Structure Constants in $\mathcal{N} = 4$ SYM and Separation of Variables

Based on 2210.04923 with C. Bercini, P. Vieira

Alexandre Homrich

IGST 2023

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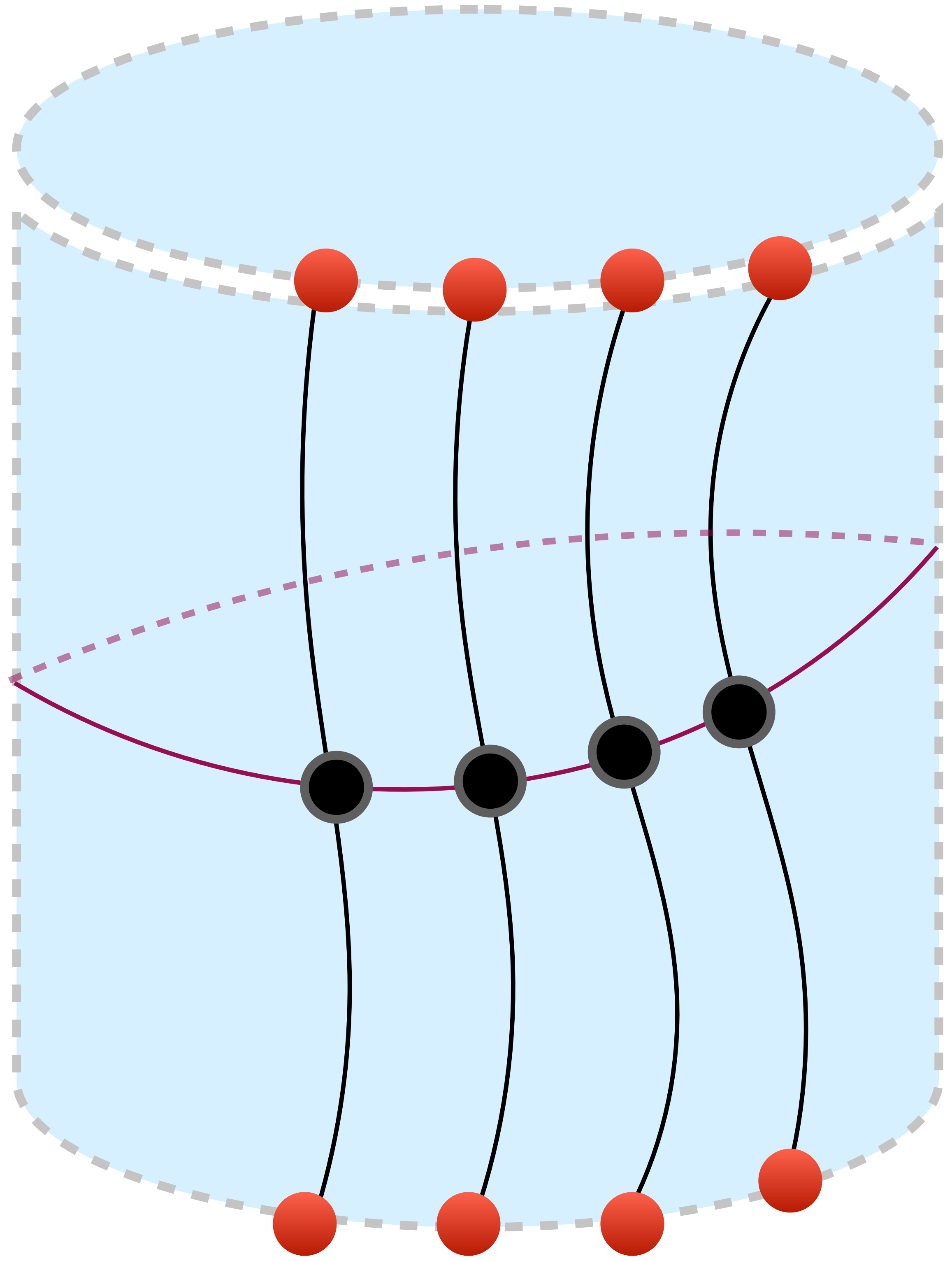

[Minaham, Zarembo], [Beisert], ..., [Gromov et. al - Arutyunov et. al. - Bombardelli et. al.], ...

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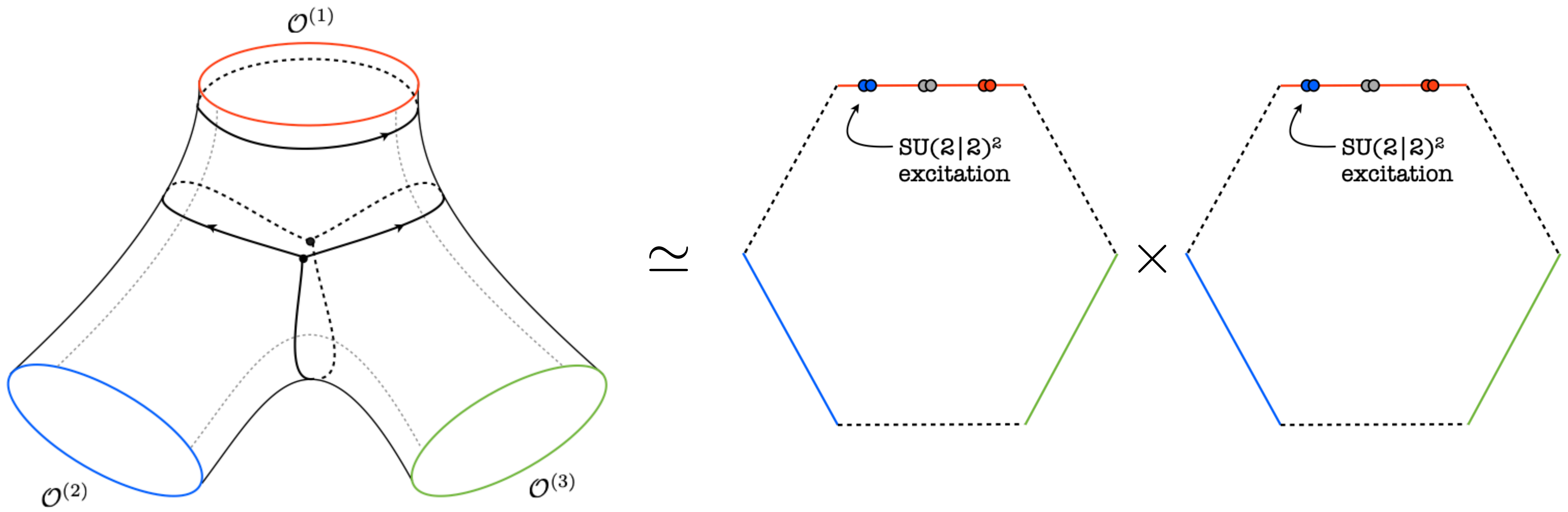

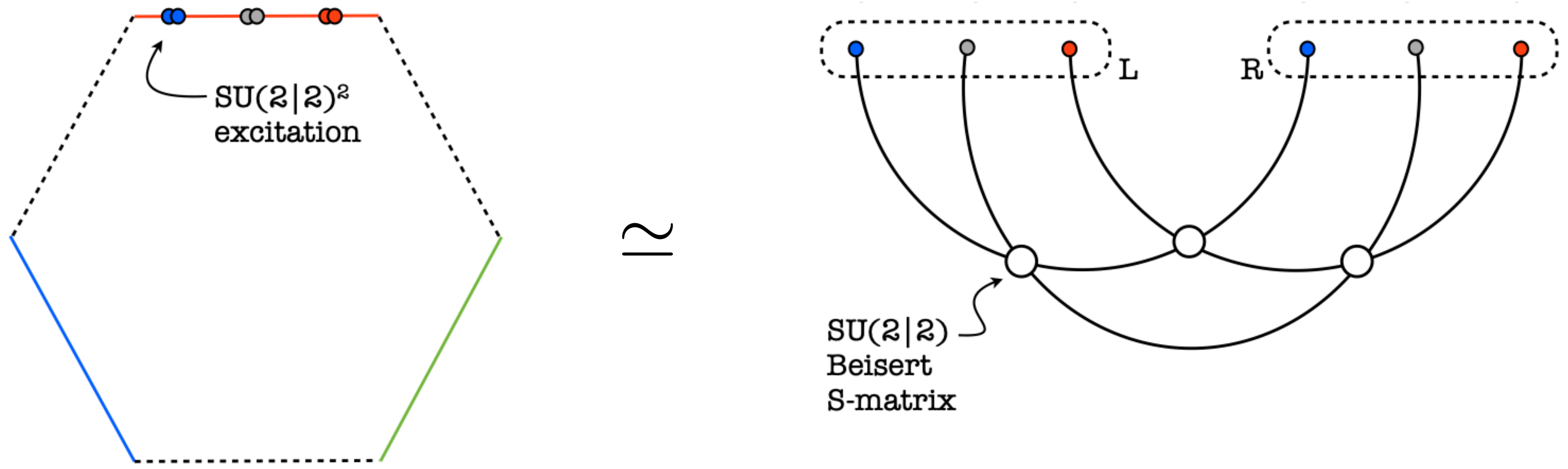

[Basso, Komatsu, Vieira][Komatsu, Fleury]

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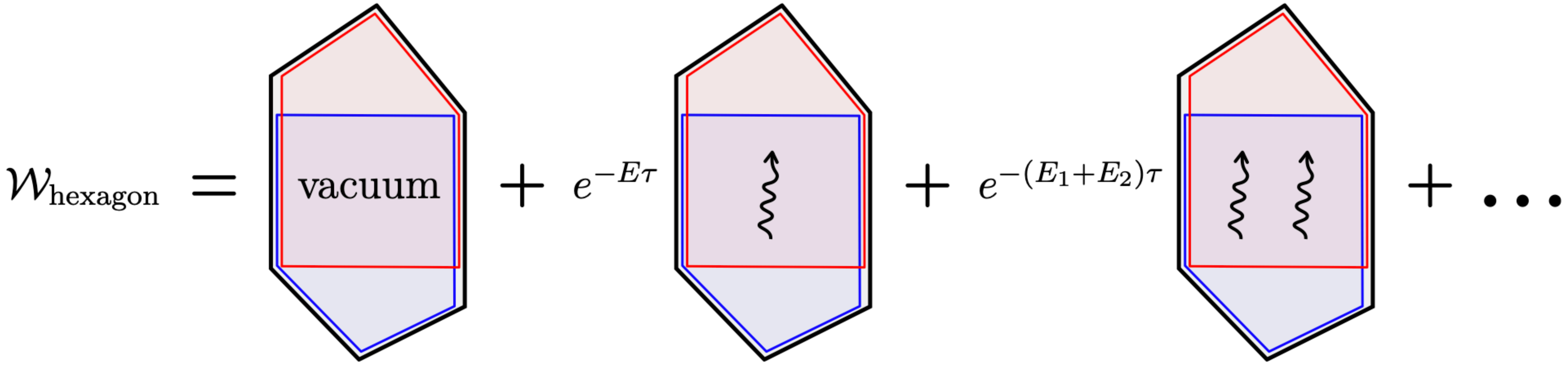

[Basso, Sever, Vieira]

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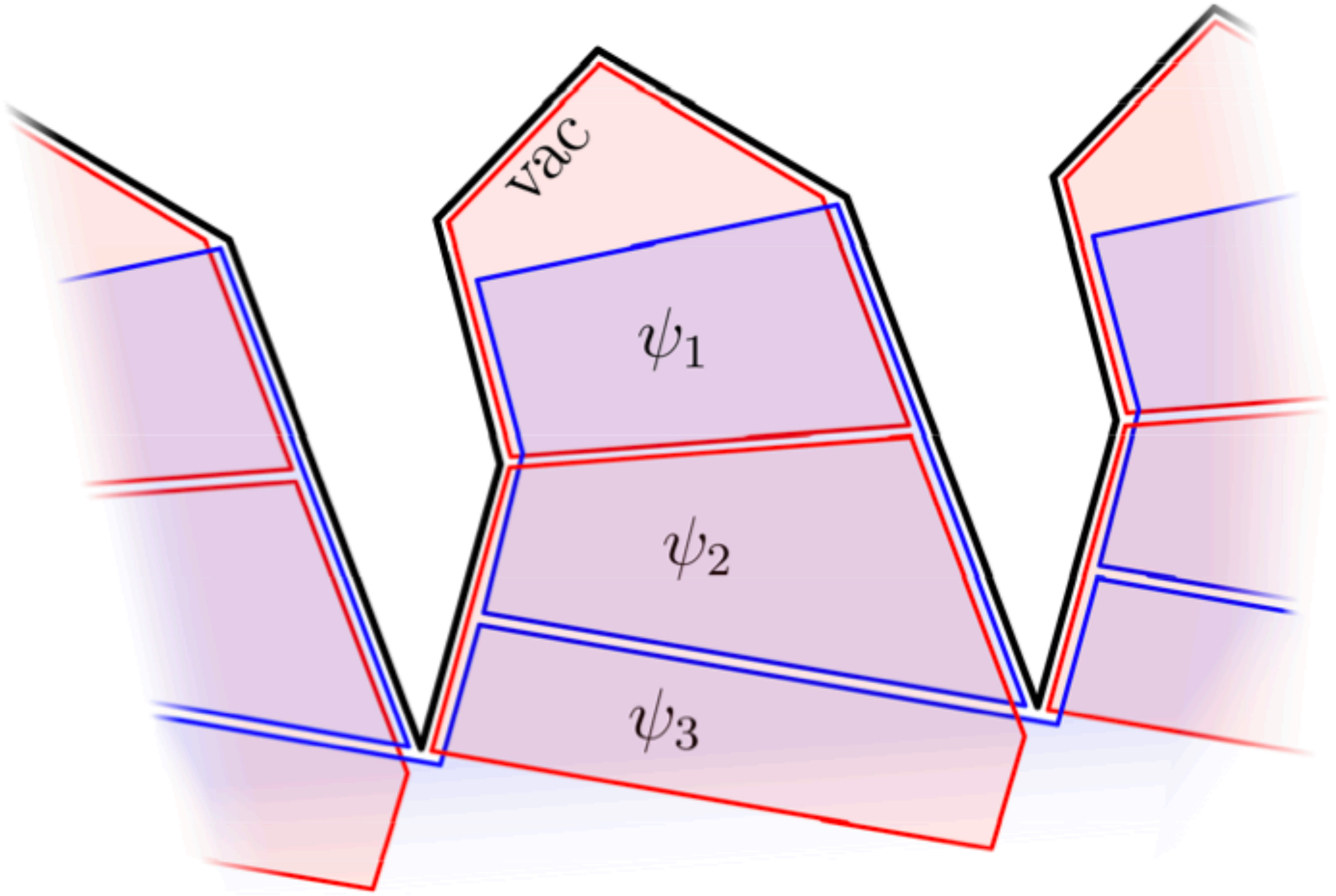

[Sever, Tumanov, Wilhelm]

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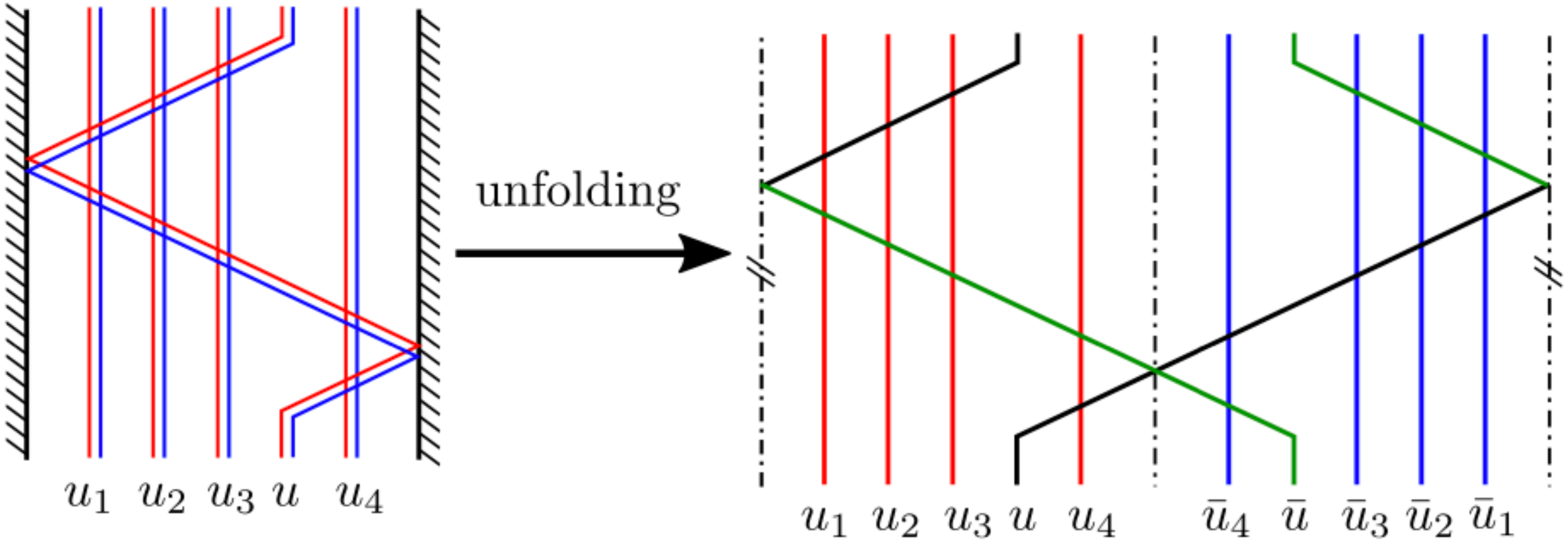
  $\ldots$



[Jiang, Komatsu, Vescovi]

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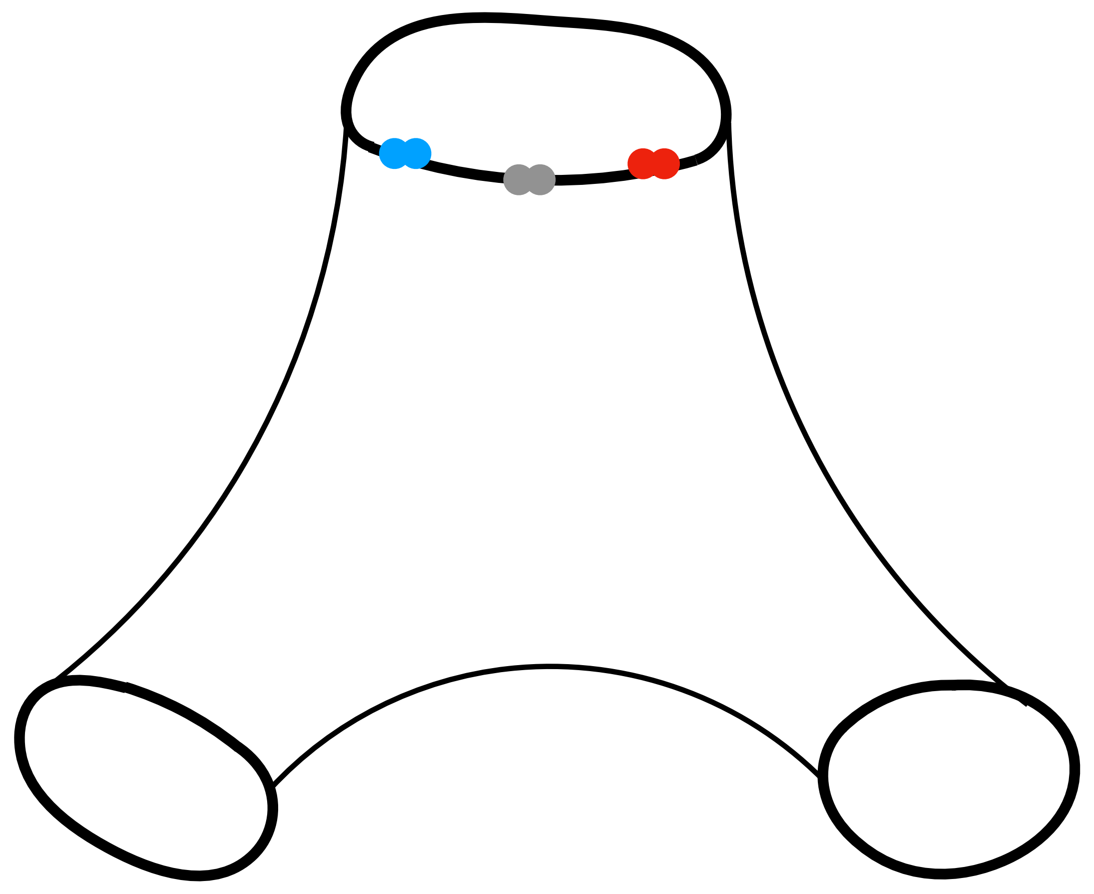

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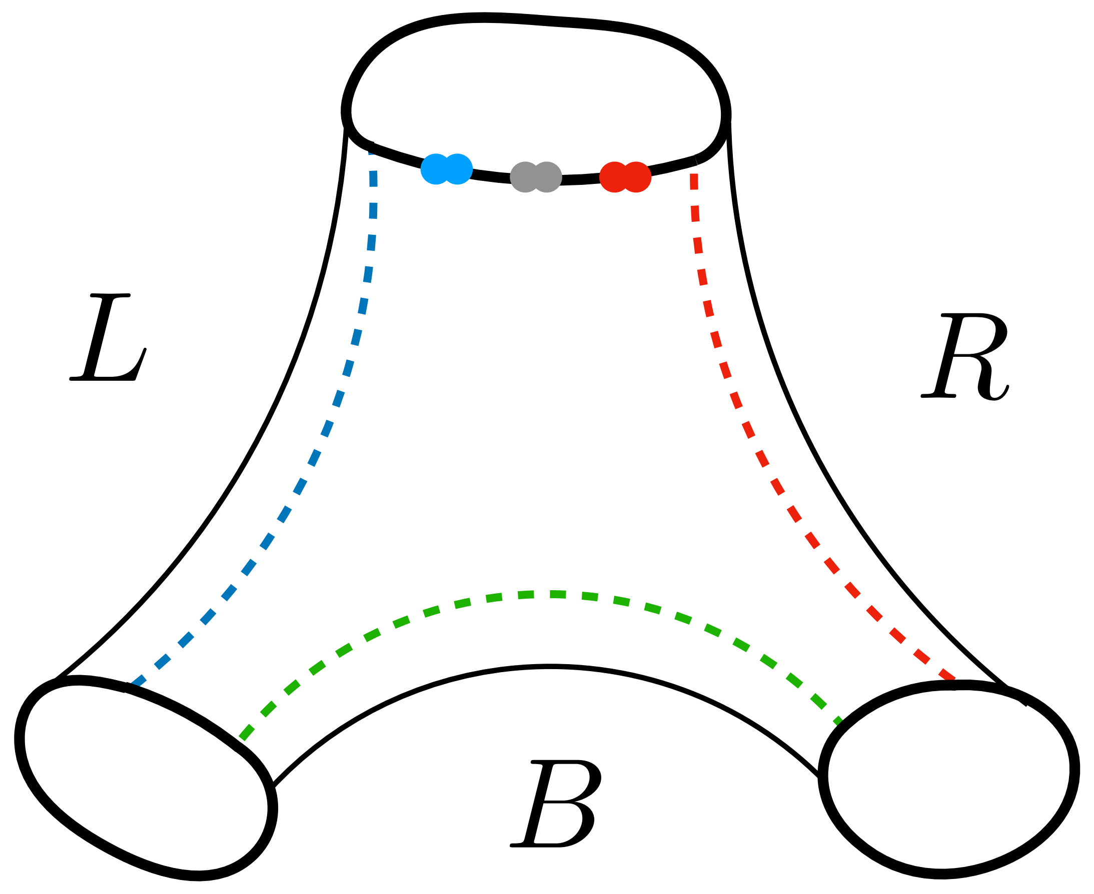

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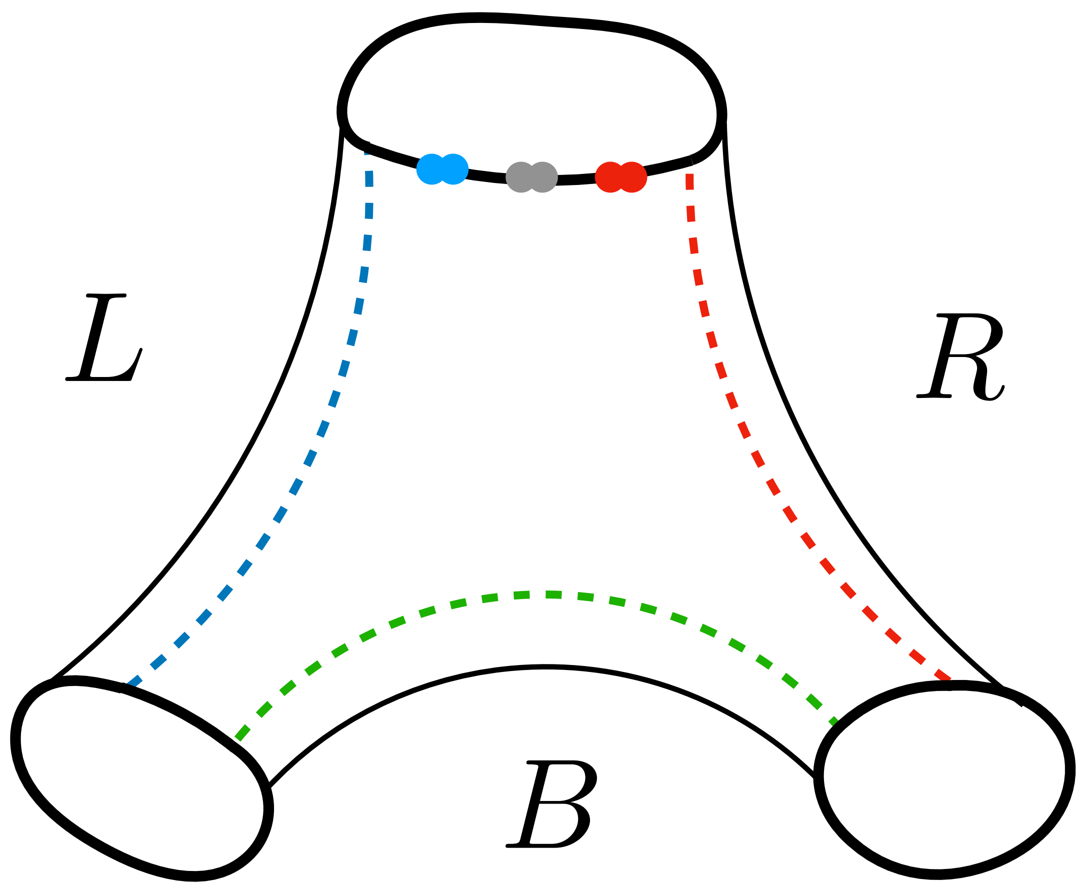
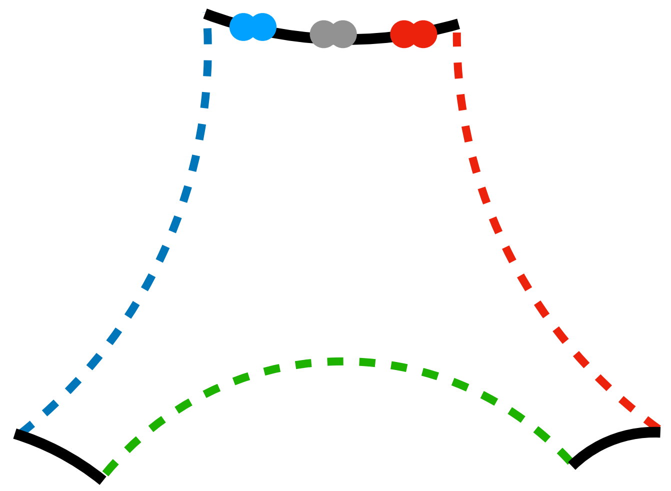
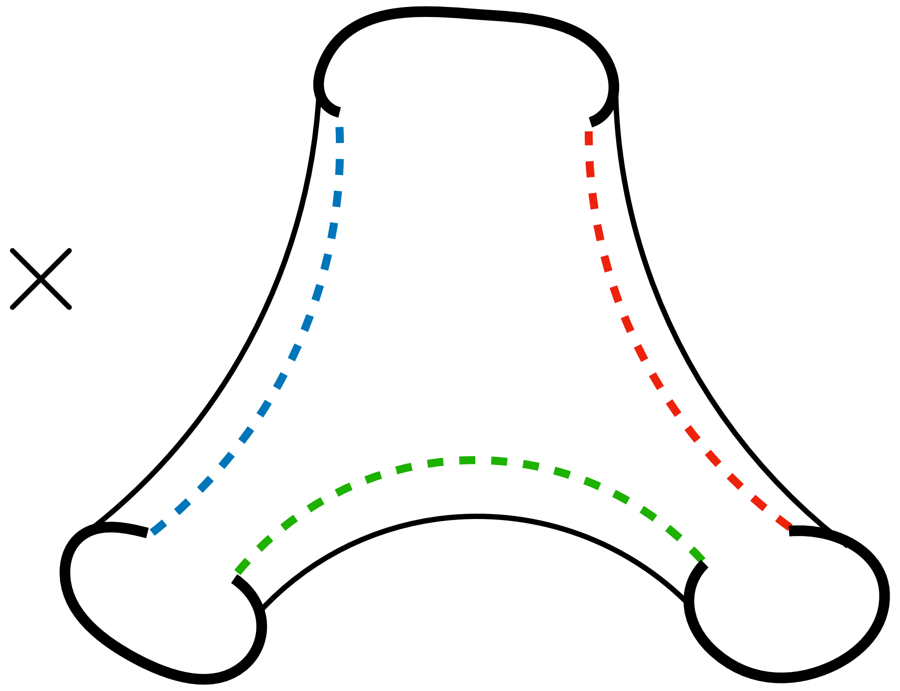


$$\simeq \sum_{\psi_L,\psi_R,\psi_B} e^{-\ell_L E(\psi_L)-\ell_R E(\psi_B)-\ell_B E(\psi_R)}$$



$\times$



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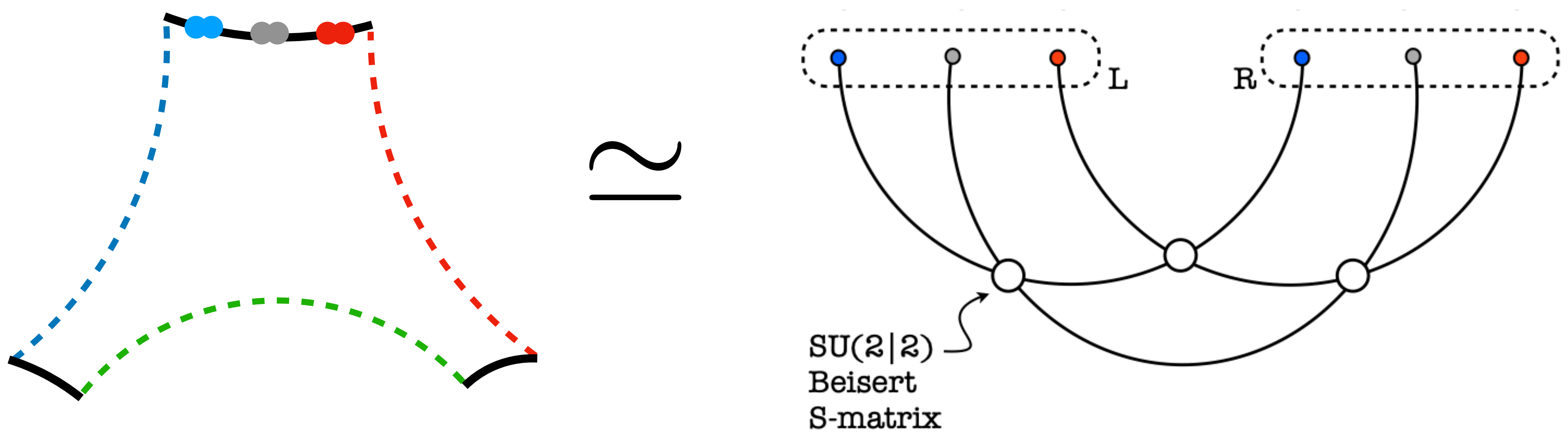

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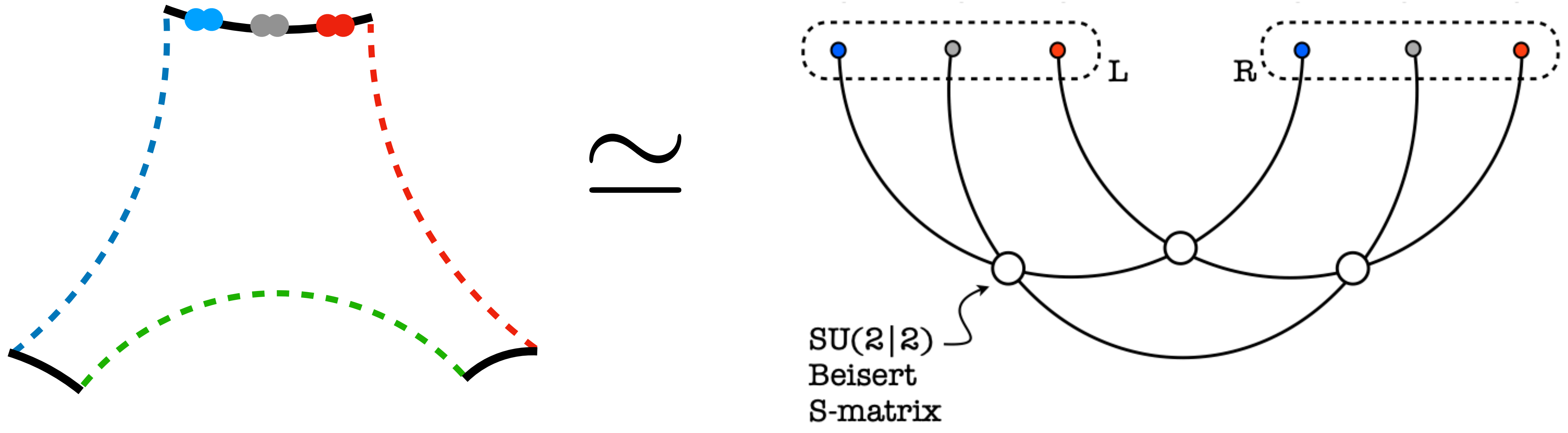

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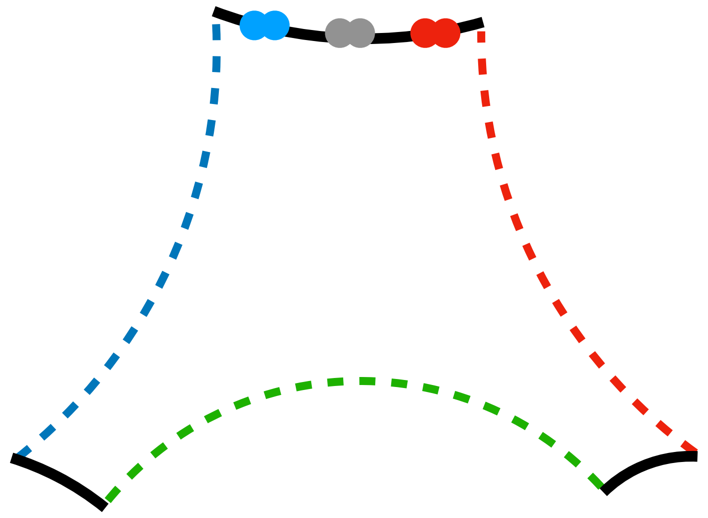

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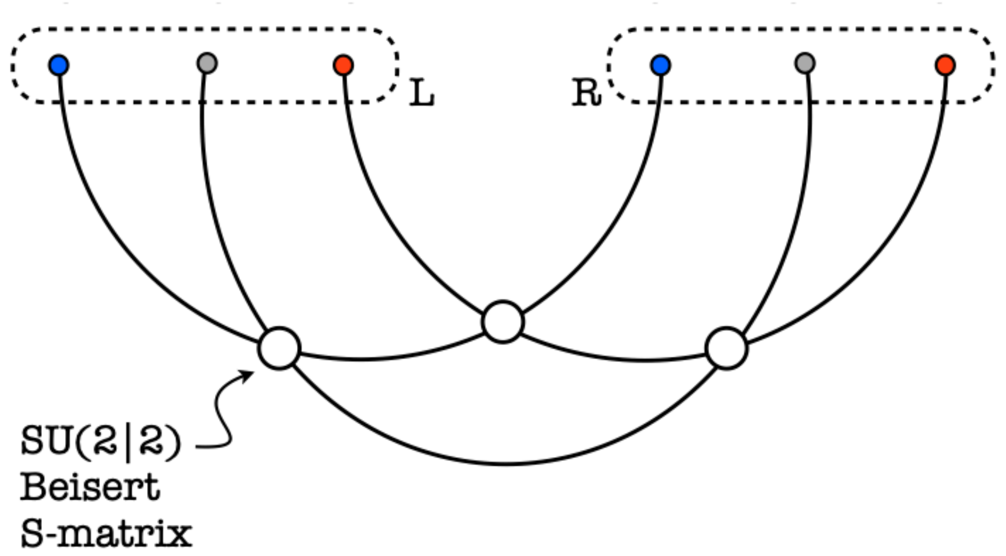

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  - ↳ Need truncating limits: weak coupling, large charge, collinear kinematics…

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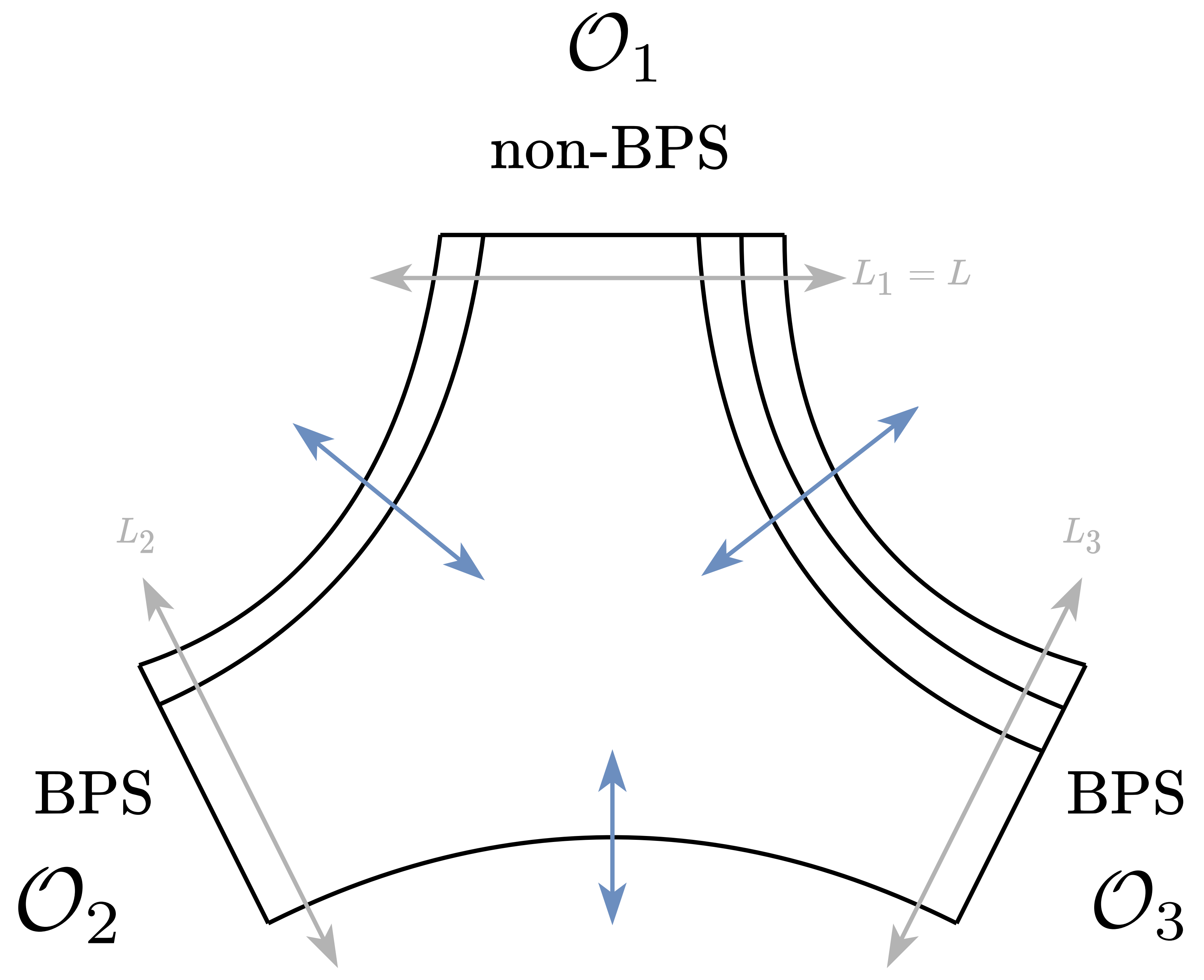

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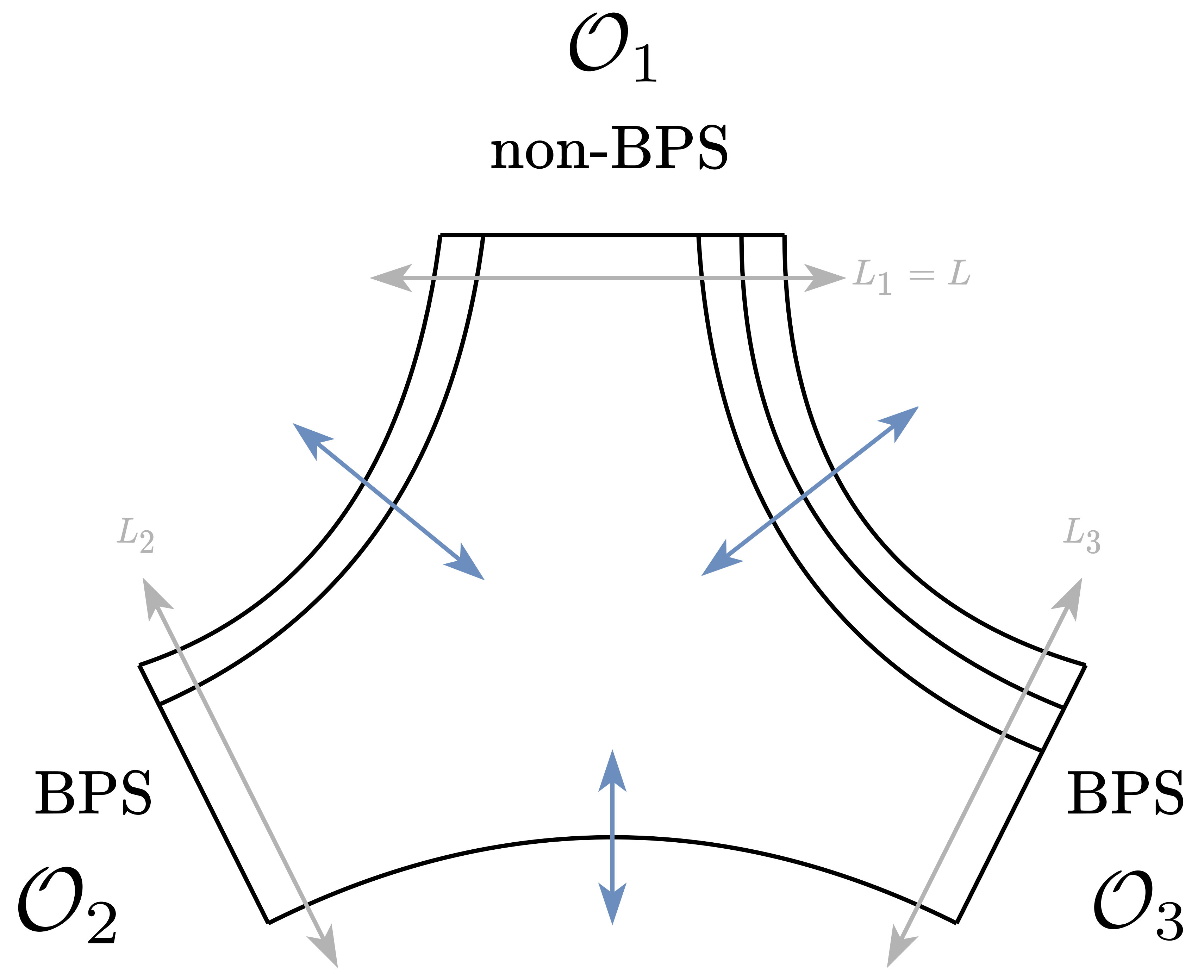

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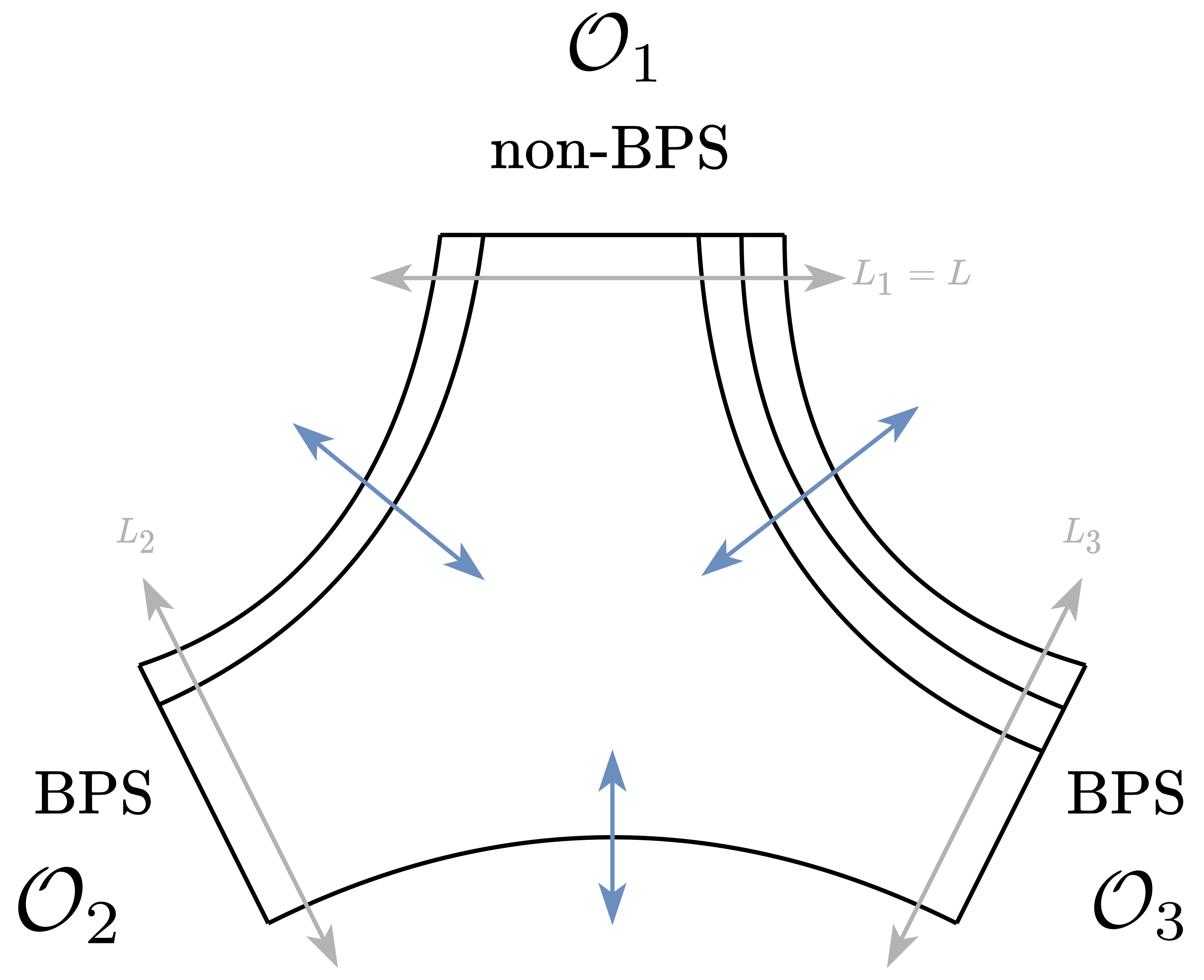

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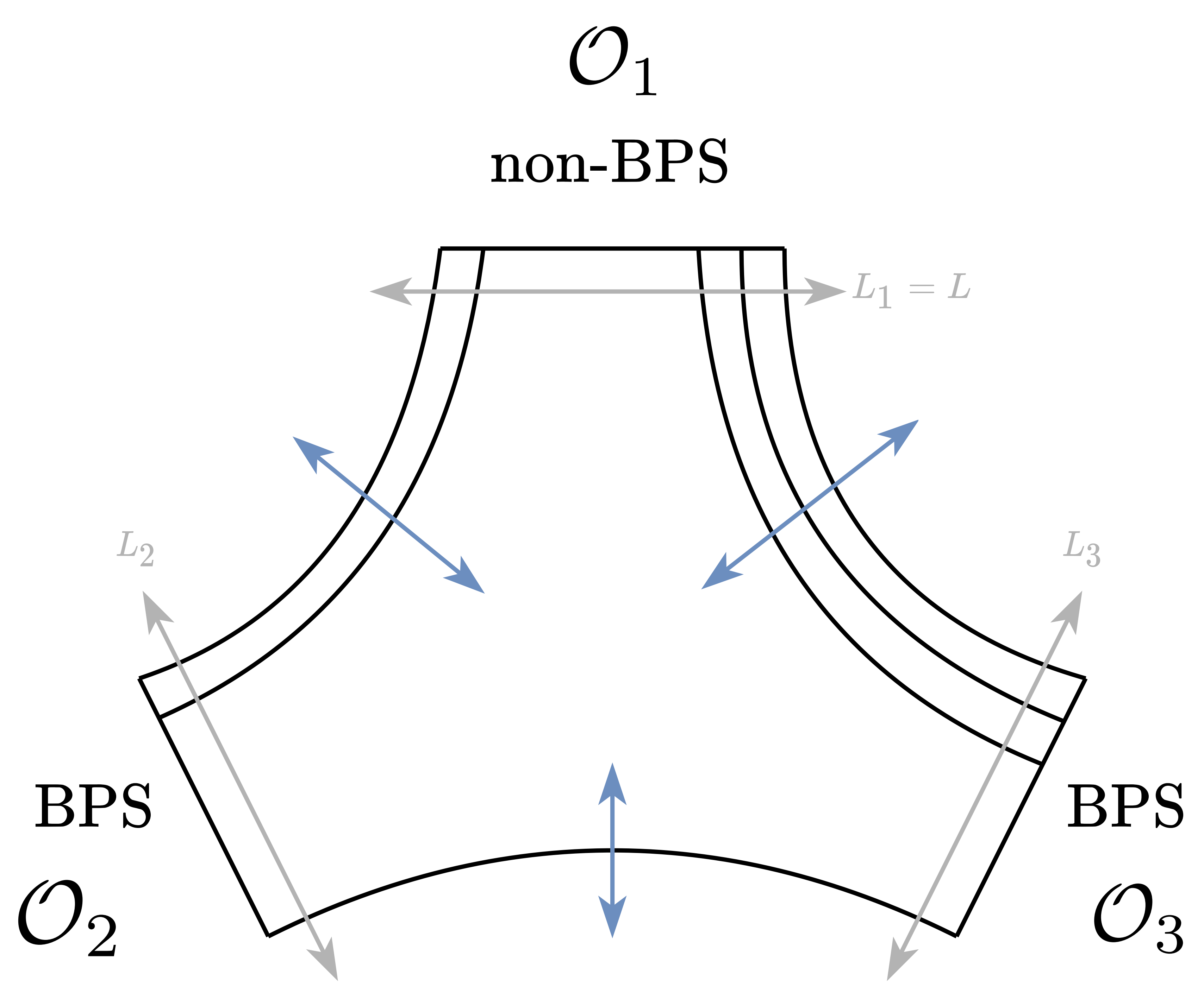

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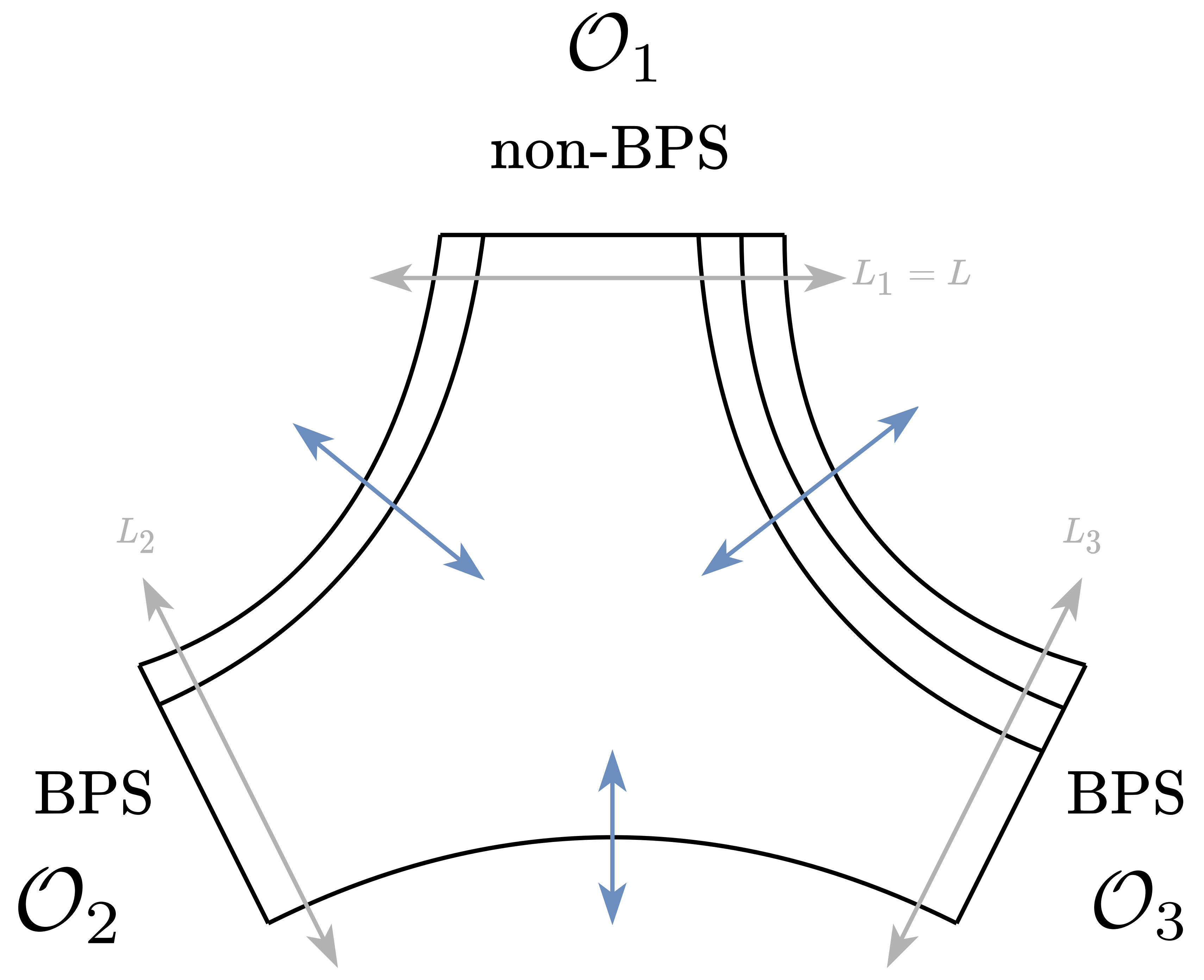

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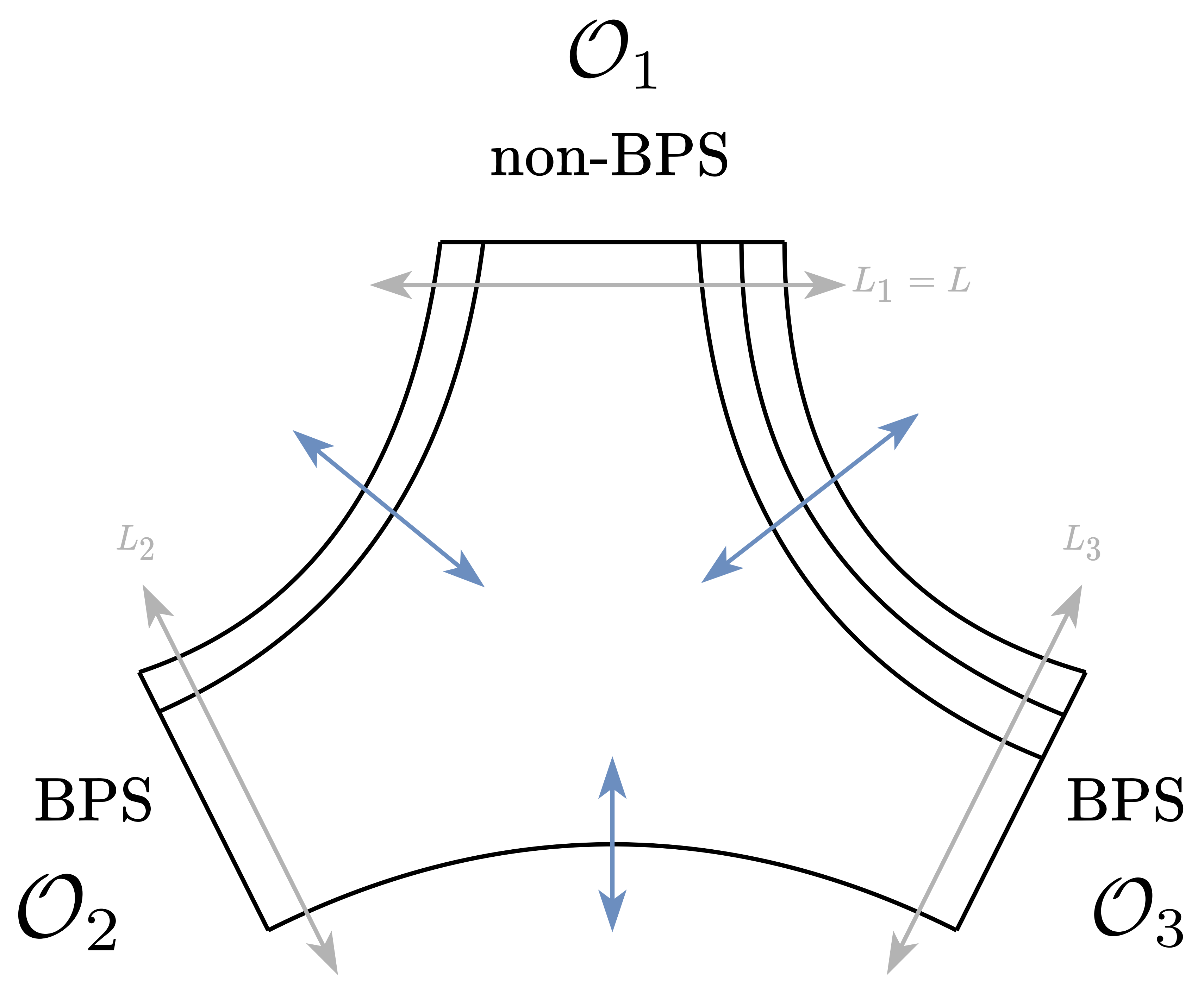

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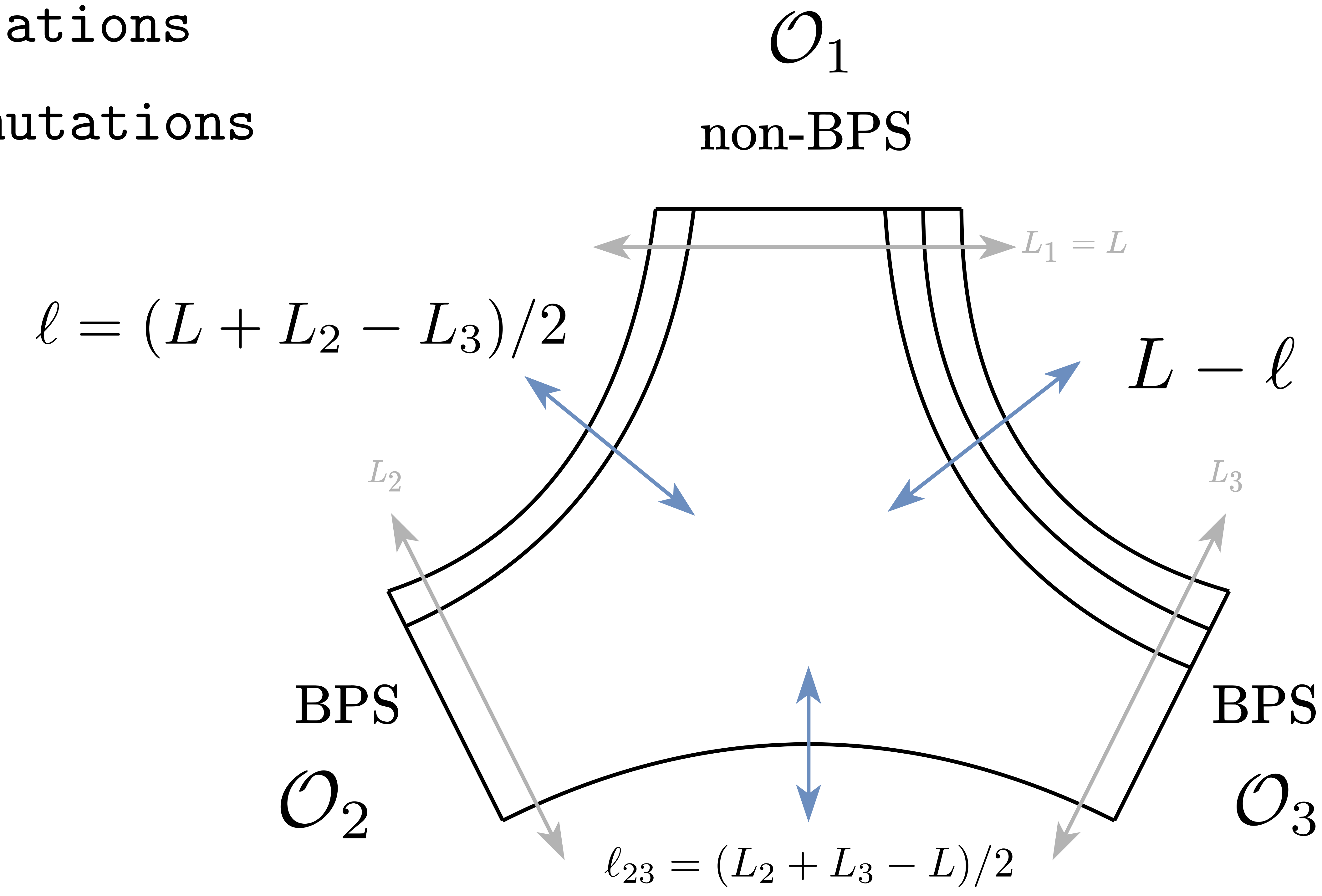

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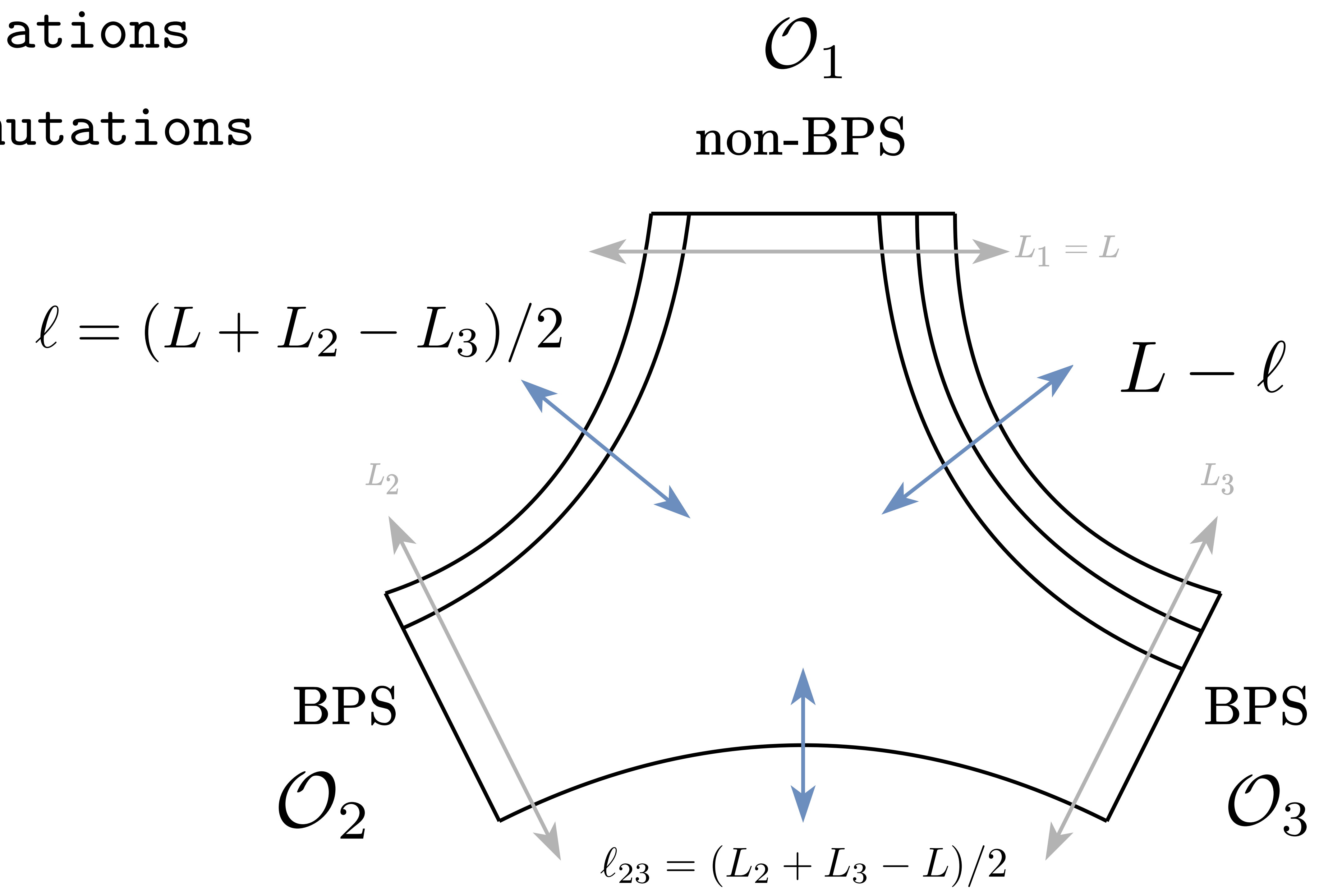

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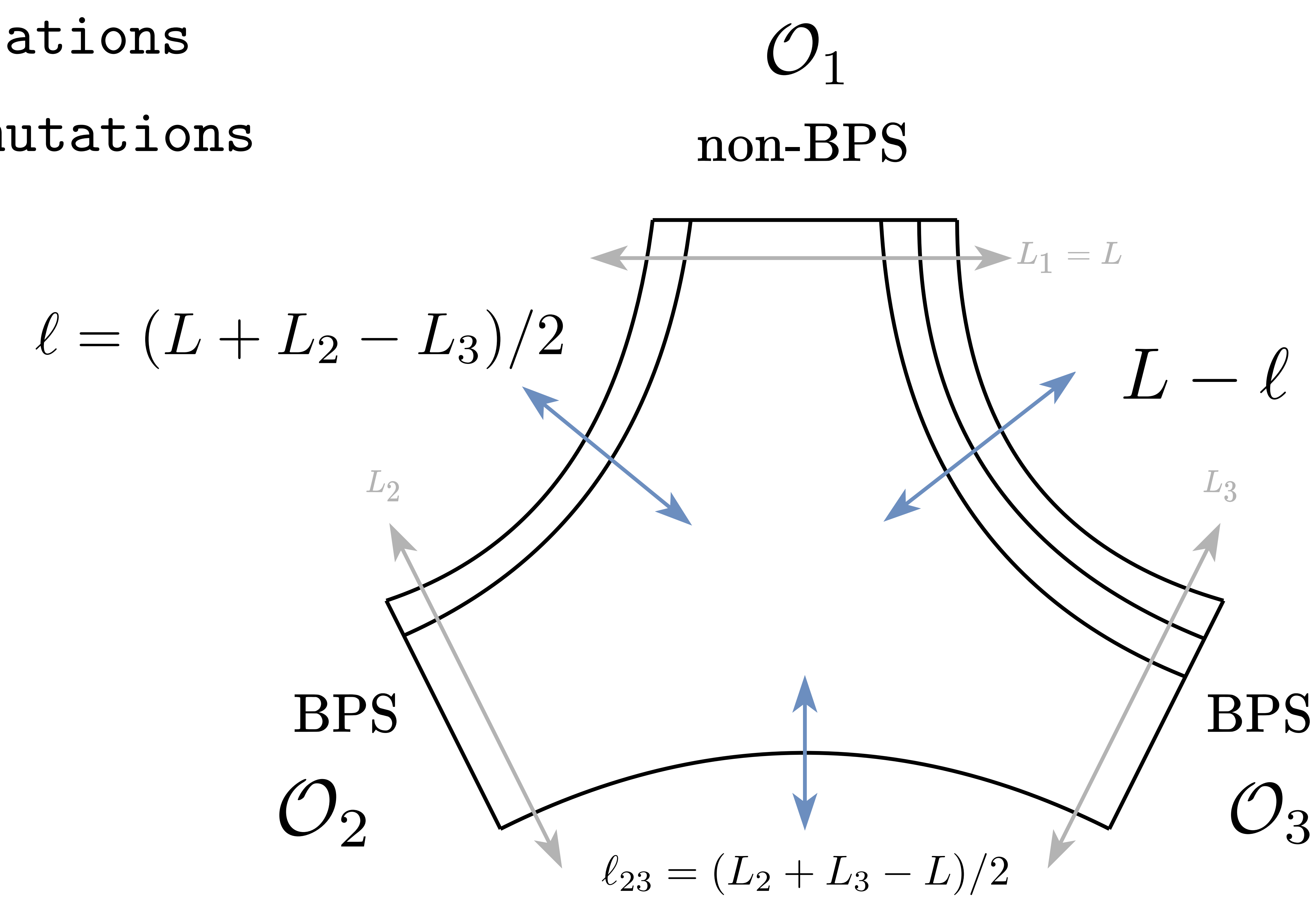

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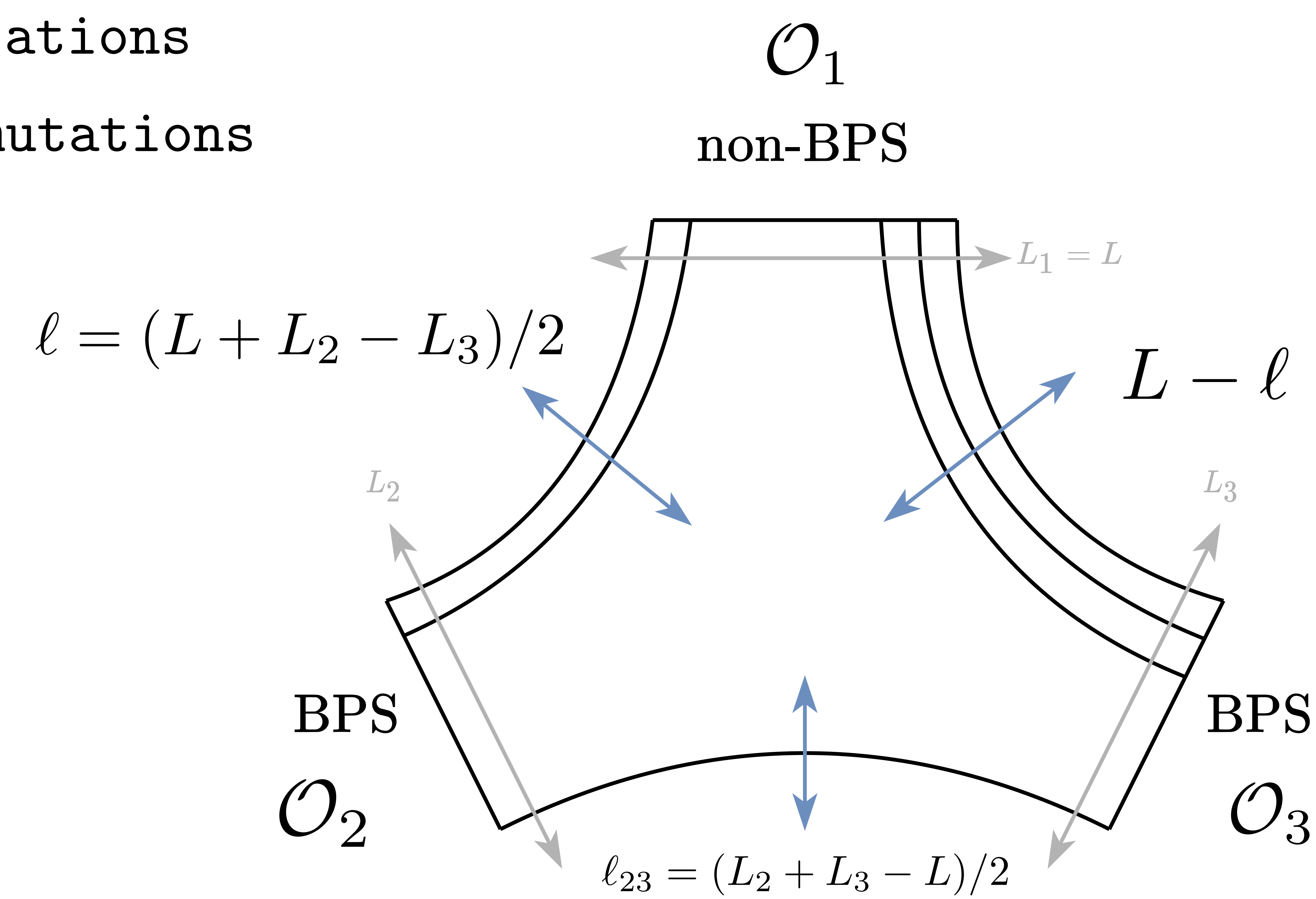

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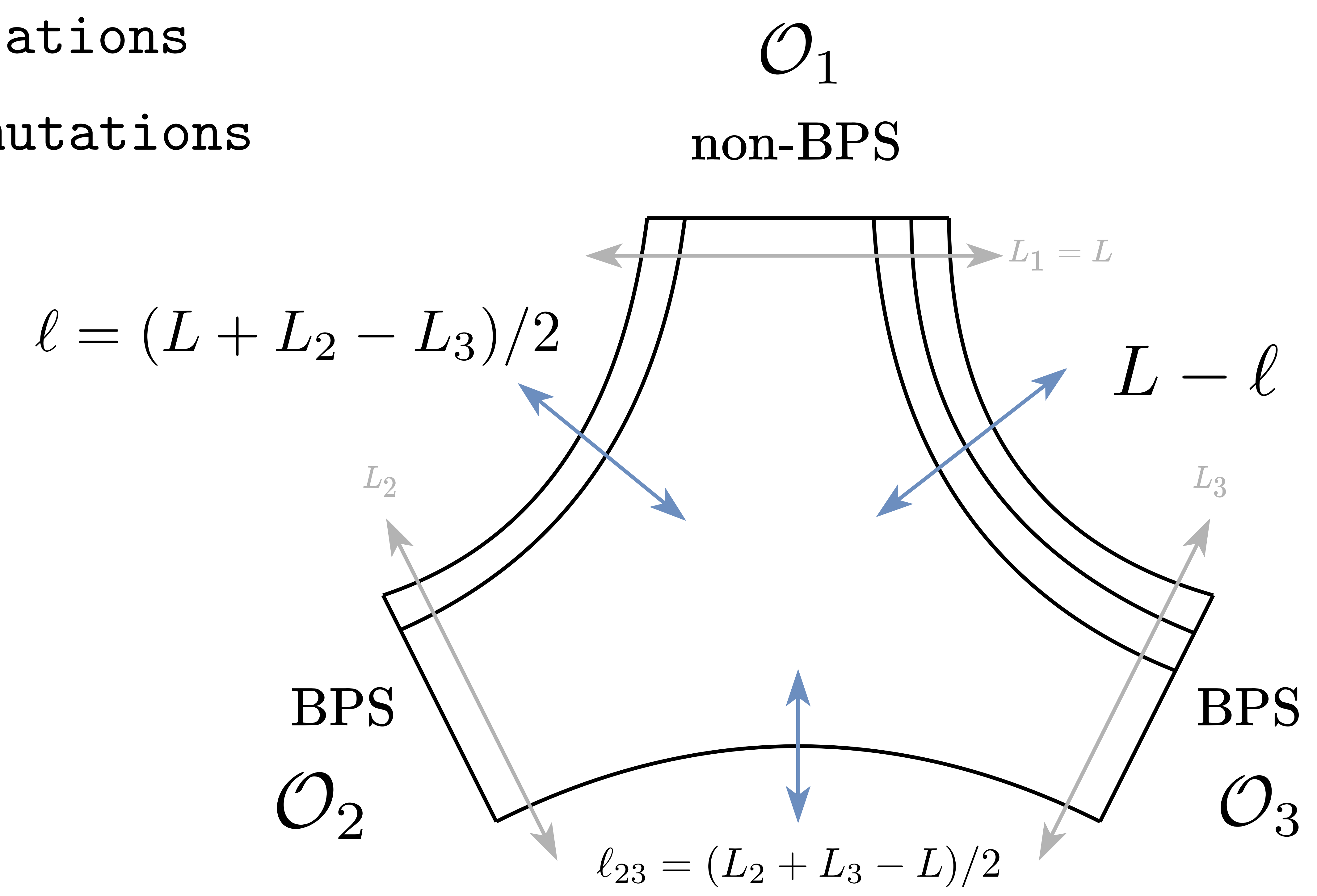

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- Consider simplest case of $L = 2$ $SL(2)$. Want to determine (μ, Γ). $\int_\Gamma du \mu Q_A Q_B$
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"Zhukowsky-fy tree level"

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Q gets dressed (no longer polynomial)

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Orthogonality remains key

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e.g. $(Q_1^+)^2\partial^2_{\theta_{L-1}}$

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$\mathcal{B}$ is given by an $L - 1$ dimensional integral with factorized measure.

$\mathcal{Q}$ is dressed by a function of higher integrable charges. $\frac{g^4 (Q_1^+)^2}{x^+ x^-} \subset B(u)$

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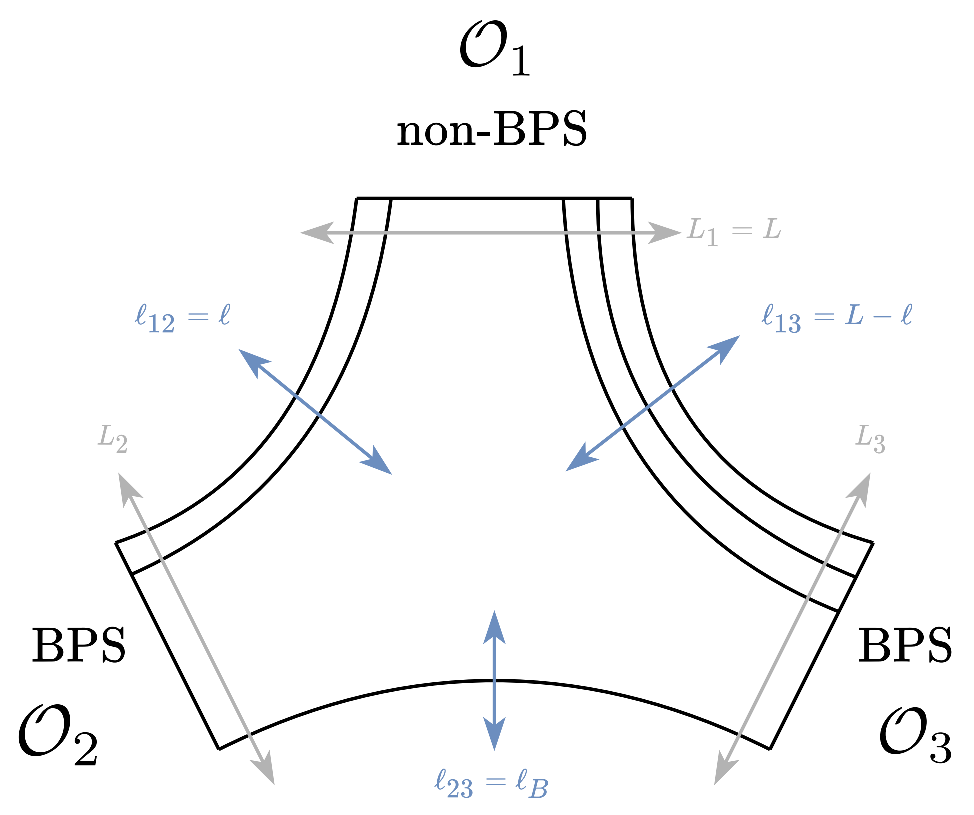

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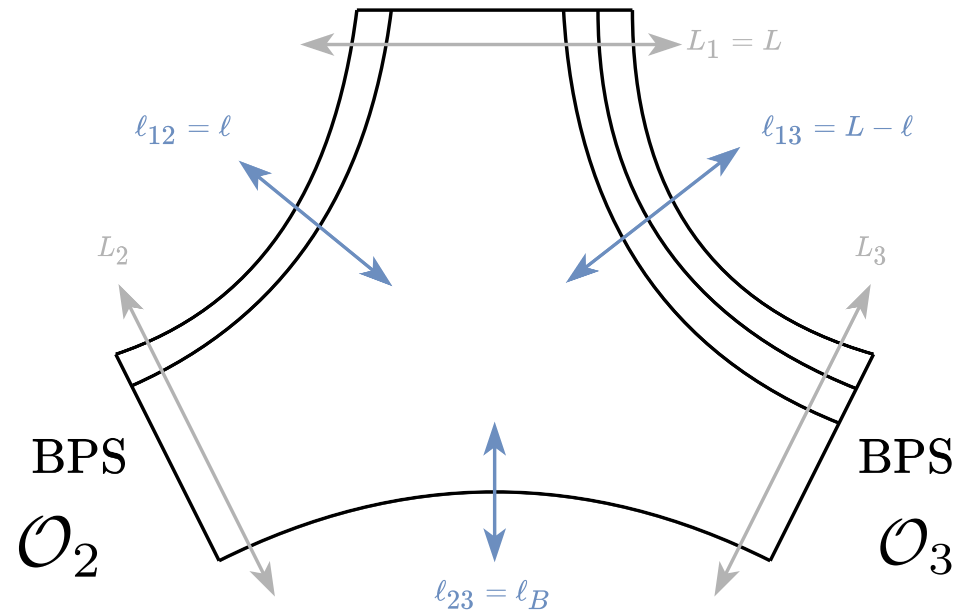

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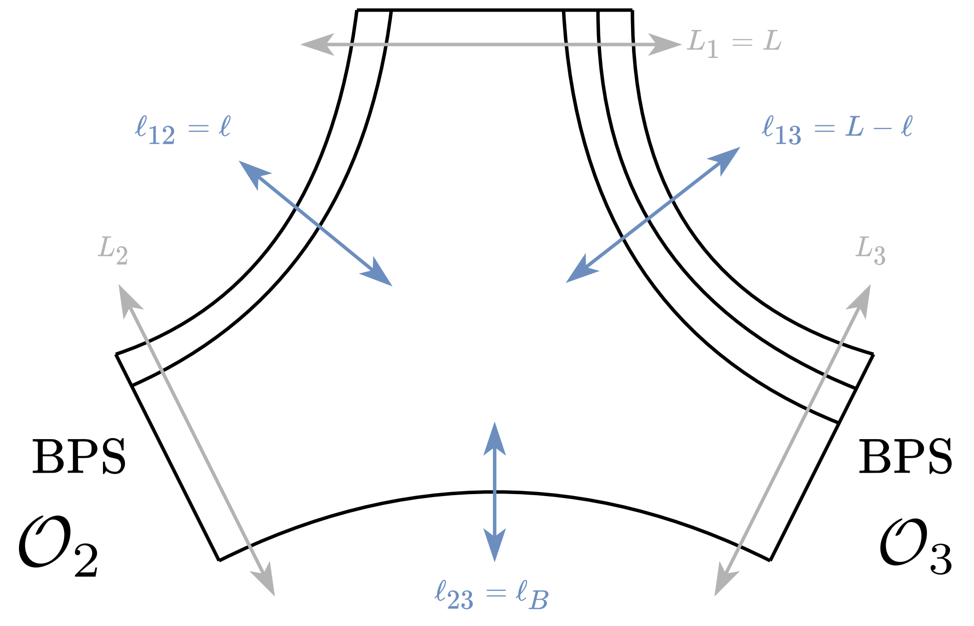

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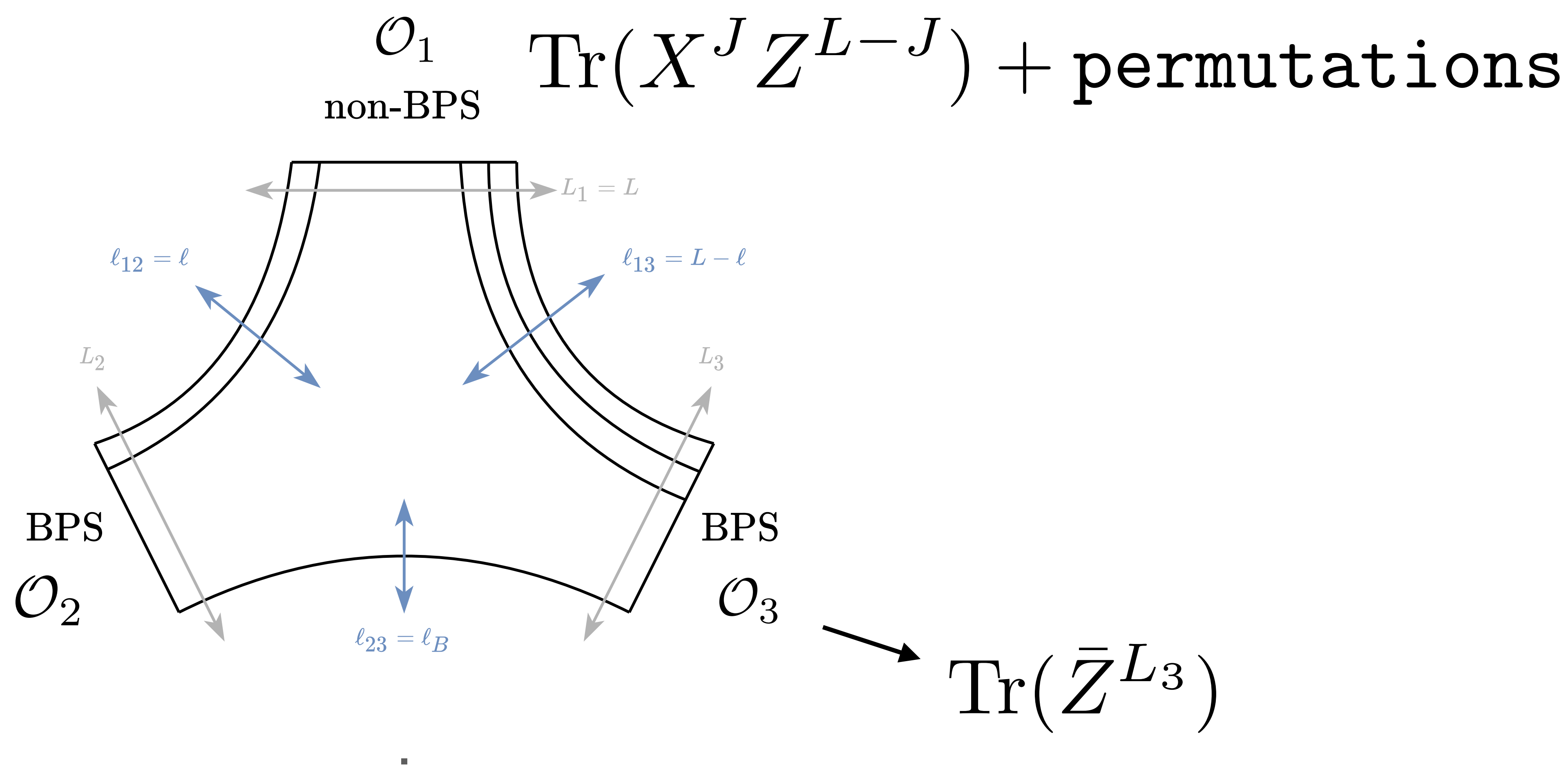

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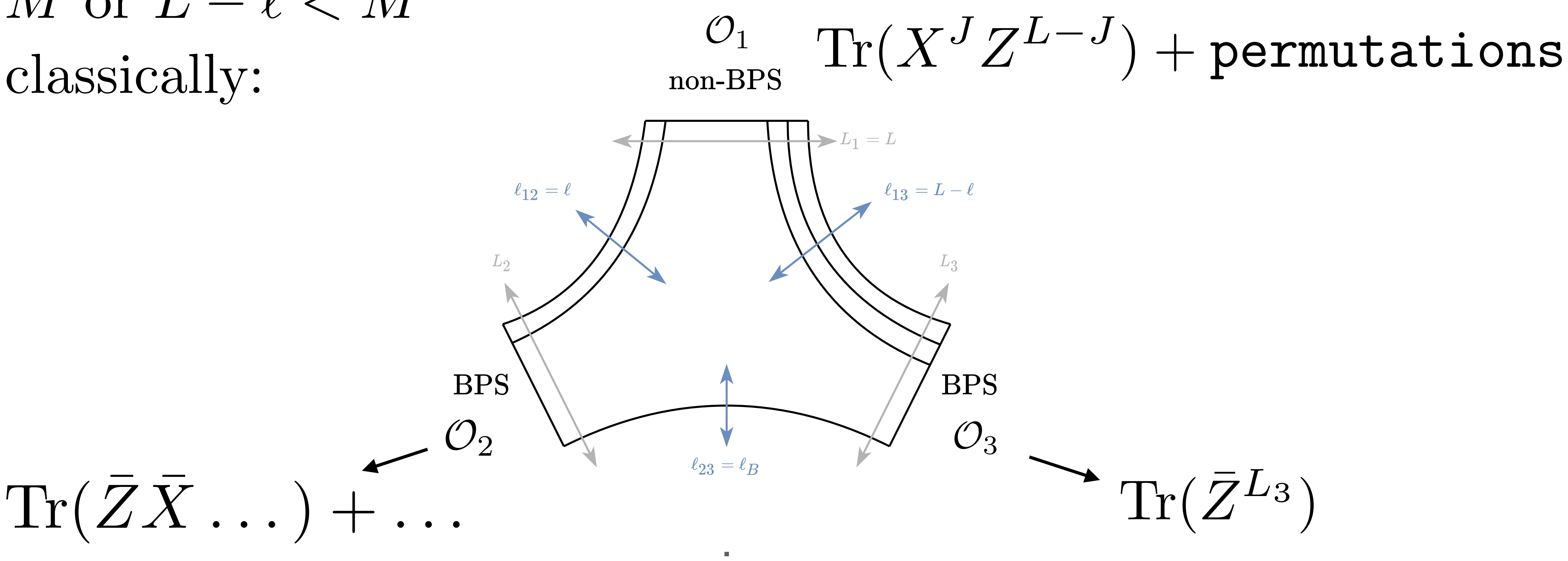

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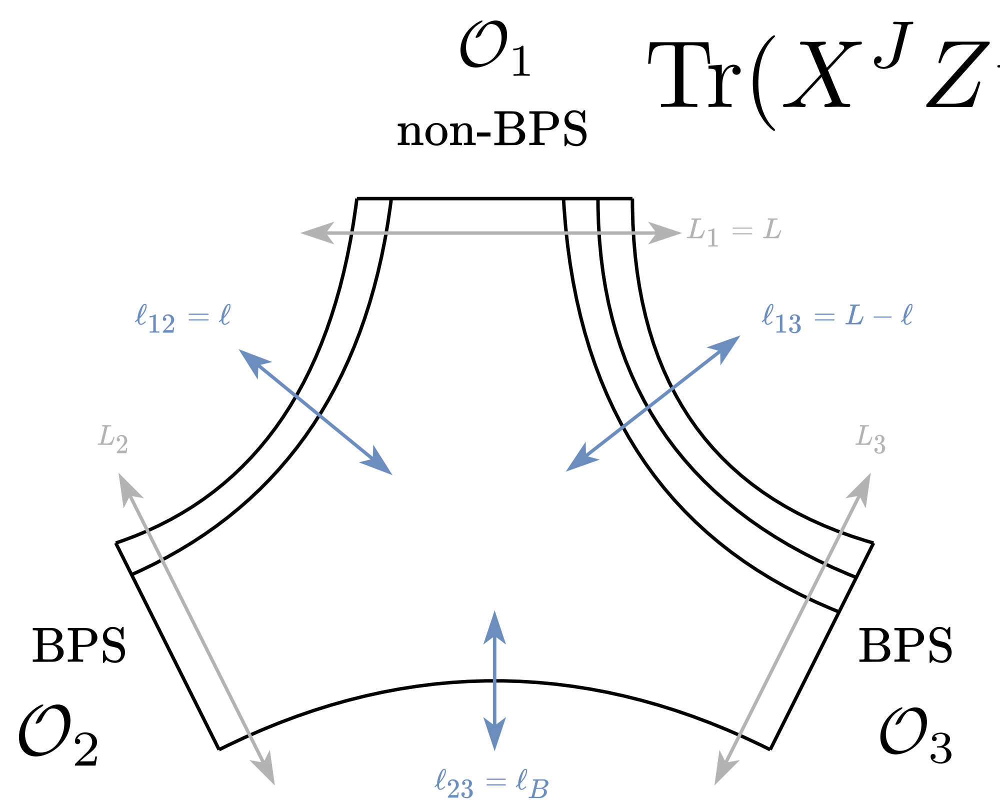

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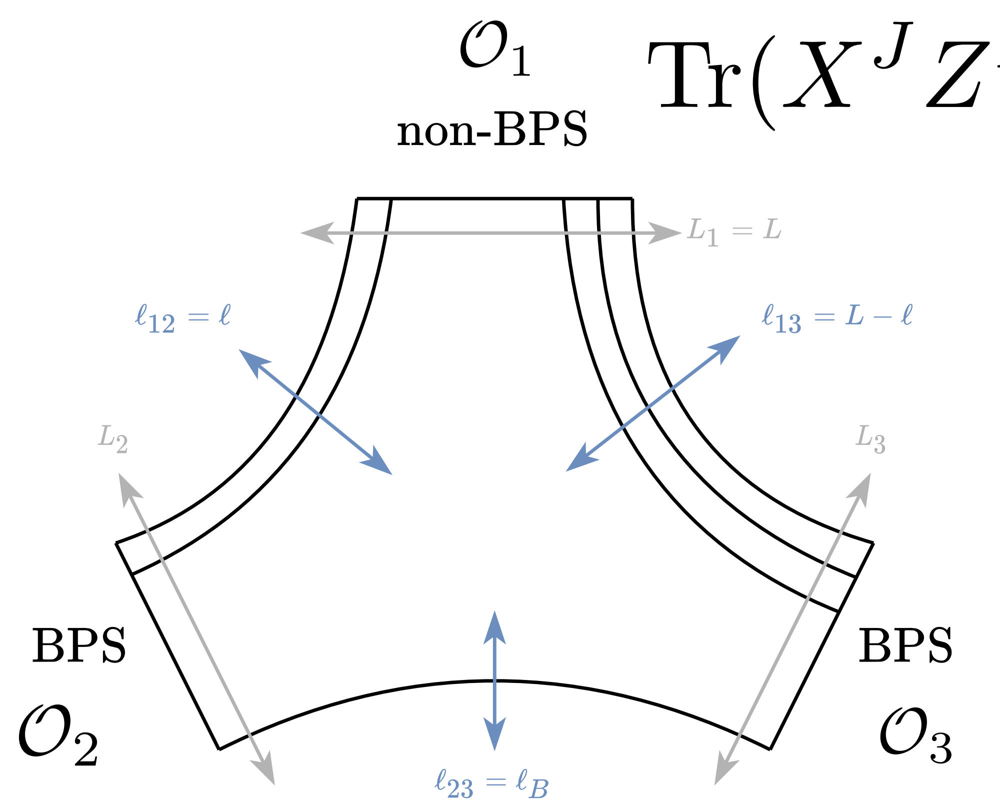

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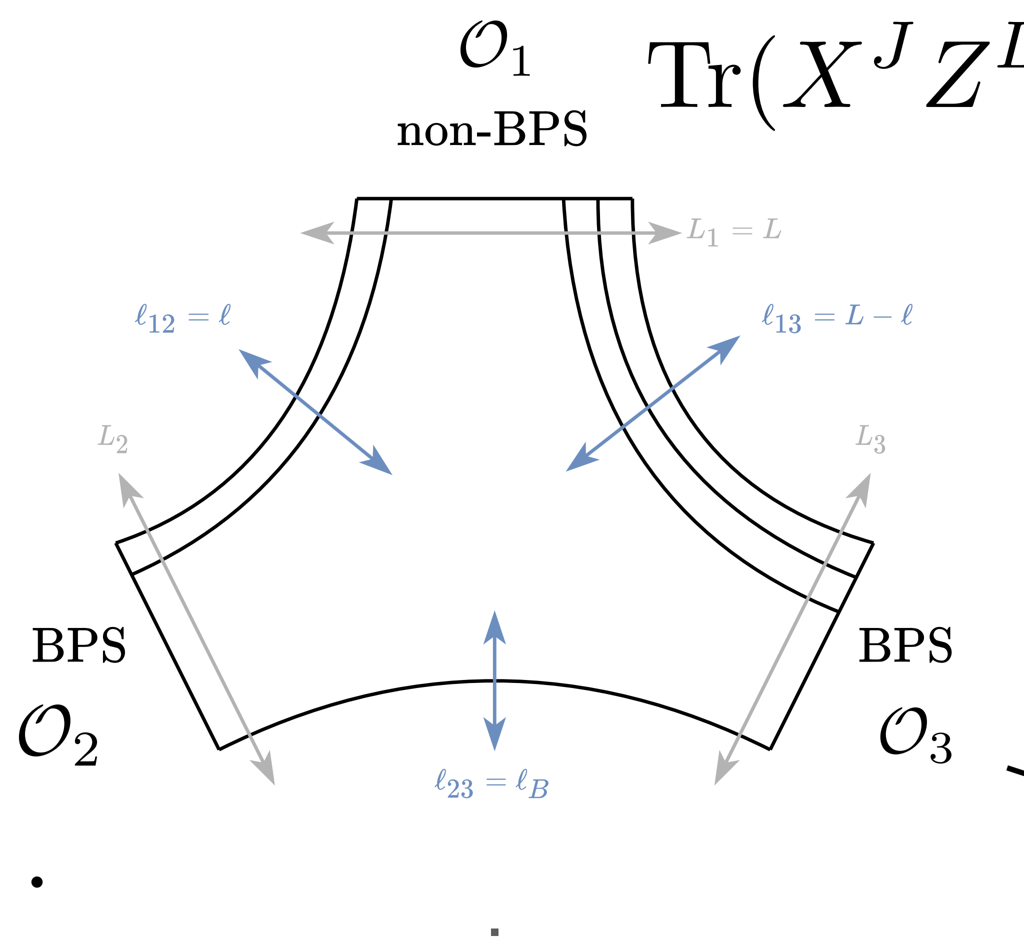

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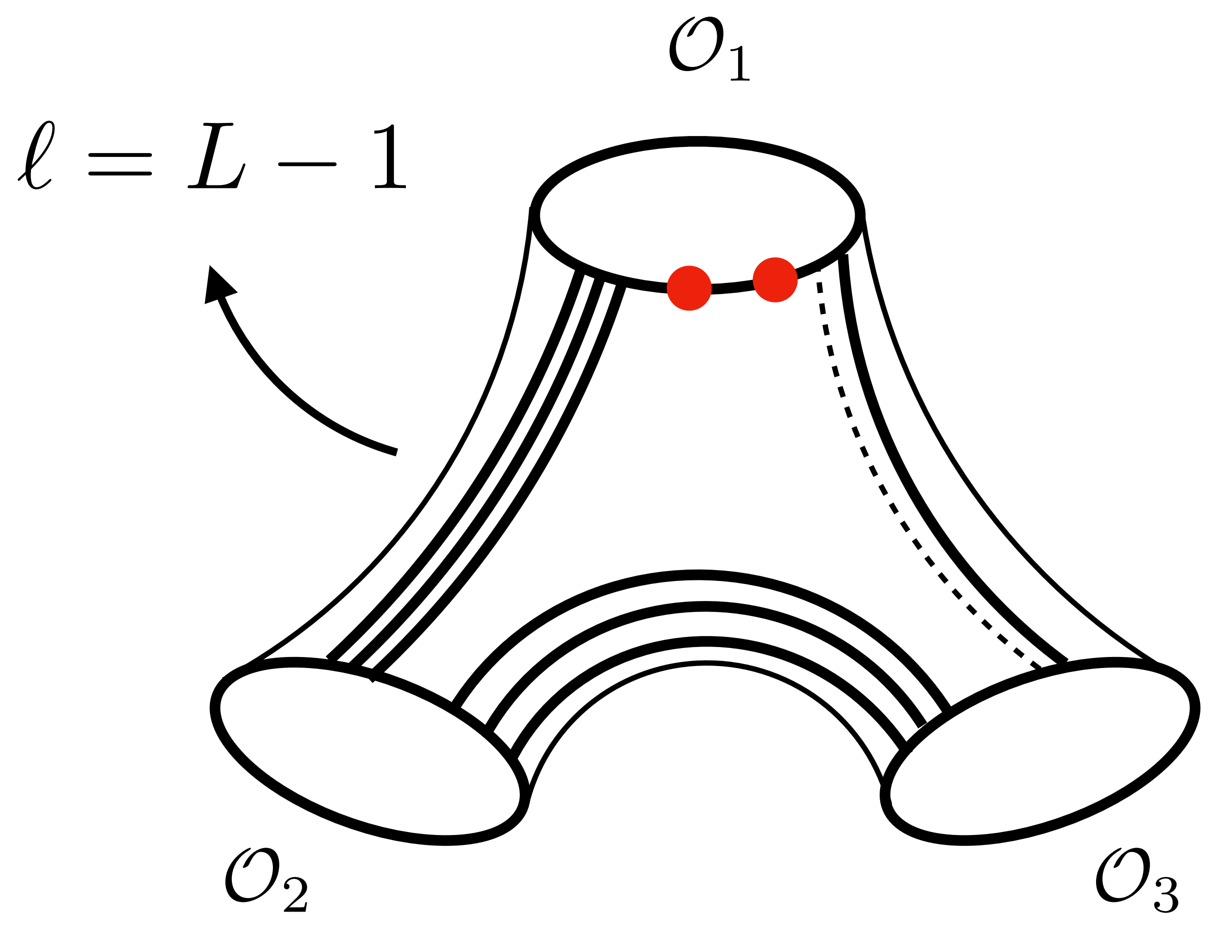

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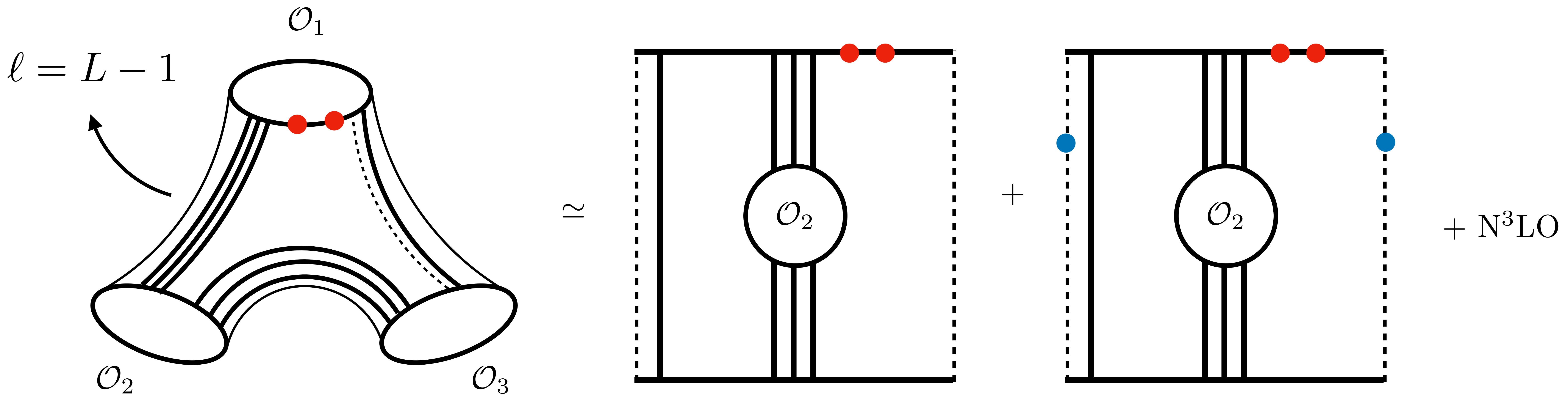

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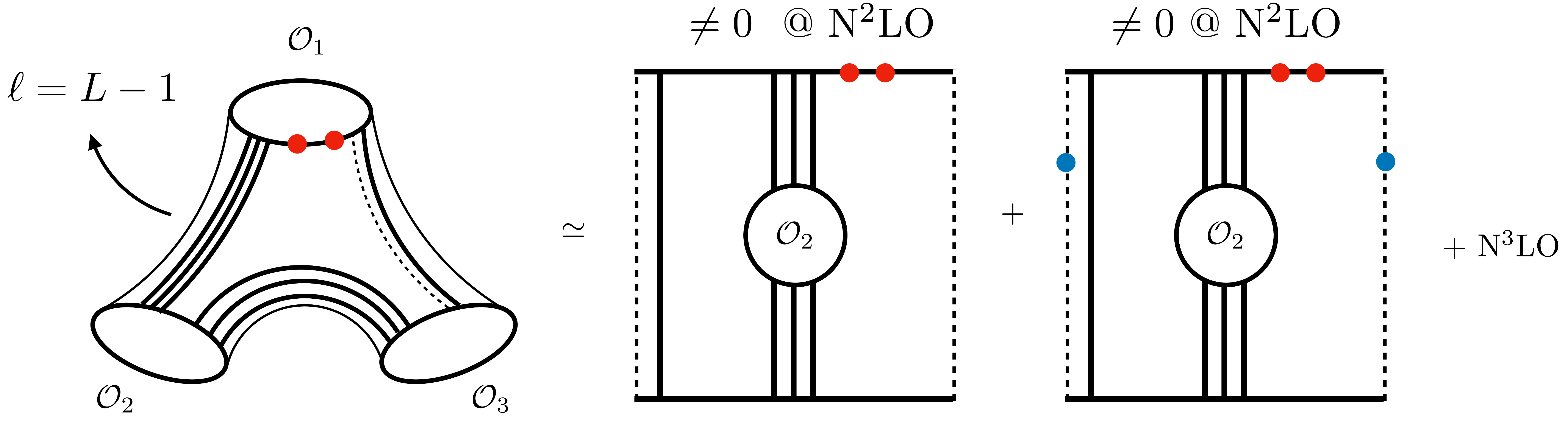

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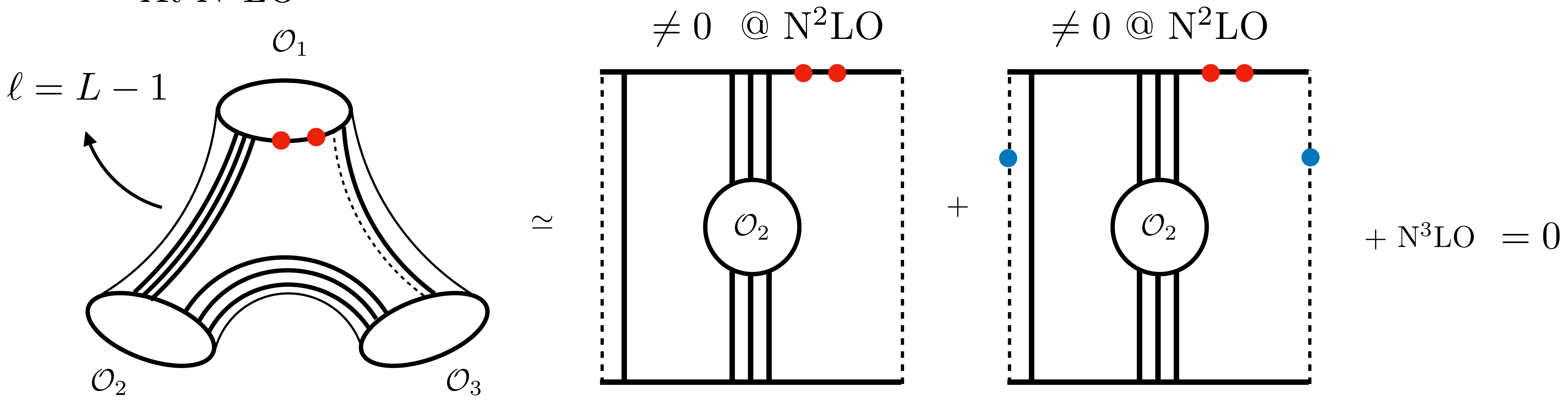

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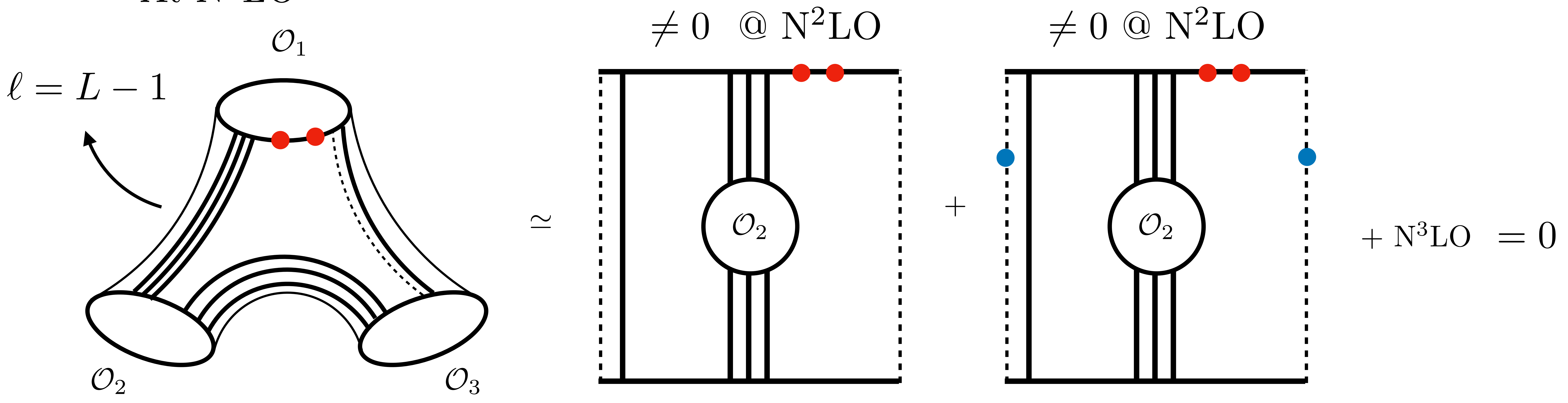

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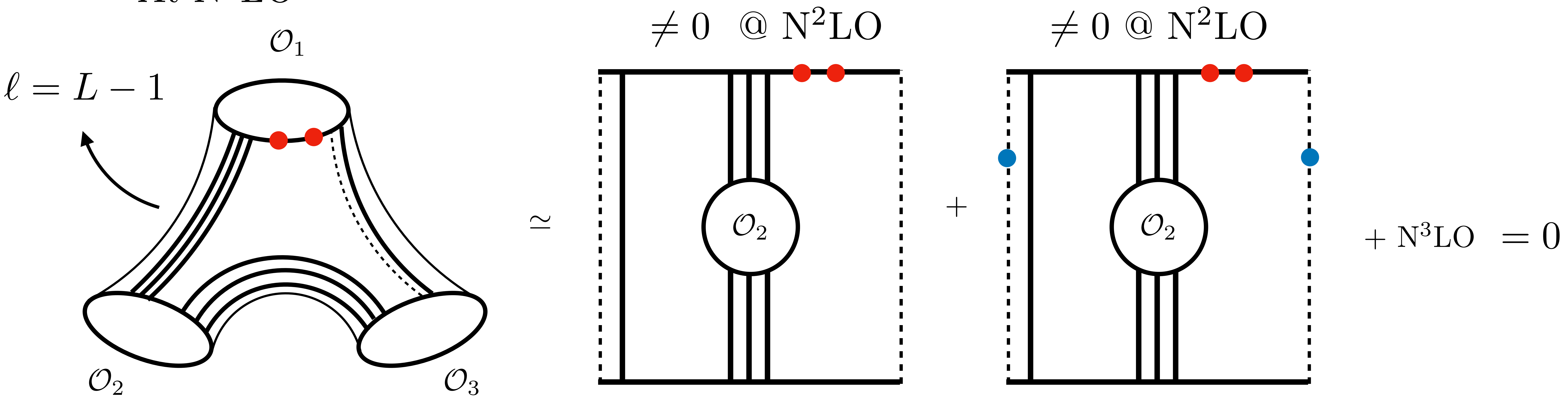

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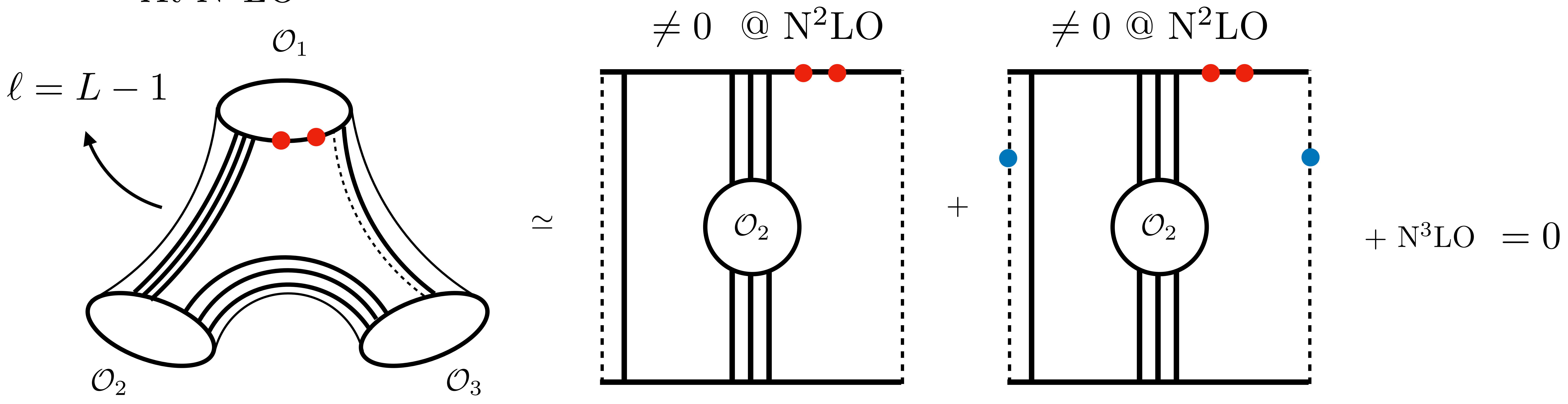

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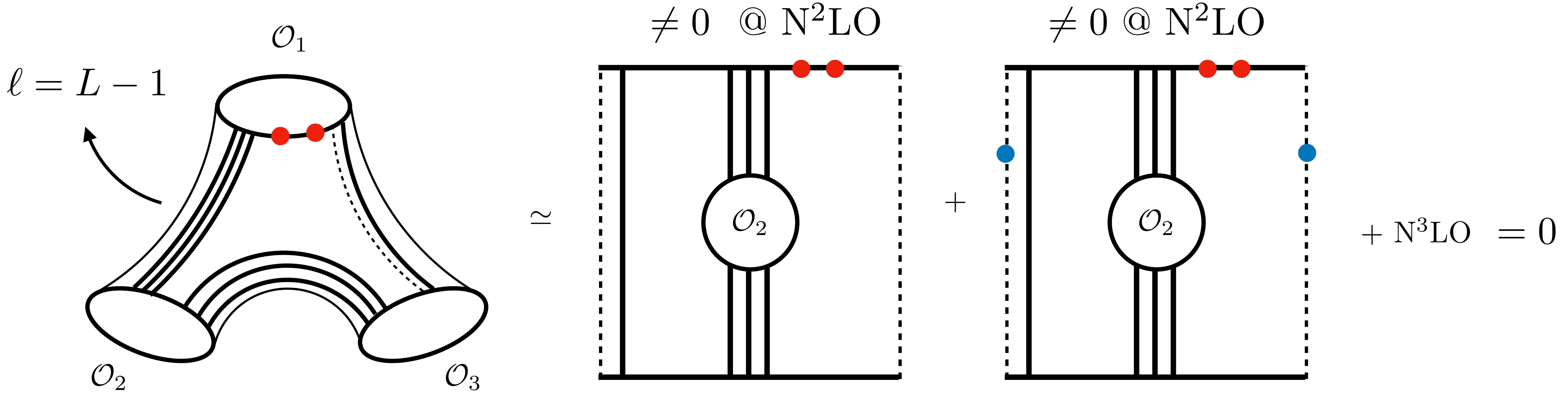

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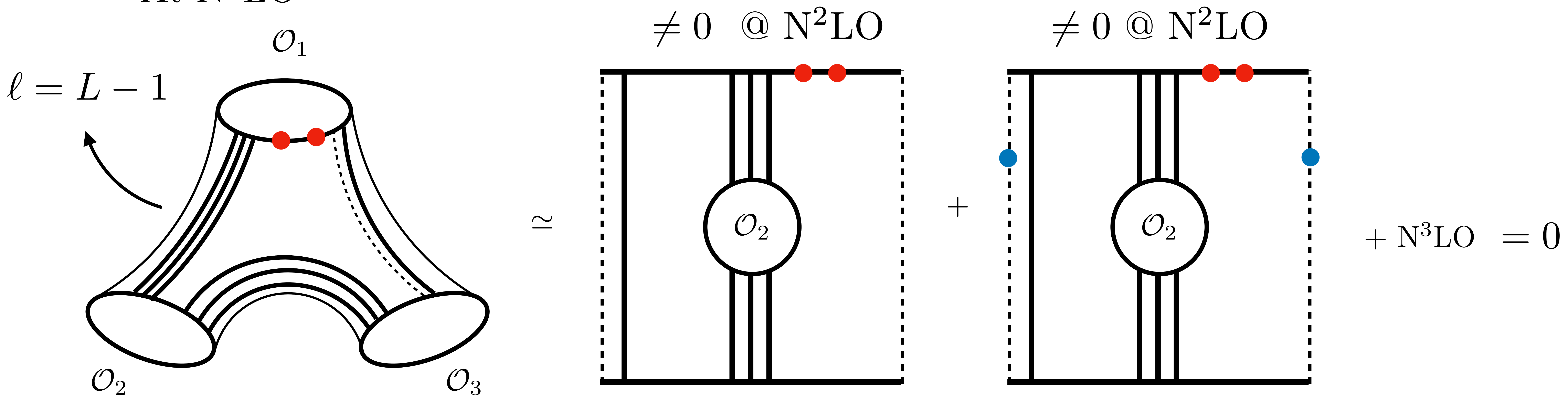

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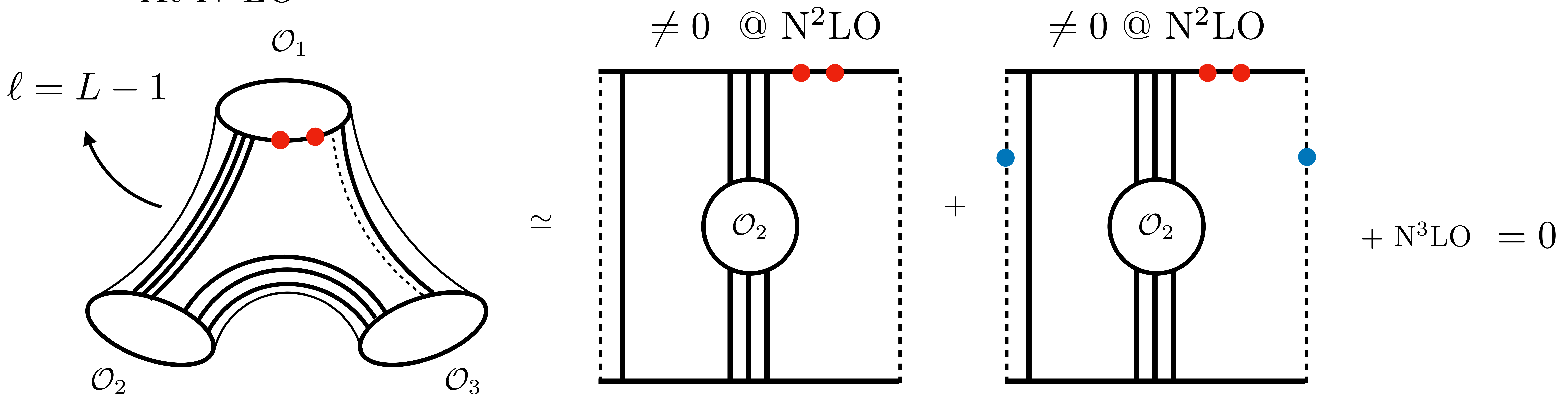

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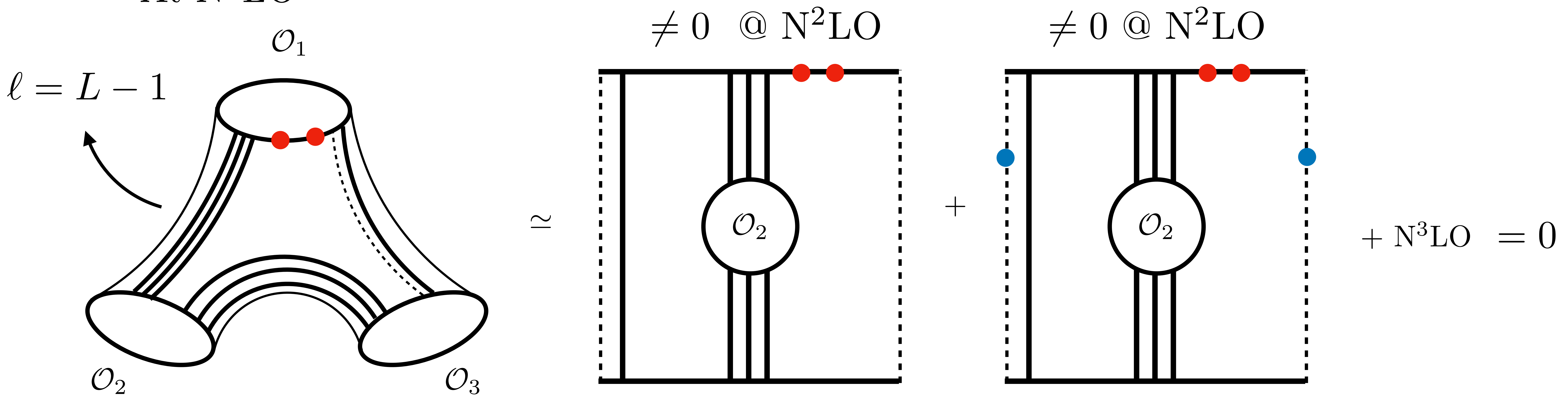

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Formulas

$$\mathcal{Q}(u) \equiv \left(\prod_{k=1}^{J} \frac{u - v_k}{\sqrt{x_k^+ x_k^-}}\right) e^{\frac{g^2}{2} Q_1^+ H_1^+(u) + \frac{g^2}{2} Q_1^- H_1^-(u) + O(g^4)}$$

$$\mathcal{Q}(u) \equiv \prod_{k=1}^{J} \frac{u - v_k}{\sqrt{x_k^+ x_k^-}} \times e^{\frac{1}{2} g^2 Q_1^+ H_1^+ + \frac{1}{8} g^4 Q_1^+ Q_2^+ H_1^+ - \frac{1}{2} g^4 Q_1^+ H_3^+}$$

and

$$\mu_1(u) = \frac{\pi}{2c_u^2}\Big(1 + \pi^2 g^2 (3t_u^2 - 1) + \pi^4 g^4 \big(\tfrac{5}{6} - 7t_u^2 + \tfrac{11}{2} t_u^4\big) - \frac{g^4}{8} H_1^+ \big(Q_1^+(\mathbf{v}_1) Q_2^+(\mathbf{v}_2) + Q_1^+(\mathbf{v}_2) Q_2^+(\mathbf{v}_1)\big)\Big), \quad (14)$$

$$\begin{aligned}
\mathtt{A}_{\ell,L}(u) &= g^2 (q_1^-)^2 + g^4 \left(\tfrac{1}{2}(q_2^+)^2 + 4\alpha (q_1^-)^2 - 6\pi^2 q_2^+ \delta_{\ell=2}\right), \\
\mathtt{A}_{\ell,L}(u,v) &= g^2 q_1^- \tilde{q}_1^- + g^4 \left(\tfrac{1}{2} q_2^+ \tilde{q}_2^+ + 2\alpha q_2^+ + 4\alpha q_1^- \tilde{q}_1^-\right), \\
\mathtt{B}(u) &= g^2 q_1^- Q_1^- \qquad (25) \\
&+ g^4 \Big(\tfrac{1}{2} q_2^+ Q_2^+ + \alpha Q_2^+ + 4\alpha q_1^- Q_1^- - \pi^2 q_1^- Q_1^- \delta_{\ell=2} + \\
&+ \big(\big(\tfrac{1}{8}(Q_1^+)^2 - \tfrac{1}{4} Q_2^+ - Q_1^+ + \tfrac{3}{8}(Q_1^-)^2\big) q_2^+ - \tfrac{1}{2} q_3^- Q_1^-\big) \delta_{\ell,L-1}\Big),
\end{aligned}$$

$$\begin{aligned}
q_k^+(u) &= i^{k+0} (x^+(u))^{-k} + (-i)^{k+0} (x^-(u))^{-k}, \\
q_k^-(u) &= i^{k-1} (x^+(u))^{-k} + (-i)^{k-1} (x^-(u))^{-k},
\end{aligned}$$

$$Q_1^+ = \oint \frac{u\, du}{4\pi g^2 \sqrt{u^2 - 4g^2}} \log \frac{\mathbb{Q}(u + i/2)}{\mathbb{Q}(u - i/2)}$$

$$\begin{aligned}
H_n^+(u) &\equiv H_n(-1/2 + iu) + H_n(-1/2 - iu), \\
iH_n^-(u) &\equiv H_n(-1/2 + iu) - H_n(-1/2 - iu).
\end{aligned}$$

formalism counter parts. For SL(2) it is simply

$$\mathcal{A}_\ell = \langle \mathcal{Q}, \mathbf{1} \rangle_\ell, \qquad \mathcal{B} = \frac{(2\mathbb{J})!}{(\mathbb{J}!)^2} \langle \mathcal{Q}, \mathcal{Q} \rangle_L, \qquad (A13)$$

and for SU(2) it reads

$$\mathcal{A}_\ell = \Lambda_{\mathcal{A}} \langle\!\langle \mathcal{Q}, \mathbf{1} \rangle\!\rangle_{\ell,L}, \quad \mathcal{B} = \Lambda_{\mathcal{B}} \frac{(2J)!}{(J!)^2} \langle\!\langle \mathcal{Q}, \mathcal{Q} \rangle\!\rangle_{L,L}. \quad (A14)$$

where

$$\Lambda_{\mathcal{A}} = e^{g^4 (\alpha - \delta_{\ell=2} \pi^2)((Q_1^-)^2 + Q_2^+)} \qquad (A15)$$

$$\Lambda_{\mathcal{B}} = \prod_{i,j}^{J} \left(1 - \frac{g^2}{x^+(v_i) x^+(v_j)}\right) \prod_{i,j}^{J} \left(1 - \frac{g^2}{x^-(v_i) x^-(v_j)}\right)$$

$$\begin{aligned}
\mathcal{M}_{\text{NNLO-b}} = {} & \tfrac{1}{2}\left(2(Q_1^+)^2 - iQ_2^- - Q_1^- Q_1^+\right) \partial_1^2 + iQ_1^+ \partial_1^3 + \\
& \tfrac{1}{2}\left(Q_1^- Q_1^+ - (Q_1^+)^2\right) \partial_2^2 - \tfrac{1}{2} i Q_1^+ \partial_2^3 - \tfrac{i}{2} Q_1^+ \partial_1 \partial_2^2 + \\
& \tfrac{1}{2}(iQ_2^- - (Q_1^+)^2) \partial_1 \partial_2 - (\partial_1 \leftrightarrow \partial_L, \partial_2 \leftrightarrow \partial_{L-1}). \qquad (C3)
\end{aligned}$$

$$\Big(\sum_\ell C_\ell^{\bullet\bullet\bullet\circ}\Big)^2 = \frac{(\mathbb{J}_1 + \mathbb{J}_2)!^2}{(2\mathbb{J}_1)!(2\mathbb{J}_2)!} \frac{\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle_\ell^2}{\langle \mathcal{Q}_1, \mathcal{Q}_1 \rangle_{L_1} \langle \mathcal{Q}_2, \mathcal{Q}_2 \rangle_{L_2}}. \quad (26)$$

$$\begin{aligned}
\mathcal{A}_\theta &= \sum_{\mathbf{u} = \alpha \cup \bar{\alpha}} (-1)^{|\bar{\alpha}|} \prod_{n=1}^{\ell} \prod_{i \in \bar{\alpha}} \frac{u_i - \theta_n + i/2}{u_i - \theta_n - i/2} \prod_{j \in \alpha} \frac{u_j - u_i + i}{u_j - u_i}, \\
\mathcal{B}_\theta &= \det \left[\partial_{u_i} \log \left(\prod_{n=1}^{L} \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} \prod_{k \neq j} \frac{u_j - u_k - i}{u_j - u_k + i}\right)\right] \\
&\quad \times \prod_{i<j} \frac{(u_i - u_j)^2}{1 + (u_i - u_j)^2}, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\mu_1(u) &= \frac{\pi}{2c_u^2}\left(1 + g^2 \pi^2 (3t_u^2 - 1) + \ldots\right) \\
\mu_2(u,v) &= \frac{\pi(u - v) s_{u-v}}{c_u c_v}\left(1 + g^2 \pi^2 \left((t_u + t_v)^2 - \tfrac{4}{3}\right) + \ldots\right)
\end{aligned}$$