Scaling behaviour of spin chains related to the inhomogeneous 6V model

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Introduction

Integrable lattice models possess important applications
to the study of critical phenomena and QFT

- They provide a laboratory for testing and developing our understanding of concepts such as renormalization group flow, universality, marginal deformations, ...
- May exhibit interesting phenomena (large degeneracies, exotic symmetries), which force us to refine our understanding of the scaling limit and QFT

Basic example: Heisenberg XXZ spin chain

$$\mathbb{H}_{XXZ} = -\sum_{m=1}^{N} J_x\big(\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y\big) + J_z\sigma_m^z\,\sigma_{m+1}^z$$

σ_m^a – Pauli matrices acting on m-th site of lattice

Lattice system critical in disordered regime:

$$\big|J_z/J_x\big| < 1\ :\qquad J_z/J_x = \cos(\gamma)\qquad \text{with}\qquad \gamma\in(0,\pi]$$

- Periodic/twisted BCs: scaling limit governed by free compact boson of radius $\sqrt{\frac{2\gamma}{\pi}}$ [Luther, Peschel '75; Kadanoff, Brown '79; Alcaraz, Barber, Batchelor '87]
- Other BCs, e.g., anti-diagonal: (sector of) $\mathbb{S}^1/\,\mathbb{Z}_2$ orbifold theory [Alcaraz, Baake, Grimm, Rittenberg '87]

Integrable spin chain with continuous spectrum

$$\mathbb{H} = \frac{1}{\sin(2\gamma)} \sum_{m=1}^{N} \Big(2\sin^2(\gamma)\, \sigma_m^z \sigma_{m+1}^z - (\sigma_m^x \sigma_{m+2}^x + \sigma_m^y \sigma_{m+2}^y + \sigma_m^z \sigma_{m+2}^z) + \mathrm{i}\, (-1)^m \sin(\gamma)(\sigma_m^x \sigma_{m+1}^y - \sigma_m^y \sigma_{m+1}^x)(\sigma_{m-1}^z - \sigma_{m+2}^z) \Big)$$

Hamiltonian is not Hermitian!

Two regimes of critical behaviour

$$\text{Regime I}: \ \gamma \in (0, \tfrac{\pi}{2}) \qquad \qquad \text{Regime II}: \ \gamma \in (\tfrac{\pi}{2}, \pi)$$

Continuous spectrum of conformal dimensions observed in Regime I [Jacobsen, Saleur '06]

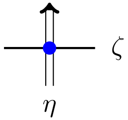
Scaling limit governed by 2D black hole sigma models [Ikhlef, Jacobsen, Saleur '11; Bazhanov, GK, Koval, Lukyanov '21]

Heisenberg XXZ spin - $\frac{1}{2}$ chain	$\longrightarrow$	Compact Gaussian field
	Scaling limit with $N \to \infty$	
Spin chain with Hamiltonian $\mathbb{H}$	$\longrightarrow$	2D black hole sigma models

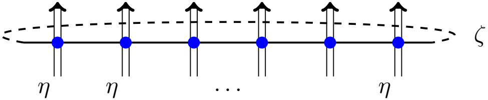
Both spin chains obtained from special cases
of transfer-matrix of inhomogeneous 6V model

Homogeneous 6V model

Trigonometric R - matrix:

$R(\zeta/\eta\,|\,q) =$ 

Row-to-row transfer-matrix ($\eta = 1$):

$\mathbb{T}(\zeta) =$ 

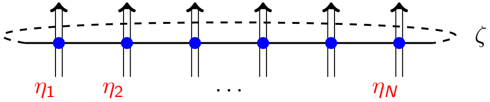
Yang-Baxter equation for $R(\zeta) \implies$

$$\big[\mathbb{T}(\zeta), \mathbb{T}(\zeta')\big] = 0\,, \qquad \mathbb{T}(\zeta) = \mathbb{T}(\zeta\,|\,q\,)$$

XXZ spin chain Hamiltonian:

$$\mathbb{H}_{XXZ} = 2\mathrm{i}\partial_\zeta \log\big(\mathbb{T}(\zeta)\big)\Big|_{\zeta=1} + \mathrm{const} \qquad \text{with} \qquad q = \mathrm{e}^{\mathrm{i}\gamma}$$

Inhomogeneous 6V model [Baxter '71]

$\mathbb{T}(\zeta) =$ 

η_J - 'inhomogeneities'

Yang-Baxter equation for $R(\zeta) \implies$

$$\big[\mathbb{T}(\zeta), \mathbb{T}(\zeta')\big] = 0\,, \qquad\qquad \mathbb{T}(\zeta) = \mathbb{T}(\zeta\,|\,q\,, \eta_1, \eta_2, \ldots, \eta_N)$$

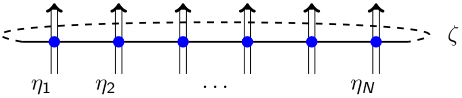
Hamiltonian for spin chain with non-compact spectrum:

$$\mathbb{H} = 2\mathrm{i} \sum_{\ell=1,2} \partial_\zeta \log\big(\mathbb{T}(\zeta)\big)\Big|_{\zeta=\eta_\ell} + \text{const}$$

with

$$\eta_J = \mathrm{i}\,(-1)^{J-1} \qquad \text{and} \qquad q = \mathrm{e}^{\mathrm{i}\gamma}$$

Inhomogeneous 6V model

$\mathbb{T}(\zeta) =$ 

Multi-parametric integrable system depending on $\{\eta_J\}_{J=1}^{N}$ and q

Framework of Yang-Baxter integrability allows for:

- Changing irreps. in each factor of quantum space:

$$\mathbb{T}(\zeta)\ :\ V_N \mapsto V_N \qquad \text{with} \qquad V_N = \mathbb{C}_1^{2j_1+1} \otimes \mathbb{C}_2^{2j_2+1} \otimes \ldots \otimes \mathbb{C}_N^{2j_N+1}$$

- Imposing different families of open BCs [Sklyanin '88] (in this talk we focus on quasi-periodic BCs)

Study of critical behaviour of inhomogeneous 6V model, including identification of critical surfaces and description of universality classes has been mainly unexplored

My research

1.) Developing methods for study of scaling limit of inhomogeneous 6V model based on:

- Bethe ansatz solution of model [Baxter '71]
- Baxter Q operator [Baxter '72] (open BCs [Frassek, Szecsenyi '15; Baseilhac, Tsuboi '17; Vlaar, Weston '20; Tsuboi '20])
- ODE/IQFT correspondence [Voros'92; Dorey-Tateo'98; BLZ'98,03]
- Integrable structures of CFT [BLZ '94,'96,'98]

2.) Applications to study of critical phenomena, e.g.,

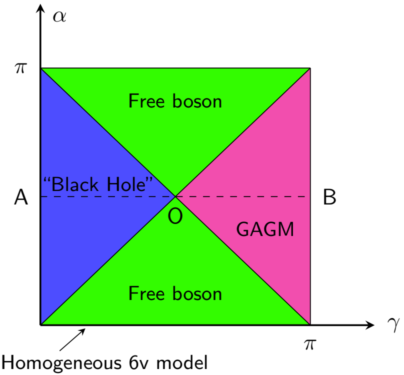
results for density of states of Euclidean black hole CFT from analysis of spin chain with continuous spectrum [Bazhanov, GK, Lukyanov '20]

Previously studied cases

'Staggered' inhomogeneous 6V model

$$\eta_{2J} = \mathrm{e}^{\mathrm{i}\alpha}\,, \qquad \eta_{2J-1} = \mathrm{e}^{-\mathrm{i}\alpha} \quad \text{and} \quad q = \mathrm{e}^{\mathrm{i}\gamma}$$

with $\alpha, \gamma \in [0, \pi)$



- Line AO: [(Ikhlef), Jacobsen, Saluer '05; '06,'11; Frahm, Martins'12; Candu, Ikhlef'13; Bazhanov, GK, Koval, Lukyanov '19,'20]

  Whole BH region [Frahm, Seel'13]

- Line OB: [Ikhlef, Jacobsen, Saluer'09]

  Whole GAGM region [GK, Lukyanov'21] (compact boson + 2 Majorana fermions)

Model with r-site translational invariance

$$\eta_{J+r} = \eta_J \qquad\qquad (r \text{ divides } N)$$

Hamiltonian:

$$\mathbb{H} = 2\mathrm{i} \sum_{\ell=1}^{r} \partial_\zeta \log\big(\mathbb{T}(\zeta)\big)\Big|_{\zeta=\eta_\ell} + \text{const}$$

Special cases:

- $r = 1$ – homogeneous 6V model (XXZ spin chain)
- $r = 2$ – staggered 6V model (spin chain with non-compact spectrum)

General r: different types of universal behaviour depending on γ:

$\mathcal{Z}_r$ invariant spin chain

Red region with

$$\eta_J \approx (-1)^r \, e^{\frac{i\pi}{r}\,(2J-1)} \qquad\qquad \pi\,(1-\tfrac{1}{r}) < \gamma < \pi$$

falls under conjecture for scaling limit from [GK, Lukyanov '21]

This talk: blue region

$$0 < \gamma < \frac{\pi}{r} \qquad \Longleftrightarrow \qquad \frac{\pi}{n+r} \qquad \text{with} \qquad n > 0$$

Impose

$$\eta_J = (-1)^r \, e^{\frac{i\pi}{r}\,(2J-1)}$$

model possesses additional $\mathcal{Z}_r$ symmetry:

$$\big[\hat{\mathcal{D}}, \mathbb{H}\big] = 0\,, \qquad\qquad \hat{\mathcal{D}}^r = 1$$

Study of $\mathcal{Z}_r$ invariant spin chain shows [GK, Lukyanov '23]

- continuous component in spectrum for even r
- infinite degeneracy of conformal primary states in scaling limit

Plan

As $N \to \infty$ low energy states of $1D$ critical spin chain organize into conformal towers

$$|\Psi_N\rangle \mapsto |\psi\rangle \otimes |\bar{\psi}\rangle \in \mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_{\bar{\Delta}}$$

Cardy formula for low energy spectrum [Cardy '86]:

$$\mathcal{E} \asymp N\, e_\infty + \frac{2\pi v_{\mathrm{F}}}{N} \left(\Delta + \bar{\Delta} + \mathtt{L} + \bar{\mathtt{L}} - \frac{c}{12} \right) + o(N^{-1})$$

e_∞, v_{F} - non-universal constants

$\Delta, \bar{\Delta}$ - conformal dimensions

$\mathtt{L}, \bar{\mathtt{L}} = 0, 1, 2, 3, \ldots$ - 'level' of states $|\psi\rangle$, $|\bar{\psi}\rangle$ in conformal tower

To be discussed:

- Low energy spectrum for $\mathcal{Z}_r$ invariant spin chain with $r = 1, 2$
- Low energy spectrum for general r and comments on underlying CFT

The case $r = 1$ (XXZ spin chain)

$$\mathbb{H}_{XXZ} = -\sum_{m=1}^{N} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \cos(\gamma)\, \sigma_m^z\, \sigma_{m+1}^z \right)$$

with

$$\gamma = \frac{\pi}{n+r} \in (0, \pi) \qquad\qquad (r = 1)$$

$\mathrm{U}(1)$ symmetry:

$$\left[\mathbb{H}_{XXZ}, \mathbb{S}^z\right] = 0 \qquad \text{with} \qquad \mathbb{S}^z = \tfrac{1}{2} \sum_{m=1}^{N} \sigma_m^z \ .$$

$\Longrightarrow$ space of states breaks up into sectors labeled by integer $S^z = 0, \pm 1, \pm 2, \ldots$

Quasi-periodic BCs:

$$\sigma_{N+1}^x \pm \mathrm{i}\sigma_{N+1}^y = \mathrm{e}^{\pm 2\pi \mathrm{i} \mathtt{k}} \left(\sigma_1^x \pm \mathrm{i}\sigma_1^y \right), \qquad \sigma_{N+1}^z = \sigma_1^z \qquad \left(\mathtt{k} \in (-\tfrac{1}{2}, \tfrac{1}{2}]\right)$$

Bethe Ansatz solution

Integrability: eigenstates of Hamiltonian labeled
by solutions of algebraic system

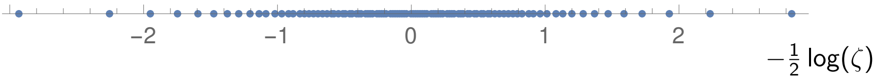
$$\left(\frac{1+q^{+1}\,\zeta_m}{1+q^{-1}\,\zeta_m}\right)^N = -\mathrm{e}^{2\mathrm{i}\pi\mathrm{k}}\,q^{2S^z}\prod_{j=1}^{\frac{N}{2}-S^z}\frac{\zeta_j - q^{+2}\,\zeta_m}{\zeta_j - q^{-2}\,\zeta_m}$$

with

$$\mathcal{E} = \sum_{m=1}^{\frac{N}{2}-S^z}\frac{2\mathrm{i}\left(q-q^{-1}\right)}{\zeta_m+\zeta_m^{-1}+q+q^{-1}}$$

Spectrum at $N \gg 1$ can be studied by finding solutions of BA equations

Ground state: pattern of Bethe roots distributed along positive real axis



Low energy state: pattern of Bethe roots differs from ground state pattern only at edges of distribution

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Spectrum at $N \gg 1$ can be determined via study of solutions of BA equations

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Low energy state: pattern of Bethe roots differs from ground state pattern only at edges of distribution

Low energy excitations

$$\left(\frac{1+q^{+1}\,\zeta_m}{1+q^{-1}\,\zeta_m}\right)^N = -\mathrm{e}^{2\mathrm{i}\pi\mathrm{k}}\,q^{2S^z}\prod_{j=1}^{\frac{N}{2}-S^z}\frac{\zeta_j-q^{+2}\,\zeta_m}{\zeta_j-q^{-2}\,\zeta_m}$$

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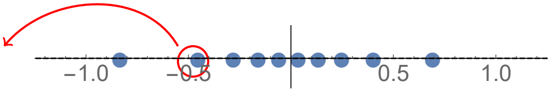


$-\frac{1}{2}\log(\zeta)$

Low energy excitations constructed by:

- S^z: removing Bethe root from distribution
- $(\mathtt{L},\ \bar{\mathtt{L}})$: creating holes at left/right edges of Bethe root distribution

Low energy spectrum

[Luther, Peschel '75; Kadanoff, Brown '79; Alcaraz, Barber, Batchelor '87]:

$$\mathcal{E} = e_\infty N + \frac{2\pi v_{\mathrm{F}}}{N}\left(\frac{p^2+\bar{p}^2}{n+r} - \frac{1}{12} + \mathtt{L} + \bar{\mathtt{L}}\right) + O\big(N^{-3}, N^{-4n-1}\big)$$

$$p = \frac{1}{2}\left(S^z + \sqrt{n+r}\,(\mathtt{k}+\mathtt{w})\right)$$

$$\bar{p} = \frac{1}{2}\left(S^z - \sqrt{n+r}\,(\mathtt{k}+\mathtt{w})\right)$$

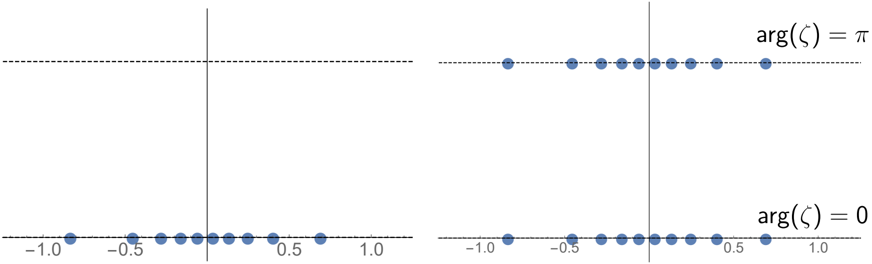
- $S^z = 0, \pm 1, \pm 2, \ldots$ – $\mathrm{U}(1)$ charge
- $\mathtt{w} = 0, \pm 1, \pm 2, \ldots$ 'winding number'

Scaling limit governed by free compact boson

Case $r = 2$ (spin chain with continuous spectrum)

$$\left(\frac{1+q^{+r}\ \zeta_m^r}{1+q^{-r}\ \zeta_m^r}\right)^{N/r} = -e^{2i\pi k}\, q^{2S^z} \prod_{j=1}^{\frac{N}{2}-S^z} \frac{\zeta_j - q^{+2}\,\zeta_m}{\zeta_j - q^{-2}\,\zeta_m}$$

Bethe roots in complex $-\frac{1}{2}\log(\zeta)$ plane:

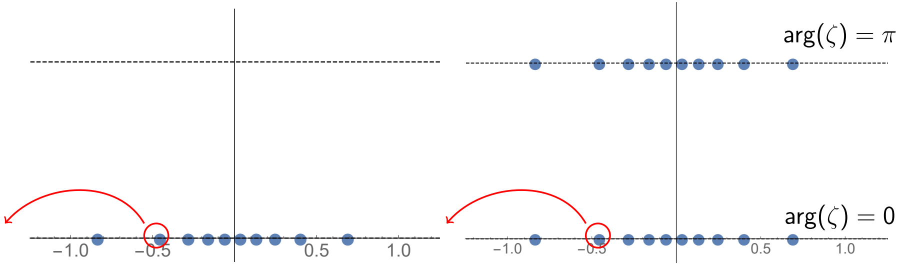


homogeneous ($N = 20$) staggered ($\frac{N}{2} = 20$)

Low lying excitations

- S^z : total number of Bethe roots $= N/2 - S^z$
- $\mathtt{L}, \bar{\mathtt{L}}$: holes at edges of the Bethe roots distribution

Bethe roots in complex $-\frac{1}{2}\log(\zeta)$ plane:

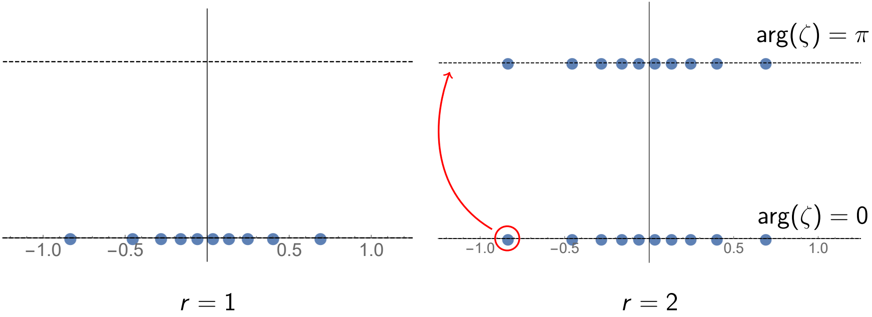


$r = 1$ $r = 2$

Low lying excitations

- S^z : total number of Bethe roots $= N/2 - S^z$
- $\mathtt{L}, \bar{\mathtt{L}}$: holes at edges of the Bethe roots distribution
- $\mathfrak{m}_2 - \mathfrak{m}_1$: difference btw No. of roots with $\arg(\zeta) = \pi$ and 0 ($r = 2$ only)

Bethe roots in complex $-\frac{1}{2}\log(\zeta)$ plane:



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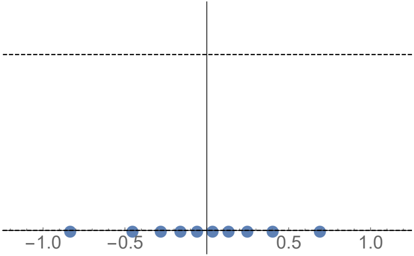
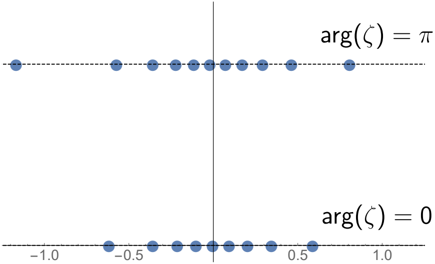


$r = 1$



$r = 2$

Low energy spectrum for case $r = 2$

[(Ikhlef), Jacobsen, Saluer '06'08;11; Frahm, Martins '12; Candu, Ikhlef '13; Frahm, Seel '14]:

$$\mathcal{E} \;=\; e_\infty\, N + \frac{2\pi v_{\mathrm{F}}}{N}\left(\frac{p^2+\bar{p}^2}{n+r} + 2n\, b^2 - \frac{1}{6} + \mathtt{L} + \bar{\mathtt{L}}\right) + O\big(N^{-3}, N^{-2n}\big)$$

$b = b(N)$ related to eigenvalue of "quasi-shift" operators:

$$\mathbb{K}^{(\ell)} = \mathbb{T}(-q^{-1}\,\eta_\ell) \;: \qquad\qquad \mathbb{K}^{(1)}\,\mathbb{K}^{(2)} \propto \text{ 2 site translation operator}$$

$\big(\ell = 1, 2\,;\; \eta_1 = \eta_2^{-1} = \mathrm{i}\big)$

Asymptotics for $\mathcal{E}$ obeyed with [Ikhlef, Jacobsen, Saleur '11]

$$b(N) = \frac{1}{4\pi}\,\log\big(\mathcal{K}^{(1)}/\mathcal{K}^{(2)}\big)\,, \qquad\qquad \mathbb{K}^{(\ell)}|\Psi_N\rangle = \mathcal{K}^{(\ell)}|\Psi_N\rangle$$

Low energy spectrum for $\mathcal{Z}_r$ invariant spin chain with $r = 2$

$$\mathcal{E} \;=\; e_\infty\, N + \frac{2\pi v_{\rm F}}{N}\,\left(\frac{p^2+\bar{p}^2}{n+r} + 2n\, b^2 - \frac{1}{6} + \mathtt{L} + \bar{\mathtt{L}}\right) + O\big(N^{-3}, N^{-2n}\big)$$

For class of low energy states labeled by $\mathfrak{m}_2 - \mathfrak{m}_1$:

$$b(N) \asymp \frac{\pi\,(\mathfrak{m}_2-\mathfrak{m}_1)}{4\log(N)} + \ldots\,, \qquad \mathfrak{m}_2-\mathfrak{m}_1 = \begin{cases} 0, \pm 2, \pm 4 \ldots & N/2 - S^z \ \text{even} \\ \pm 1, \pm 3, \pm 5 \ldots & N/2 - S^z \ \text{odd} \end{cases}$$

In taking scaling limit one should increase $\mathfrak{m}_2 - \mathfrak{m}_1$ together with N such that limiting value of $b(N)$ is held fixed as $N \to \infty$

Spectrum of conformal dimensions possesses continuous component parameterized by

$$s = \operatorname*{slim}_{N\to\infty} b(N)\,, \qquad\qquad s \in (-\infty, +\infty)$$

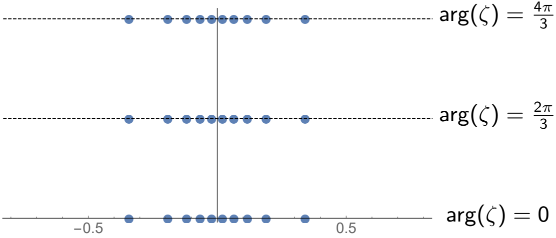
Ground state for general r

For ground state Bethe roots lie on r rays (N/r even):

$$\arg(\zeta) = \frac{2\pi \mathrm{i}}{r}(\ell - 1) \qquad \text{with} \qquad \ell = 1, 2, \ldots r$$

Example: ground state with $r = 3$

Bethe roots in complex $-\frac{1}{2}\log(\zeta)$ plane



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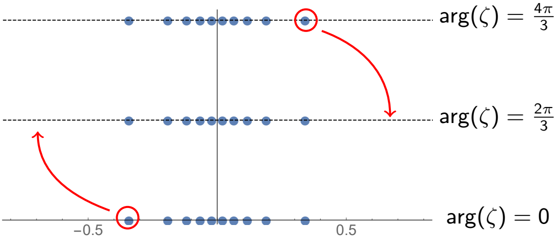

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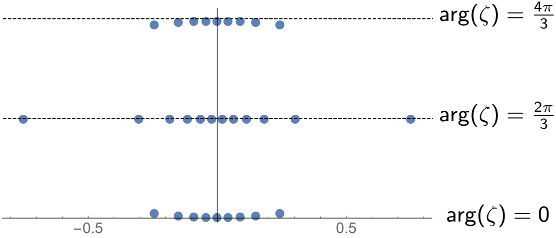

Low energy exciations for general r

- S^z : total number of Bethe roots $= N/2 - S^z$
- $\mathtt{L}, \bar{\mathtt{L}}$: holes at edges of the Bethe roots distribution
- $r-1$ differences $\mathfrak{m}_\ell - \mathfrak{m}_{\ell'}$ btw No. of roots with

$$\arg(\zeta) \approx \tfrac{2\pi \mathrm{i}}{r}\,(\ell-1) \qquad \text{and} \qquad \arg(\zeta) \approx \tfrac{2\pi \mathrm{i}}{r}\,(\ell'-1)$$

Example with $r=3$:

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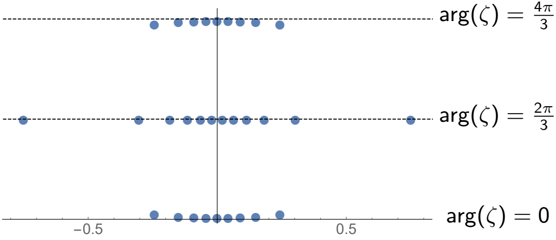

Quasi-shift operators

To take into account extra $r-1$ degrees of freedom:

$$\mathbb{K}^{(\ell)} = \mathbb{T}(-q^{-1}\,\eta_\ell) \qquad\qquad (\ell = 1, 2, \ldots, r)$$

$$\mathbb{K}^{(1)}\mathbb{K}^{(2)}\ldots\mathbb{K}^{(r)} \propto \ \text{r site translation}\,, \qquad \hat{\mathcal{D}}^{-1}\,\mathbb{K}^{(\ell)}\,\hat{\mathcal{D}} = \mathbb{K}^{(\ell+1)}$$

Define $r-1$ independent quantities via Fourier transform

$$b_a \equiv \frac{N^{1-\frac{2|a|}{r}}}{2\pi \mathrm{i} r}\ \sum_{\ell=1}^{r} \mathrm{e}^{\frac{\mathrm{i}\pi}{r}a(r+1-2\ell)}\ \log\left(\mathcal{K}^{(\ell)}\right) \qquad\qquad \left(1 \leq |a| \leq [\tfrac{r}{2}]\right)$$

with $b_{-\frac{r}{2}} = b_{\frac{r}{2}}$

- b_a has $\mathcal{Z}_r$ charge a
- b_a generically tends to finite, non-vanishing number as $N \to \infty$

Low energy spectrum for general r

$\equiv 0$ for r odd

$$\mathcal{E} = N\, e_\infty + \frac{2\pi r v_{\mathrm{F}}}{N}\left[\frac{p^2+\bar{p}^2}{n+r} + 2n\,\left(b_{\frac{r}{2}}\right)^2 - \frac{r}{12} + \mathtt{L} + \bar{\mathtt{L}}\right.$$

$$\left. - \sum_{a=1}^{[\frac{r-1}{2}]} 2\pi\,(r-2a)\,\cot\left(\tfrac{\pi\,(r-2a)}{2n}\right)\,\underbrace{\frac{b_a\, b_{-a}}{N^{2-\frac{4a}{r}}}}_{\mathcal{Z}_r \text{ neutral}} + O\left(N^{-2}, N^{-\frac{4n}{r}}\right)\right]$$

Decays faster than N^{-1}

Scaling limit of low energy states

States appearing in scaling limit of $|\Psi_N\rangle$
labeled by p, $\bar{p}$, $\mathtt{L}$, $\bar{\mathtt{L}}$ as well as

$$s_a = C_a \operatorname*{slim}_{N\to\infty} b_a \qquad\qquad 1 \leq |a| \leq [\tfrac{r}{2}]$$

(C_a inessential and chosen for convenience)

Conformal dimensions:

$$\begin{aligned} \Delta - \frac{c}{24} &= \frac{p^2}{n+r} + n\left(s_{\frac{r}{2}}/C_{\frac{r}{2}}\right)^2 - \frac{r}{24} + \mathtt{L} \\ \bar{\Delta} - \frac{c}{24} &= \frac{\bar{p}^2}{n+r} + n\left(s_{\frac{r}{2}}/C_{\frac{r}{2}}\right)^2 - \frac{r}{24} + \bar{\mathtt{L}} \end{aligned}$$

independent of s_a for $|a| = 1, 2, \ldots [\frac{r-1}{2}]$

$\Longrightarrow$ large (infinite) number of conformal towers $\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_{\bar{\Delta}}$ with same pair of conformal dimensions

Scaling limit of low energy states

Our analysis suggests

$$\big|\Psi_N\big\rangle \mapsto \big|\psi^{(\mathtt{L})}_{p,\boldsymbol{s}}\big\rangle \otimes \big|\bar{\psi}^{(\bar{\mathtt{L}})}_{\bar{p},\bar{\boldsymbol{s}}}\big\rangle$$

with

$$\boldsymbol{s} = \big(s_1, \ldots, s_{[\frac{r}{2}]}\big)\,, \qquad \bar{\boldsymbol{s}} = \big(s_{-1}, \ldots, s_{-[\frac{r}{2}]}\big)$$

It is expected that chiral states organize into irreps of algebra of extended conformal symmetry

Open questions:

- What is the algebra of extended conformal symmetry?
- What are conditions on $\boldsymbol{s}$, $\boldsymbol{s}'$ such that $\big|\psi^{(\mathtt{L})}_{p,\boldsymbol{s}}\big\rangle$, $\big|\psi^{(\mathtt{L}')}_{p,\boldsymbol{s}'}\big\rangle$ belong to same irrep?
- What are selection rules for admissible values of $\boldsymbol{s}$ and $\bar{\boldsymbol{s}}$?

Scaling limit of low energy states

Our analysis suggests

$$\big|\Psi_N\big\rangle \mapsto \big|\psi^{(\mathtt{L})}_{p,\boldsymbol{s}}\big\rangle \otimes \big|\bar{\psi}^{(\bar{\mathtt{L}})}_{\bar{p},\bar{\boldsymbol{s}}}\big\rangle$$

with

$$\boldsymbol{s} = \big(s_1, \ldots, s_{[\frac{r}{2}]}\big)\,, \qquad \bar{\boldsymbol{s}} = \big(s_{-1}, \ldots, s_{-[\frac{r}{2}]}\big)$$

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Quantization condition ($r = 2$)

Chiral states labeled by $(p, \bar{p}, \mathtt{L}, \bar{\mathtt{L}})$ and $s \equiv s_1 = \bar{s}_1$

$$|\Psi_N\rangle \mapsto \big|\psi^{(\mathtt{L})}_{p,s}\big\rangle \otimes \big|\bar{\psi}^{(\bar{\mathtt{L}})}_{\bar{p},\bar{s}}\big\rangle$$

'Quantization Condition' (QC) is satisfied

$$\left(\frac{N}{2\tilde{N}_0}\right)^{2\mathrm{i}s} \mathrm{e}^{\frac{\mathrm{i}}{2}\delta(s)} = \sigma + O\big((\log N)^{-\infty}\big)$$

$\sigma =$ sign factor, $\tilde{N}_0 = \mathrm{const.}$

Originally obtained for $\mathtt{L} = \bar{\mathtt{L}} = 0$ in [Ikhlef, Jacobsen, Saleur '11] with (note that s from that work $= -\frac{s}{2}$)

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\delta} = \frac{\Gamma\big(\frac{1}{2} + p - \frac{\mathrm{i}s}{2}\big)}{\Gamma\big(\frac{1}{2} + p + \frac{\mathrm{i}s}{2}\big)} \frac{\Gamma\big(\frac{1}{2} + \bar{p} - \frac{\mathrm{i}s}{2}\big)}{\Gamma\big(\frac{1}{2} + \bar{p} + \frac{\mathrm{i}s}{2}\big)} \qquad (\mathtt{L} = \bar{\mathtt{L}} = 0)$$

Extension to all low energy states using ODE/IQFT correspondence [Bazhanov, GK, Koval, Lukyanov'19]

Quantization condition ($r = 2$)

QC allows one to determine admissible values of s.
Example for $\mathtt{L} = \bar{\mathtt{L}} = 0$:

$$\left(\frac{N}{2\tilde{N}_0}\right)^{2\mathrm{i}s} \mathrm{e}^{\frac{\mathrm{i}}{2}\delta} = \sigma + O\big((\log N)^{-\infty}\big)\,, \qquad \mathrm{e}^{\frac{\mathrm{i}}{2}\delta} = \frac{\Gamma\big(\frac{1}{2}+p-\frac{\mathrm{i}s}{2}\big)}{\Gamma\big(\frac{1}{2}+p+\frac{\mathrm{i}s}{2}\big)}\,\frac{\Gamma\big(\frac{1}{2}+\bar{p}-\frac{\mathrm{i}s}{2}\big)}{\Gamma\big(\frac{1}{2}+\bar{p}+\frac{\mathrm{i}s}{2}\big)}$$

- Discrete spectrum: s pure imaginary and tends to a pole and zero of $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}$ for $\Im m(s) > 0$ and $\Im m(s) < 0$, respectively
- Continuous spectrum:

$$2s\log\big(N/(2\tilde{N}_0)\big) + \tfrac{1}{2\pi}\,\partial_s\delta = \pi\,(\mathfrak{m}_2 - \mathfrak{m}_1) + O\big((\log N)^{-\infty}\big)$$

$\Longrightarrow$ s is real and densely distributed along real line with density

$$\rho(s) = \tfrac{1}{\pi}\,\log\big(N/(2\tilde{N}_0)\big) + \tfrac{1}{4\pi}\,\partial_s\,\delta$$

QC was key to identifying scaling limit of lattice model with 2D black hole CFTs

Quantization condition (general r) [GK, Lukyanov '23]

General form:

$$\left(\frac{2^{\frac{r}{n}}\,N}{rN_0}\right)^{\frac{4\mathrm{i}}{r}\,(-1)^{\ell}\,s}\quad \frac{F_p^{(\ell+1)}(\boldsymbol{s})}{F_p^{(\ell)}(\boldsymbol{s})}\,\frac{F_{\bar{p}}^{(\ell)}(\bar{\boldsymbol{s}})}{F_{\bar{p}}^{(\ell+1)}(\bar{\boldsymbol{s}})} = \sigma\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{r}\,S^z} + O\big((\log N)^{-\infty}\big) \qquad (\star)$$

with

$$s \equiv 0 \qquad \text{for} \qquad r \text{ odd}\,, \qquad\qquad s \equiv s_{\frac{r}{2}} \qquad \text{for} \qquad r \text{ even}$$

and

$$\ell = 1, 2, \ldots, r$$

Functions $F_p^{(\ell)}(\boldsymbol{s})$ explained on next slide

$(\star) = r-1$ independent relations for $r-1$ variables

- Odd r: no N dependent term (in red). Discrete set of solutions expected
- Even r: continuous spectrum parameterized by s, while s_a with $|a| = 1, 2, \frac{r}{2}-1$ belongs to discrete set

Quantization condition (general r)

For the case $\mathtt{L} = \bar{\mathtt{L}} = 0$:

$$F_p^{(\ell)}(\boldsymbol{s}) = F_p(\boldsymbol{s}^{(\ell)})\,, \qquad\qquad F_{\bar{p}}^{(\ell)}(\bar{\boldsymbol{s}}) = F_{\bar{p}}(\bar{\boldsymbol{s}}^{(\ell)})$$

with

$$s_a^{(\ell)} = (-1)^{ar}\ \mathrm{e}^{+\frac{\mathrm{i}\pi a}{r}(2\ell-1)}\ s_a\,, \qquad\qquad \bar{s}_a^{(\ell)} = (-1)^{ar}\ \mathrm{e}^{-\frac{\mathrm{i}\pi a}{r}(2\ell-1)}\ s_{-a}$$

$F_p(\boldsymbol{s}) \equiv F_p\big(s_1, \ldots s_{[\frac{r}{2}]}\big)$ is a certain connection coefficient for the ODE

$$\Big[-\partial_v^2 + \mathrm{e}^{rv} + p^2 + \sum_{a=1}^{[\frac{r}{2}]} s_a\ \mathrm{e}^{av}\Big]\,\psi = 0$$

Explicit analytic formula for $F_p(\boldsymbol{s})$ exists only for $r = 2$

Generalization to any $\mathtt{L}, \bar{\mathtt{L}} \geq 0$ along the lines of ODE/IQFT correspondence contained in [GK, Lukyanov '23]

Conclusion

- Scaling limit of $\mathcal{Z}_r$ invariant spin chain in regime $0 < \gamma = \frac{\pi}{n+r} < \frac{\pi}{r}$
- Odd r: spectrum of conformal dimensions discrete with large (infinite) degeneracies
- Even r: continuous component in spectrum appears
- Important result: 'quantization condition' that is expected to determine admissible values of $\boldsymbol{s}$ and $\bar{\boldsymbol{s}}$ labeling states. Involves connection coefficient of certain class of ODEs.
- Description of the CFT remains an open problem