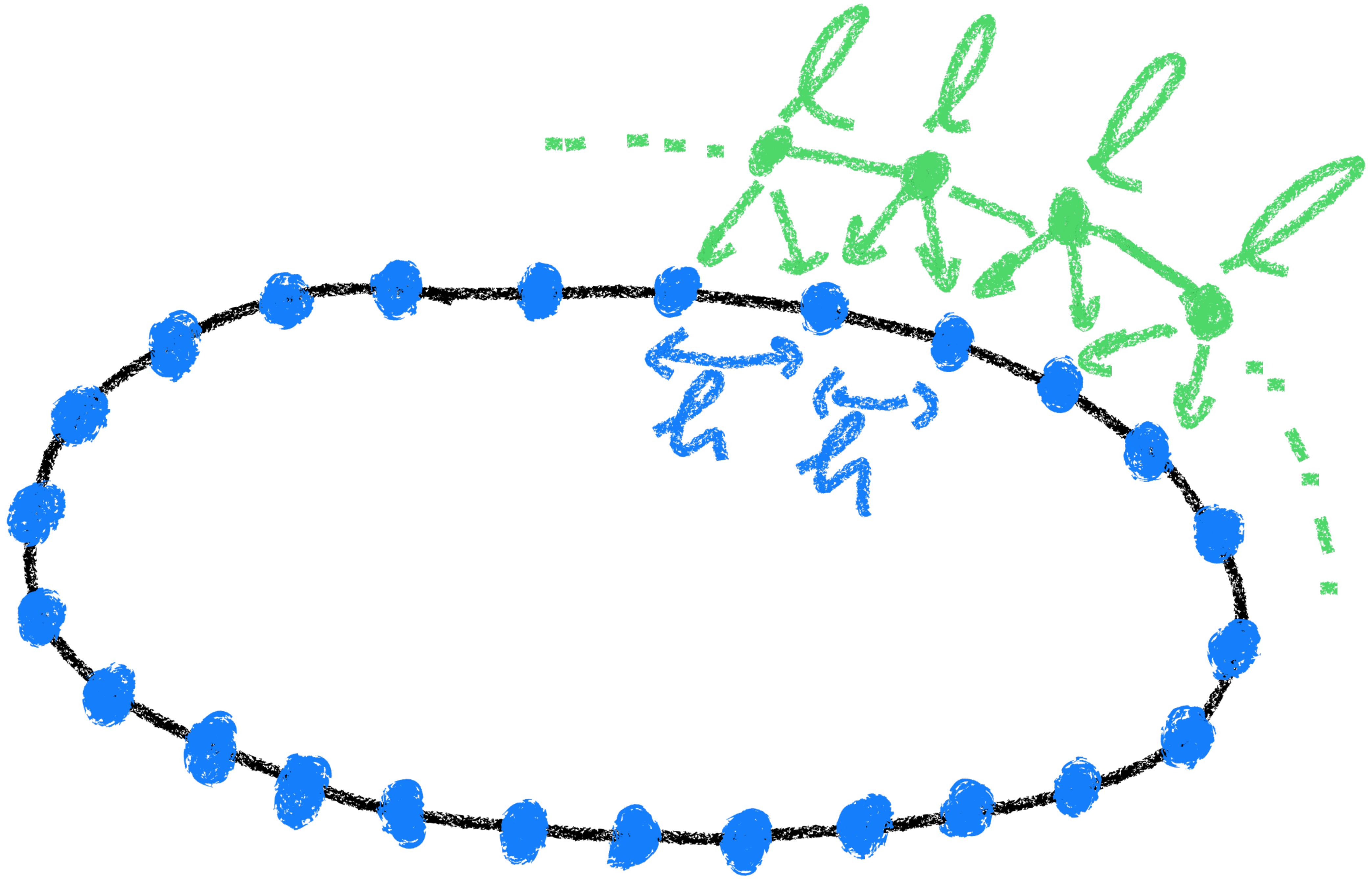


$\Longrightarrow \rho$ takes values in chain with $V \otimes V^*$

Spin ladder



Jump operators act on neighbouring sites



Hamiltonian is NN

For regular spin chain

$$\Longrightarrow \quad \mathcal{H} = \sum_i \mathcal{h}_{i,i+1}$$

$$\boxed{\mathcal{h}_{ij} = -i h^{(1)}_{ij} + i h^{(2)*}_{ij} + \ell^{(1)}_{ij} \ell^{(2)}_{ij} - \frac{1}{2} \ell^{(1)+}_{ij} \ell^{(1)}_{ij} - \frac{1}{2} \ell^{(2)T}_{ij} \ell^{(2)*}_{ij}}$$

We look for h, ℓ s.t. this is integrable.

Consider spin ladder system

$$\dot{\rho} = \mathcal{L}\rho$$

<u>Idea</u> $\mathcal{L} \sim \mathcal{H}$ integrable [Medvedyeva, Eßler, Prosen 16]

Example Hubbard with imaginary coupling

$\longrightarrow$ XX chain with dephasing.

$$\mathcal{H} = \sum_i \overset{h}{c^\dagger_\uparrow \otimes c_\uparrow} + \overset{h^*}{c^\dagger_\downarrow \otimes c_\downarrow} + u\, \overset{\ell}{n_\uparrow}\, \overset{\ell^*}{n_\downarrow}$$

Example

$$h = h_1\, \sigma^+ \otimes \sigma^- + h_2\, \sigma^- \otimes \sigma^+$$

$$\ell = \sigma^- \otimes \sigma^+ + \lambda_{1/4} (1-\sigma^z) \otimes (1-\sigma^z)$$

boost $\Rightarrow$ $\lambda + 2ih_1 = 0 \qquad \lambda^* - 2ih_2 = 0 \qquad |\lambda|^2 = 1$

Integrable and new

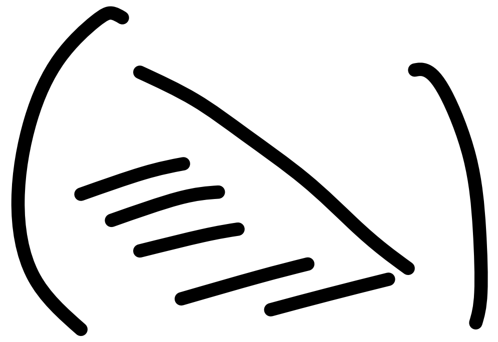
Not very interesting $\rightarrow$ no coupling

$\Uparrow$

fine tuned

Integrable Lindblad

Classification

Type A : $h = 6V$ $\ell =$ 

2 new models, both fine-tuned

Type B : $h = 6V$ $\ell = 6V$

- Hubbard model
- Generalised SU(4)

- new model

[Ziolkowska, Eßler '19]

Type C

- $h = XX$

$$\ell = u \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$$

(ℓ is integrable h in itself)

- No manifest symmetries, except at $\theta = 0, \pi/2$

R-matrix

$$R = \begin{pmatrix}
r_8 & 0 & 0 & 0 & 0 & -r_{12} & 0 & 0 & 0 & 0 & -r_{12} & 0 & 0 & 0 & 0 & r_1 \\
0 & r_6 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 & r_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & r_6 & 0 & 0 & 0 & 0 & r_{11} & r_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_2 & 0 & 0 & -r_9 & 0 & 0 & -r_9 & 0 & 0 & r_4 & 0 & 0 & 0 \\
0 & r_7 & 0 & 0 & -r_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{11} & 0 \\
-r_{12} & 0 & 0 & 0 & 0 & r_{10} & 0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & r_{12} \\
0 & 0 & 0 & -r_9 & 0 & 0 & r_3 & 0 & 0 & r_4 & 0 & 0 & r_9 & 0 & 0 & 0 \\
0 & 0 & r_{11} & 0 & 0 & 0 & 0 & r_5 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 \\
0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & -r_5 & 0 & 0 & 0 & 0 & r_{11} & 0 & 0 \\
0 & 0 & 0 & -r_9 & 0 & 0 & r_4 & 0 & 0 & r_3 & 0 & 0 & r_9 & 0 & 0 & 0 \\
-r_{12} & 0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & r_{10} & 0 & 0 & 0 & 0 & r_{12} \\
0 & r_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_5 & 0 & 0 & r_7 & 0 \\
0 & 0 & 0 & r_4 & 0 & 0 & r_9 & 0 & 0 & r_9 & 0 & 0 & r_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & r_{11} & 0 & 0 & 0 & 0 & -r_6 & 0 & 0 \\
0 & 0 & 0 & 0 & r_{11} & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & -r_6 & 0 \\
r_1 & 0 & 0 & 0 & 0 & r_{12} & 0 & 0 & 0 & 0 & r_{12} & 0 & 0 & 0 & 0 & r_8
\end{pmatrix},$$

$$r_1 = -\frac{2ikg_{u,v}}{\mathrm{dn}_u + \mathrm{dn}_v}, \qquad r_4 = f_{u,v}, \qquad r_7 = 1,$$

$$i\,r_9 = \frac{\mathrm{cn}_u - \mathrm{cn}_v}{\mathrm{sn}_u + \mathrm{sn}_v}, \qquad i\,k\,r_{11} = \frac{\mathrm{dn}_u - \mathrm{dn}_v}{\mathrm{sn}_u + \mathrm{sn}_v}, \qquad \frac{r_{12}}{k} = \frac{\mathrm{cn}_u - \mathrm{cn}_v}{\mathrm{dn}_u + \mathrm{dn}_v},$$

$$r_5 + r_6 = -2ig_{u,v}, \qquad k\,(r_5 - r_6) = (\mathrm{dn}_v - \mathrm{dn}_u)f_{u,v},$$

$$r_3 + r_{10} + r_8 - r_2 = 2f_{u,v}, \qquad r_3 + r_{10} + r_2 - r_8 = \frac{4\,i\,k(\mathrm{cn}_u - \mathrm{cn}_v)}{(\mathrm{dn}_u + \mathrm{dn}_v)(\mathrm{sn}_u + \mathrm{sn}_v)f_{u,v}},$$

$$\frac{r_{10}(\mathrm{dn}_u + i\,k\,\mathrm{cn}_u\mathrm{sn}_u) + r_8(\mathrm{dn}_u - i\,k\,\mathrm{cn}_u\mathrm{sn}_u)}{r_4} = \mathrm{sn}_u^2(\mathrm{dn}_v - \mathrm{dn}_u) + \frac{2\mathrm{dn}_u}{f_{u,v}^2} + \frac{2k^2\mathrm{sn}_u(\mathrm{cn}_u - \mathrm{cn}_u)g_{u,v}}{(\mathrm{dn}_u + \mathrm{dn}_v)f_{u,v}},$$

$$\frac{r_3(\mathrm{dn}_u + i\,k\,\mathrm{cn}_u\mathrm{sn}_u) + r_2(\mathrm{dn}_u - i\,k\,\mathrm{cn}_u\mathrm{sn}_u)}{r_4} =$$
$$r_{12}\left(\frac{4\mathrm{dn}_u}{(\mathrm{sn}_u + \mathrm{sn}_v)f_{u,v}^2} + \mathrm{sn}_u\,(\mathrm{dn}_v - \mathrm{dn}_u)\right) + \frac{f_{u,v}g_{u,v}}{k}\left(\frac{2k^2\mathrm{sn}_u^2}{f_{u,v}^2} - \mathrm{dn}_u\mathrm{dn}_v + \mathrm{dn}_u^2\right),$$

Not quite elliptic:

$$g_{u,v} = \sin\left(\frac{1}{2}(\mathrm{Am}_u - \mathrm{Am}_v)\right), \qquad f_{u,v} = \sec\left(\frac{1}{2}(\mathrm{Am}_u - \mathrm{Am}_v)\right).$$

Close to Hubbard model

- related to known model [Murakami '98]

$\rightsquigarrow$ contains Hubbard after limit
and "bond-site" transformation

Bond-site transformation:

- map products of Pauli matrices to products of Pauli matrices preserving operator algebra

[Jones, Linden '22]

$$Z_j \longrightarrow X_{j-1} X_j$$

$$X_j \longrightarrow \prod_{k=1}^{j} Z_k$$

(Y via X, Z)

- changes range, preserves integrability

Bond-site + basis trans:

$$H_3 = h^{xx} + h^{*xx} + \frac{u}{4} \ell_{i\,i+1\,i+2} \ell^{*}_{i\,i+1\,i+2}$$

$$\ell = Z_{j+1} + \gamma (X_j + X_{j+2}) X_{j+1} - \gamma^2 X_j Z_{j+1} X_{j+2}$$

- $\gamma = 0 \Rightarrow$ Hubbard
- Exactly integrable
- There is no range 2 operator commuting with H_3
- No spin conservation

H_3 follows from R-matrix and Lax operator

[Pozsgay, Gombor '22
MdL, Retore '22]

Map of conserved charges via MPS

$$Q_\gamma = T_\gamma^\dagger Q_0 T \qquad (T^\dagger T = 1 + \gamma^L \pi Z)$$

$$T_\gamma = \mathrm{tr}_a M_{a L} \cdots M_{a 1}$$

$$\rightsquigarrow [T_\gamma, T_{\gamma'}] = 0$$

Why is this interesting:

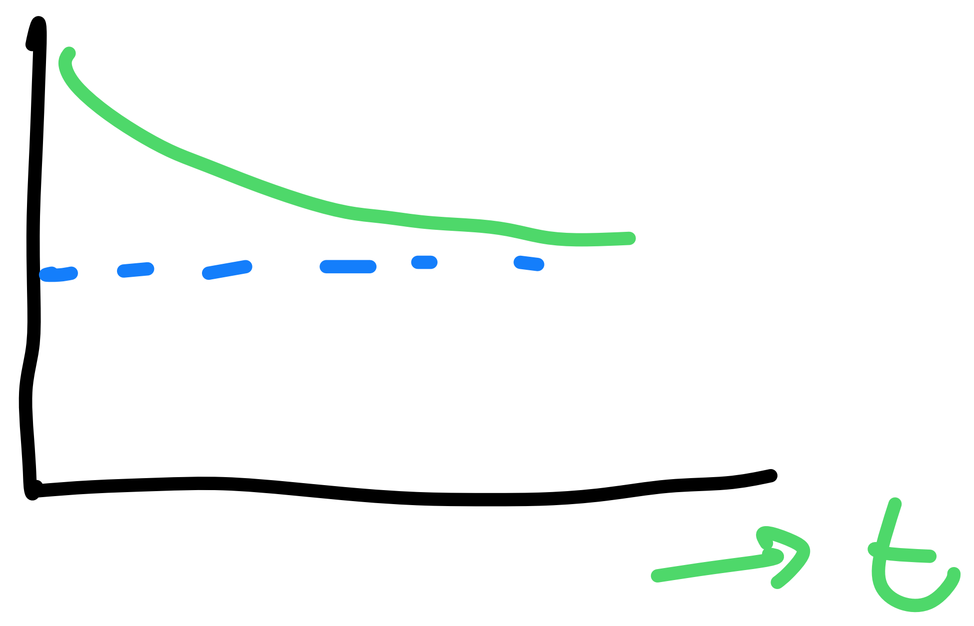
- Degenerate NESS

  $\hookrightarrow$ $\mathcal{L}(\rho_{NESS}) = 0$ (ground state)

  But no clear symmetry

- States thermalize to NESS

  $\rho^{NESS}_{Hubb} = 1$ $L \to \infty$

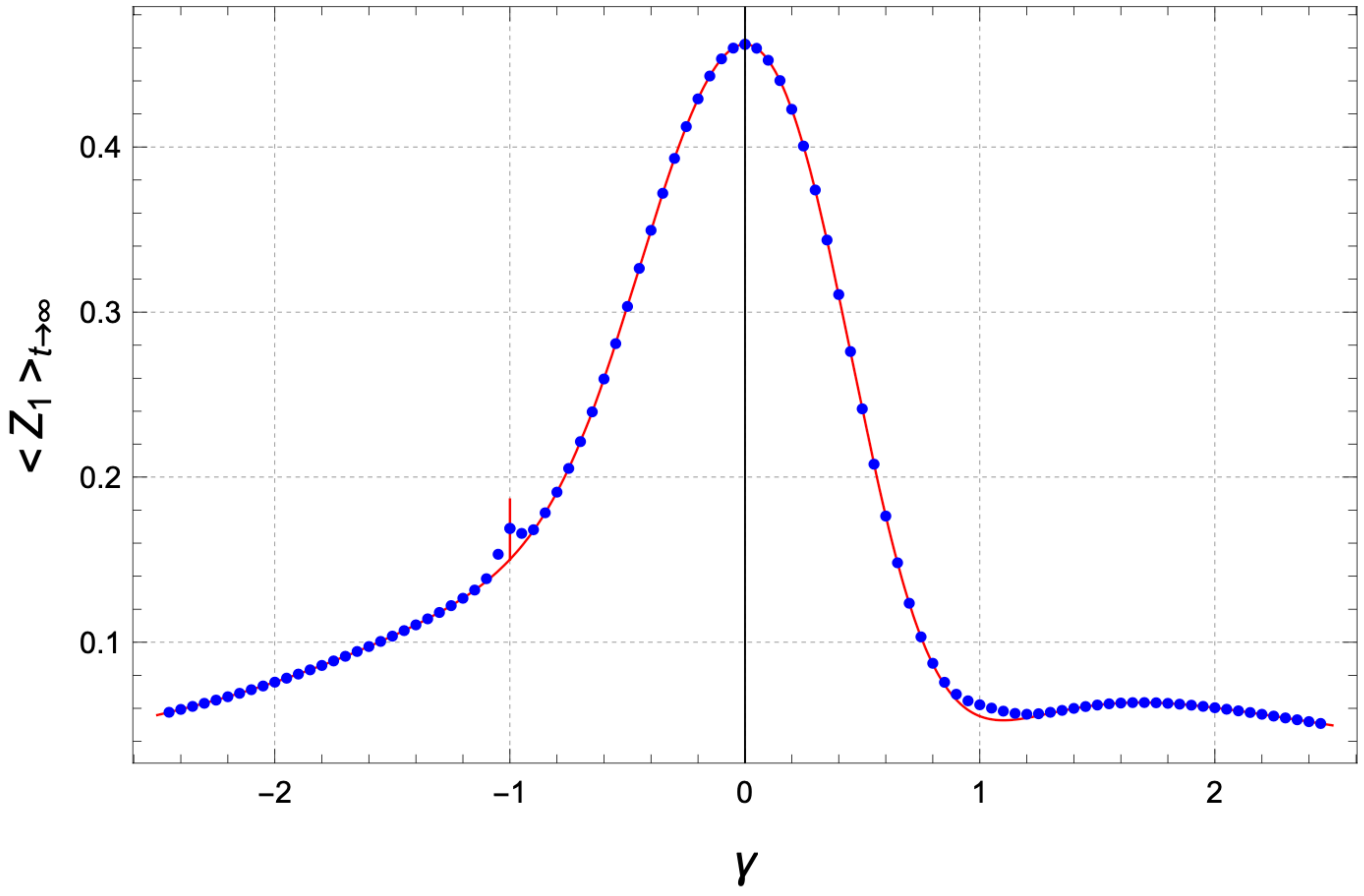


- State remembers initial state

  $L+1$ NESS (more at $\gamma = 1$)

Can compute some overlaps & expectation values:

$$\mathcal{P} = e^{\beta Z}$$



$$\lim_{t\to\infty} \langle Z_j \rangle = \frac{(\gamma^2-1)^2 \tanh\beta (1 - 2\gamma^L \tanh^{L-2}\beta + \gamma^{2L})}{(1-\gamma^{2L})^2}.$$

Interesting because:

Integrability techniques to study an exotic Lindbladian.

Duality:

$$\gamma \overset{\sim}{\longleftrightarrow} 1/\gamma$$

Conclusions

- Find integrable Lindbladians
- New medium range def of Hubbard
- interesting properties

Open questions:

- Bethe Ansatz
  - spectrum
  - gap?
- Physical properties
- Long range integrable models

Thank

you