# Spin chains and quantum symmetries in $\mathcal{N} = 2$ SCFT

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Integrability in Gauge and String Theory

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Based on arXiv: 2106.08449 (with E. Pomoni and R. Rabe) and ongoing work

# Motivation

- Integrable spin chains are a useful tool in studying the spectrum of planar gauge theory
- The primary examples are maximally supersymmetric:
  - 4d $\mathcal{N}=4$ SYM: One loop $\leftrightarrow$ XXX Heisenberg chain [Minahan, Zarembo '02]
  - $3d$ $\mathcal{N}=6$ ABJM theory: Two loops $\leftrightarrow$ Alternating chain [Minahan, Zarembo '09]
- Some marginal deformations are integrable e.g. $\beta$-deformation of $\mathcal{N}=4$ SYM
- How about more general planar QFT's in four dimensions? What types of spin chains do we expect?
- We need the dilatation operator to be part of the symmetry algebra $\Rightarrow$ Conformal Field Theory
- A very large class of CFT's can be obtained by orbifolding the $\mathcal{N}=4$ theory and then marginally deforming
- In this talk I will focus mainly on $\mathcal{N}=2$ SCFT's

# Outline

- Review of the $\mathbb{Z}_2$ orbifold SCFT $\leftrightarrow$ dynamical spin chains
- Coordinate Bethe ansatz for dynamical chains
- Quantum symmetries for orbifold theories
- Interpretation as RSOS models

## $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM

- Start with $\mathcal{N} = 4$ SYM with $\mathrm{SU}(2N)$ gauge group
- Project $(V, X, Y, Z) \to (V, -X, -Y, Z)$ in $R$-symmetry space
- Project by $[\cdots] \to \gamma[\cdots]\gamma$ in colour space, where

$$\gamma = \begin{pmatrix} I_{N\times N} & 0 \\ 0 & -I_{N\times N} \end{pmatrix}$$

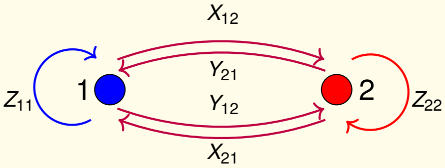
- End up with $\mathcal{N} = 2$ SYM with $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2$ gauge group

$$Z = \begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}, \; X = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & Y_{12} \\ Y_{21} & 0 \end{pmatrix}$$

- $Z$'s adjoints, $X$, $Y$ bifundamentals

## $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM

- Represent using a quiver diagram:




- Superpotential: $\mathcal{W}_{N=4} = ig\mathrm{Tr}(X[Y, Z]) \rightarrow$

$$\mathcal{W}_{N=2} = ig\left(\mathrm{Tr}_2(Y_{21}Z_{11}X_{12} - X_{21}Z_{11}Y_{12}) - \mathrm{Tr}_1(X_{12}Z_{22}Y_{21} - Y_{12}Z_{22}X_{21})\right)$$

- The $\mathbb{Z}_k$ orbifold theory is integrable [Beisert,Roiban '05]
- More general ADE orbifolds [Solovyov '07]

# Marginally deformed orbifold

- Move away from the orbifold point: $g_1 \neq g_2$

$$\mathcal{W} = ig_1 \,\mathrm{Tr}_2(Y_{21}Z_{11}X_{12} - X_{21}Z_{11}Y_{12}) - ig_2 \,\mathrm{Tr}_1(X_{12}Z_{22}Y_{21} - Y_{12}Z_{22}X_{21})$$

- Preserves $\mathcal{N} = 2$ supersymmetry
- Studied in detail in [Gadde,Pomoni,Rastelli '10].
- E.g. $X$ magnons in $Z$-vacuum:

$$\cdots Z_{11}Z_{11}\overset{\overset{p_1}{\rightarrow}}{X}_{12}Z_{22}Z_{22}\cdots Z_{22}\overset{\overset{p_2}{\rightarrow}}{X}_{21}Z_{11}Z_{11}\cdots Z_{11}\overset{\overset{p_3}{\rightarrow}}{X}_{12}Z_{22}Z_{22}\cdots$$

- The S-matrix for $X_{12}X_{21}$ scattering is XXZ-like

$$S_\kappa(p_1,p_2) = -\frac{1 - 2\kappa e^{ip_1} + e^{i(p_1+p_2)}}{1 - 2\kappa e^{ip_2} + e^{i(p_1+p_2)}} \quad \text{where} \quad \kappa = \frac{g_2}{g_1}$$

# Marginally deformed orbifold

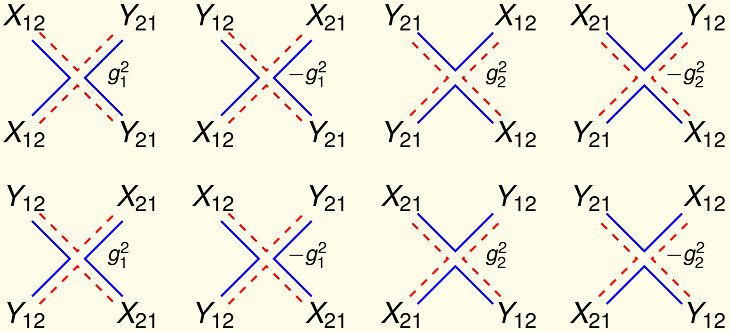
- For $X_{21}X_{12}$ scattering we have $S_\kappa \to S_{1/\kappa}$
- The YBE is not satisfied!

$$S_\kappa S_{1/\kappa} S_\kappa \neq S_{1/\kappa} S_\kappa S_{1/\kappa}$$

- Conclusion was that the deformed theory is not integrable
- We want to revisit this by better understanding the spin chain and its algebraic structure
- Will look separately at the unbroken $\mathrm{SU}(2)$ sector made up of $X, Y$ fields and the "$\mathrm{SU}(2)$-like" sector made up of $X, Z$ fields

# XY sector: Diagrams

- F-term contributions to the Hamiltonian




- Will rescale by $g_1 g_2$ and define $\kappa = g_2/g_1$.

# XY sector: Hamiltonian

- $\mathcal{N} = 2$ picture

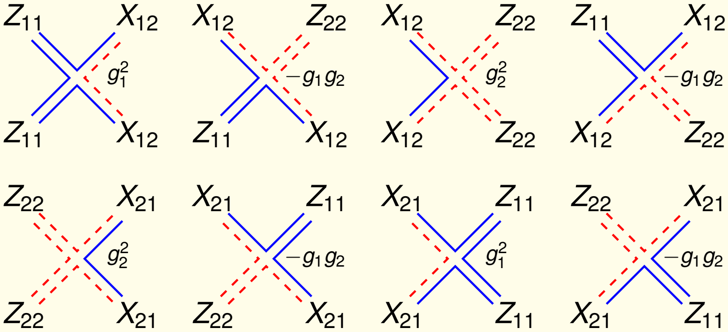
$$\mathcal{H}_{\ell,\ell+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -\kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\kappa^{-1} & \kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} X_{12}X_{21} \\ X_{12}Y_{21} \\ Y_{12}X_{21} \\ Y_{12}Y_{21} \\ X_{21}X_{12} \\ X_{21}Y_{12} \\ Y_{21}X_{12} \\ Y_{21}Y_{12} \end{pmatrix}$$

- "Dynamical $\mathcal{N} = 4$" picture

$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -\kappa^{-1} & 0 \\ 0 & -\kappa^{-1} & \kappa^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathcal{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa & -\kappa & 0 \\ 0 & -\kappa & \kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} XX \\ XY \\ YX \\ YY \end{pmatrix}_i$$

# XZ sector: Diagrams

- F-term contributions to the Hamiltonian




- Will again rescale by $g_1 g_2$ and define $\kappa = g_2/g_1$.

# XZ sector: Hamiltonian

- $\mathcal{N} = 2$ picture

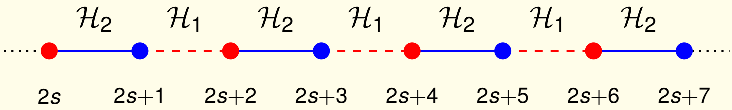
$$\mathcal{H}_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} X_{12}X_{21} \\ X_{12}Z_{22} \\ Z_{11}X_{12} \\ Z_{11}Z_{11} \\ X_{21}X_{12} \\ X_{21}Z_{11} \\ Z_{22}X_{21} \\ Z_{22}Z_{22} \end{pmatrix} .$$

- "Dynamical $\mathcal{N} = 4$" picture

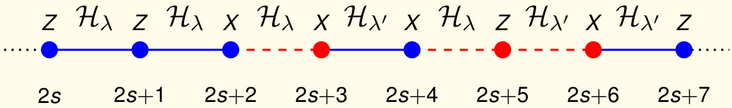
$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa & -1 & 0 \\ 0 & -1 & \kappa^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \mathcal{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -1 & 0 \\ 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} XX \\ XZ \\ ZX \\ ZZ \end{pmatrix}_i$$

# Dynamical spin chains

- The *XY* chain is strictly alternating:



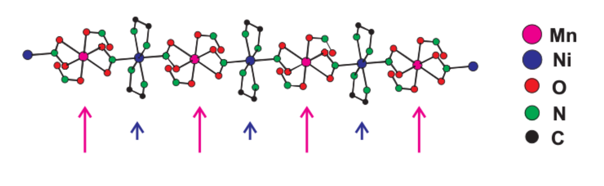

- The *XZ* chain is "dynamical": The Hamiltonian depends on the number of *X*'s crossed.




- Introduced a "dynamical" parameter taking two values $\lambda, \lambda'$ (more later)
- $\lambda \leftrightarrow \lambda'$ when crossing $X$, $Y$, unchanged when crossing $Z$

# Alternating chains

- There is extensive condensed-matter literature on alternating chains, though mostly for the antiferromagnetic case
- A ferromagnetic alternating bond example is in [Sirker et al. '08]
- Alternating spin chains are mathematically very similar
- E.g. the bimetallic chain $\mathrm{MnNi(NO_2)_4(en)_2(en = ethylenediamide)}$




[Feyerherm, Mathonière, Kahn, J. Phys. Condens. Matter 13, 2639 (2001)]

- Have been studied with various techniques such as the recursion method [Viswanath,Müller '94], also long-wavelength approximations [Huang et al. '91]

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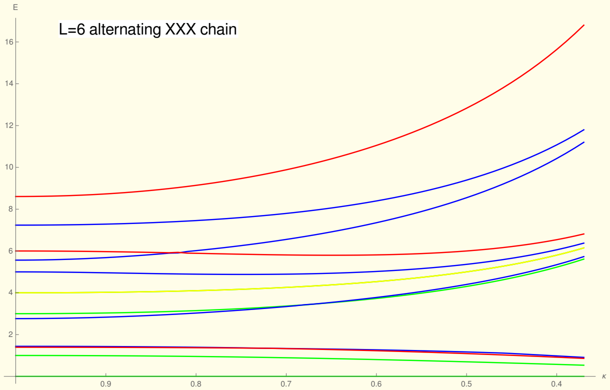
# XY sector: Spectrum

- Explicitly diagonalise the Hamiltonian for short chains




- Red: 3 magnons, Blue: 2 magnons, Green: 1 magnon
- Can we reproduce this from a Bethe ansatz?

# XY sector: One magnon

- XXX case ($\kappa = 1$): Eigenstate $\mathcal{H}\,|p\rangle = E(p)\,|p\rangle$ if

$$|p\rangle = \sum_{\ell} e^{ip\ell}\,|\ell\rangle \quad \text{with} \quad E(p) = 2 - 2\cos(p)$$

- Alternating XXX case ($\kappa \neq 1$):

$$|p\rangle = \sum_{\ell\in 2\mathbb{Z}} A_e e^{ip\ell}|\ell\rangle + \sum_{\ell\in 2\mathbb{Z}+1} A_o e^{ip\ell}|\ell\rangle$$

with

$$r(p;\kappa) = \frac{A_o(p)}{A_e(p)} = \frac{e^{ip}\sqrt{1+\kappa^2 e^{-2ip}}}{\sqrt{1+\kappa^2 e^{2ip}}}$$

- Dispersion relation:

$$E(p) = \frac{1}{\kappa} + \kappa \pm \frac{1}{\kappa}\sqrt{(1+\kappa^2)^2 - 4\kappa^2\sin^2 p}$$

- Naturally uniformised by elliptic functions

# XY sector: Two magnons

- Split equations into non-interacting and interacting
- First solve the non-interacting equations
- Normally, the interacting equations require us to add a second term with swapped momenta

$$|p_1, p_2\rangle = A(p_1, p_2)e^{i(\ell_1 p_1 + \ell_2 p_2)} + A(p_2, p_1)e^{i(\ell_1 p_2 + \ell_2 p_1)}$$

- The ratio gives the $S$-matrix

$$S = \frac{A(p_2, p_1)}{A(p_1, p_2)}$$

- Our case is more complicated!
- Similar to chains studied in [ Medved, Southern, Lavis '91]
- I will focus on the XY sector, but the XZ sector is similar

# XY sector: Additional momenta

- Given $(p_1, p_2)$ need to add all other solutions allowed by momentum and energy conservation
- For XXX that is only $(p_2, p_1)$
- Here we also have $(k_1, k_2)$, $(k_2, k_1)$

$$p_1 + p_2 = k_1 + k_2 = K \ , \ \ E(p_1) + E(p_2) = E(k_1) + E(k_2) = E_2$$

where

$$k_{1,2} = \frac{K}{2} \pm \frac{\pi}{2} \mp \frac{1}{2} \arccos\left(\cos(p_1 - p_2) + \frac{(E_2 - 2(\kappa + 1/\kappa))^2 \cos K}{2 \sin^2 K}\right)$$

- $k$'s are not just permutations of the $p$'s
- Violates Sutherland's criterion for quantum integrability
- Can be thought of as "discrete diffractive scattering" [Bibikov '16] $\Rightarrow$ Can still have solvability

# XY sector: Closed chains

- Can write a Bethe ansatz equation for two magnons
- E.g. Untwisted sector:

$$\boxed{e^{iLp_1} = -\frac{a(p_1,p_2) + \bar{x}(k_1,k_2)b(p_1,p_2)}{a(p_2,p_1) + \bar{x}(k_1,k_2)b(p_2,p_1)} \,.}$$

with

$$\bar{x}(k_1,k_2) = -\frac{a(k_1,k_2) + a(k_2,k_1)e^{iLk_1}}{b(k_1,k_2) + b(k_2,k_1)e^{iLk_1}} \,,$$

where

$$a(p_1,p_2,\kappa) = e^{i(p_1+p_2)} - 2e^{ip_1}r(p_2,\kappa) + r(p_1,\kappa)r(p_2,\kappa) \,,$$
$$b(p_1,p_2,\kappa) = 1 - 2e^{ip_1}r(p_1,\kappa) + r(p_1,\kappa)r(p_2,\kappa)e^{i(p_1+p_2)}$$

- This is an algebraic equation in the $p$'s
- Energies agree with explicit diagonalisation
- Extending to three or more magnons is challenging

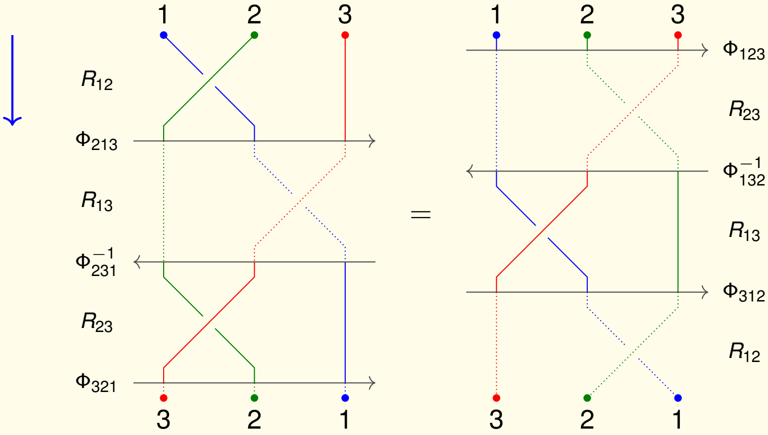
# 3 magnons in the $Z$-vacuum

- Apply the above insights to the $Z$-vacuum
- Here the dispersion relation is much simpler: $E = \kappa + \kappa^{-1} - 2\cos p$
- $1, 2$ magnons solved in [Gadde,Pomoni,Rastelli '10], 3-magnon was not known
- Can make progress in special kinematic limits, e.g: $p_1 \ , \ p_2 + p_3 = \pi$
- Introduce additional momenta $k_1 = p_1$, $k_2 + k_3 = \pi$
- $\{p_1, p_2, k_3\}$ 3-magnon problem can be solved [D. Bozkurt, E. Pomoni, ongoing]
- Interesting quasi-Hopf-like structure ($\Phi_{123}$ is a coassociator)

$$S_{12}\Phi_{312}S_{13}\Phi^{-1}_{132}S_{23}\Phi_{123} = \Phi_{321}S_{23}\Phi^{-1}_{231}S_{13}\Phi_{213}S_{12}$$

- Can be worthwhile to study non-associative scattering more generally

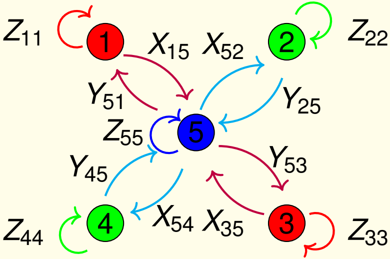
# Quasi-Hopf YBE




$$\Phi_{321} R_{23} \Phi^{-1}_{231} R_{13} \Phi_{213} R_{12} = R_{12} \Phi_{312} R_{13} \Phi^{-1}_{132} R_{23} \Phi_{123}$$

# Non-abelian orbifolds

- Simplest case: Binary dihedral group $\hat{D}_4$




- $\hat{D}_4$ orbifold of $\mathrm{SU}(8N)$

$$\mathcal{W} = 2g_1 \mathrm{Tr}_1(Z_{11}X_{15}Y_{51}) - 2g_2\mathrm{Tr}_2(Z_{22}Y_{25}X_{52}) + 2g_3\mathrm{Tr}_3(Z_{33}X_{35}Y_{53}) - 2g_4\mathrm{Tr}_4(Z_{44}Y_{45}X_{54}) \\ + g_5\mathrm{Tr}_5\left(Z_{55}(X_{52}Y_{25} + X_{54}Y_{45} - Y_{51}X_{15} - Y_{53}X_{35})\right)$$

- 13 fields, 4 ratios of coupling constants $\kappa_i = g_i/g_5$

# The $\hat{D}_4$ spin chain

- Dynamical spin chain of quite peculiar type
- No $X$ or $Y$ vacua, only $Z$ vacua. No nontrivial $\mathrm{SU}(2)$ sectors.
- Fields meeting at node 5 are purely reflected:

$$H(X_{15}Y_{51}) = 4\kappa_1^2 X_{15}Y_{51}$$

- Fields meeting at the outer nodes scatter nontrivially:

$$H(Y_{51}X_{15}) = \frac{1}{2}(Y_{51}X_{15} + Y_{53}X_{35} - X_{52}Y_{25} - X_{54}Y_{45})$$

- So far: coordinate Bethe ansatz for 2-magnon excitations around the $Z$ vacuum [with J. Bath, ongoing].

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- Review of the $\mathbb{Z}_2$ orbifold SCFT $\leftrightarrow$ dynamical spin chains
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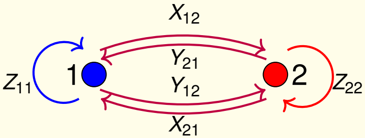
# Quantum Symmetry

[work with E. Andriolo, H. Bertle, E. Pomoni and X. Zhang]

- Let us go back to the $\mathbb{Z}_2$ case




- Naively, $\mathrm{SU}(4)_R \to \mathrm{SU}(2)_L^{i=1,2} \times \mathrm{SU}(2)_R^{i=3,4} \times \mathrm{U}(1)$
- Eight broken generators: $R^1{}_3\ ,\ R^1{}_4\ ,\ R^2{}_3\ ,\ R^2{}_4$ + conjugates
- Relate fields which now belong to different $\mathrm{SU}(N) \times \mathrm{SU}(N)$ representations
- Claim: Can upgrade them to true generators in a *quantum* version of $\mathrm{SU}(4)_R$
- E.g. want to write: $$R^3{}_2 X^a{}_{\hat{a}} = Z^a{}_a \quad , \quad R^2{}_3 Z^a{}_a = X^a{}_{\hat{a}}$$

# Quantum Symmetry

- Gauge indices of all fields to the right need to be flipped

$$\cdots Z_{11} X_{12} Z_{22} Y_{21} X_{12} \cdots \xrightarrow{\Delta(\sigma_{-}^{XZ})} \cdots Z_{11} Z_{11} Z_{11} Y_{12} X_{21} \cdots$$

- Can achieve this with a suitable coproduct $\Rightarrow$ Quantum algebra
- Structure is that of a quantum groupoid [Lu '96, Xu '99]
- Path groupoid: Like a group, but not all compositions of elements are allowed. The allowed paths are those given by the quiver.
- Unbroken generators have the usual algebraic coproduct $\Delta_o(a) = I \otimes a + a \otimes I$
- However, for the broken generators we define:

$$\Delta_o(a) = I \otimes a \; + \; a \otimes \gamma \quad , \quad \text{where} \quad \gamma(X_i) = X_{i+1}$$

- To complete the algebra we also need $\Delta(\gamma) = \gamma \otimes \gamma$

# Twist

- Can move away from the orbifold point by a Drinfeld twist

$$\Delta(a) = F\Delta_o(a)F^{-1}$$

- We require that $\Delta$ preserves the $F$-term relations:

$$\Delta(\sigma_{\pm}^{XZ}) \triangleright \left(X_{12}Z_{22} - \frac{1}{\kappa}Z_{11}X_{12}\right) = 0$$

- A suitable twist is:

$$F = I \otimes \kappa^{-\frac{s}{2}} \text{ where } s = \begin{cases} 1 & \text{if the gauge index is 1} \\ -1 & \text{if the gauge index is 2} \end{cases}$$

- Recall that $\gamma$ flips the gauge index $\Rightarrow s \circ \gamma = -\gamma \circ s$

# Twisted coproduct

- Twisting the unbroken generators has no effect:

$$\Delta(\sigma_3) = (I \otimes \kappa^{-\frac{s}{2}})(I \otimes \sigma_3 + \sigma_3 \otimes I)(I \otimes \kappa^{\frac{s}{2}}) = (I \otimes \sigma_3 + \sigma_3 \otimes I)$$

- But on the broken generators we find:

$$\Delta(\sigma_\pm) = (I \otimes \kappa^{-\frac{s}{2}})(I \otimes \sigma_\pm + \sigma_\pm \otimes \gamma)(I \otimes \kappa^{\frac{s}{2}}) = (I \otimes \sigma_\pm + \sigma_\pm \otimes \gamma\kappa^s)$$

- Defining $K = \gamma\kappa^s$, and also $\Delta_o(s) = s \otimes I$, our final coproducts are:

$$\boxed{\Delta(\sigma_\pm) = I \otimes \sigma_\pm \ + \ \sigma_\pm \otimes K \ \ , \ \ \Delta(K) = K \otimes K}$$

- $K^2 = 1 \quad \Rightarrow$ Compatibility of the coproduct with the algebra product

$$\Delta([\sigma_+, \sigma_-]) = [\Delta(\sigma_+), \Delta(\sigma_-)]$$

- The $\mathrm{SU}(2)$ commutation relations are not deformed, unlike in $U_q(\mathrm{sl}(2))$

# Iterated coproduct

- The twist satisfies the cocycle condition

$$F_{12} \circ (\Delta_o \otimes \mathrm{id})(F) = F_{23} \circ (\mathrm{id} \otimes \Delta_o)(F) =: F_{(3)}$$

  giving

$$\Delta^{(3)}(a) = F_{(3)} \Delta_o^{(3)}(a) F_{(3)}^{-1} = I \otimes I \otimes a + I \otimes a \otimes K + a \otimes K \otimes K$$

- Similarly we find the $L$-site coproduct for the broken/revived generators:

$$\boxed{\Delta^{(L)}(a) = \sum_i \cdots I \otimes I \otimes a_i \otimes K \otimes K \cdots}$$

- By construction, the coproduct preserves the quantum plane relations
- The superpotential is now invariant under all $\mathrm{SU}(3)$ generators

$$\Delta^{(3)}(\sigma^{XY}_{\pm,3}) \triangleright \mathcal{W} = \Delta^{(3)}(\sigma^{XZ}_{\pm,3}) \triangleright \mathcal{W} = \Delta^{(3)}(\sigma^{YZ}_{\pm,3}) \triangleright \mathcal{W} = 0$$

# Is this useful?

- The Hamiltonian does not commute with $\Delta(a)$ (for the broken $a$'s).
- So we do not expect $\kappa$-deformed multiplets
  to map 1-1 to eigenstates of the Hamiltonian
- Let us make an analogy to the Algebraic Bethe Ansatz and assume there exists an $R$-matrix $R(u)$, depending on a spectral parameter $u$
- Our twist is in the quantum plane limit ($u \to \infty$ for rational integrable models)
- The full twist will also be $u$-dependent, such that

$$R(u,\kappa) = F(u)_{21} R(u,\kappa{=}1) F(u)^{-1}_{12}$$

- So we expect a different twist/coproduct for each $u$ (i.e. each eigenvalue of $\mathcal{H}$)
- For BPS states, it turns out that $\Delta^{BPS}(a,\kappa) = \Delta(a, 1/\kappa)$.
- Agrees with the direct diagonalisation in [Gadde,Pomoni,Rastelli '10]

# Example: BPS spectrum

$$X_{12}X_{21}X_{12}X_{21}$$

$$\Big\downarrow \Delta^{BPS}(\sigma_{-}^{XZ})$$

$$X_{12}X_{21}X_{12}Z_{22} + \kappa X_{12}X_{21}Z_{11}X_{12} + X_{12}Z_{22}X_{21}X_{12} + \kappa Z_{11}X_{12}X_{21}X_{12}$$

$$\Big\downarrow \Delta^{BPS}(\sigma_{-}^{XZ})$$

$$\kappa X_{12}X_{21}Z_{11}Z_{11} + X_{12}Z_{22}X_{21}Z_{11} + \tfrac{1}{\kappa}X_{12}Z_{22}Z_{22}X_{21} + \kappa Z_{11}X_{12}X_{21}Z_{11} + Z_{11}X_{12}Z_{22}X_{21} + \kappa Z_{11}Z_{11}X_{12}X_{21}$$

$$\downarrow$$

$$\cdots$$

- To get a closed eigenstate, add the state with $\{1 \leftrightarrow 2, \kappa \leftrightarrow \kappa^{-1}\}$ and impose cyclicity. We find the following BPS state:

$$\kappa \mathrm{Tr}_1(X_{12}X_{21}Z_{11}Z_{11}) + \mathrm{Tr}_1(X_{12}Z_{22}X_{21}Z_{11}) + \frac{1}{\kappa}\mathrm{Tr}_1(X_{12}Z_{22}Z_{22}X_{21})$$

- This state is not protected by $\mathcal{N} = 2$ supersymmetry. The fact that it still has $E = 0$ is a consequence of the quantum symmetry

# Twisted $\mathrm{SU}(4)$ groupoid

- We have extended this to multiplets in the full deformed $\mathrm{SU}(4)$ sector
  [Andriolo, Bertle, Pomoni, Zhang, KZ, to appear]
- Mainly focused on $L=2$ $(\mathbf{20'},\mathbf{15})$ and $L=3$ $(\mathbf{50},\mathbf{10})$ etc.
- The non-BPS multiplets of the closed Hamiltonian at $\kappa=1$ break up into several multiplets as $\kappa\neq 1$
- Main idea: Can partially untwist the Hamiltonian to make the open multiplets agree with those at the orbifold point, while leaving the closed spectrum unchanged. Schematically:

$$R'(u,\kappa)=G(u)_{21}R(u,\kappa)G(u)_{12}^{-1} \quad\Rightarrow\quad H'_{\text{open}}=H_{\text{open}}+\delta H_{\text{open}} \quad (\text{but } H'_c=H_c)$$

- In this basis the splitting is only due to the closed boundary conditions
- First step towards constructing $R(u)$

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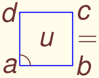
- Interpretation as RSOS models

# RSOS models




$$= W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\Big| u \Big) .$$

- Lattice models where the Boltzmann weights depend on labels around each face (called heights)
- Introduced in [Andrews, Baxter, Forrester '84]
- Original model related to elliptic XYZ spin chain

$$W\left(\begin{matrix} a & a+1 \\ a+1 & a+2 \end{matrix}\,\Big|u\right) = W\left(\begin{matrix} a & a-1 \\ a-1 & a-2 \end{matrix}\,\Big|u\right) = \frac{\theta_1(2\eta-u)}{\theta_1(2\eta)}$$

$$W\left(\begin{matrix} a & a+1 \\ a-1 & a \end{matrix}\,\Big|u\right) = W\left(\begin{matrix} a & a-1 \\ a+1 & a \end{matrix}\,\Big|u\right) = \frac{\sqrt{\theta_1(2\eta(a-1)+w_0)\theta_1(2\eta(a+1)+w_0)}}{\theta_1(2\eta a+w_0)}\,\frac{\theta_1(u)}{\theta_1(2\eta)}$$

$$W\left(\begin{matrix} a & a+1 \\ a+1 & a \end{matrix}\,\Big|u\right) = \frac{\theta_1(2\eta a+w_0+u)}{\theta_1(2\eta a+w_0)}\ ,\quad W\left(\begin{matrix} a & a-1 \\ a-1 & a \end{matrix}\,\Big|u\right) = \frac{\theta_1(2\eta a+w_0-u)}{\theta_1(2\eta a+w_0)}$$

- Neighbouring heights related by $|a-b|=1$.

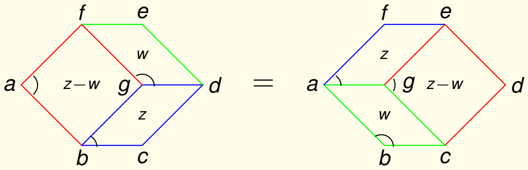
# RSOS models

- The ABF weights satisfy the Star-Triangle relation

$$\sum_g W\begin{pmatrix} f & g \\ a & b \end{pmatrix} \Big| z-w \Big) W\begin{pmatrix} g & d \\ b & c \end{pmatrix} \Big| z \Big) W\begin{pmatrix} f & e \\ g & d \end{pmatrix} \Big| w \Big)$$

$$= \sum_g W\begin{pmatrix} a & g \\ b & c \end{pmatrix} \Big| w \Big) W\begin{pmatrix} f & e \\ a & g \end{pmatrix} \Big| z \Big) W\begin{pmatrix} e & d \\ g & c \end{pmatrix} \Big| z-w \Big) \,,$$

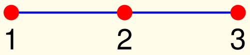
or pictorially:

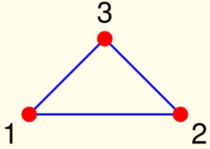



- Guarantees $[T(u), T(u')] = 0 \;\Rightarrow$ integrability of the model

# Adjacency diagrams

- SOS: Range of heights is unrestricted
- RSOS: The heights take values in an interval, $a = 1, \ldots, N$
- More generally can draw an adjacency/incidence diagram
- Any ADE Dynkin diagram can be the incidence diagram for a critical RSOS model [Pasquier '86]
- Generalisations [Di Francesco, Zuber '90, Roche '90, Fendley,Ginparg '89]
- Cyclic SOS models [Pearce, Seaton '88, '89, Kuniba, Yajima '88]: $a_{max} + 1 = a_1$
- Examples: $A_3$ (Ising model), $A_2^{(1)}$



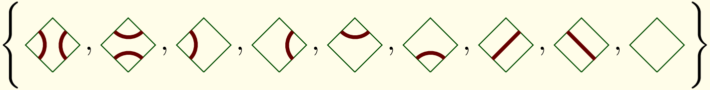



# Dilute RSOS models

- Can relax the $|a-b|=1$ condition to $|a-b|\leq 1$
- Additional Boltzmann weights

$$W\left(\begin{smallmatrix} d & d\pm1 \\ d\pm1 & d\pm1 \end{smallmatrix}\middle|u\right), W\left(\begin{smallmatrix} d & d \\ d & d\pm1 \end{smallmatrix}\middle|u\right), W\left(\begin{smallmatrix} d & d\pm1 \\ d & d \end{smallmatrix}\middle|u\right), W\left(\begin{smallmatrix} d & d \\ d\pm1 & d \end{smallmatrix}\middle|u\right),$$
$$W\left(\begin{smallmatrix} d & d \\ d\pm1 & d\pm1 \end{smallmatrix}\middle|u\right), W\left(\begin{smallmatrix} d & d\pm1 \\ d & d\pm1 \end{smallmatrix}\middle|u\right), W\left(\begin{smallmatrix} d & d \\ d & d \end{smallmatrix}\middle|u\right).$$

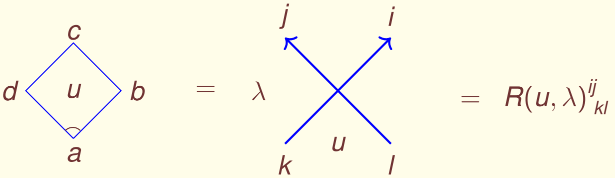
- Such models are called "dilute" [Nienhuis '90,Kostov '91, Warnaar, Nienhuis, Seaton '92, Roche '92, Warnaar '93, Warnaar, Pearce, Seaton, Nienhuis '94, Behrend, Pearce '97]
- Notation comes through the link to loop models



- "Domain walls" separating different values of $d$

# Face-vertex map

- RSOS and vertex models can be mapped to each other
- Graphically:



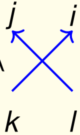

- For SU(2) ABF case introduced by [Felder '94]
- The $R$-matrix depends on dynamical parameter $\lambda = -2\eta d$
- Satisfies a dynamical Yang-Baxter equation

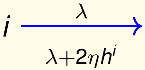
$$R_{23}(u_2 - u_3; \lambda) R_{13}(u_1 - u_3; \lambda + 2\eta h^{(2)}) R_{12}(u_1 - u_2; \lambda)$$
$$= R_{12}(u_1 - u_2; \lambda + 2\eta h^{(3)}) R_{13}(u_1 - u_3; \lambda) R_{23}(u_2 - u_3; \lambda + 2\eta h^{(1)}) \, .$$

# The dynamical YBE

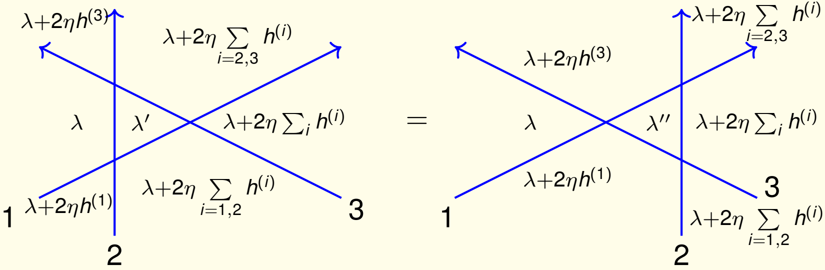
- $\lambda$ is shifted every time a line is crossed

$R^{i\,j}_{k\,l}(u;\lambda) = \lambda$




- Dynamical Yang-Baxter equation:

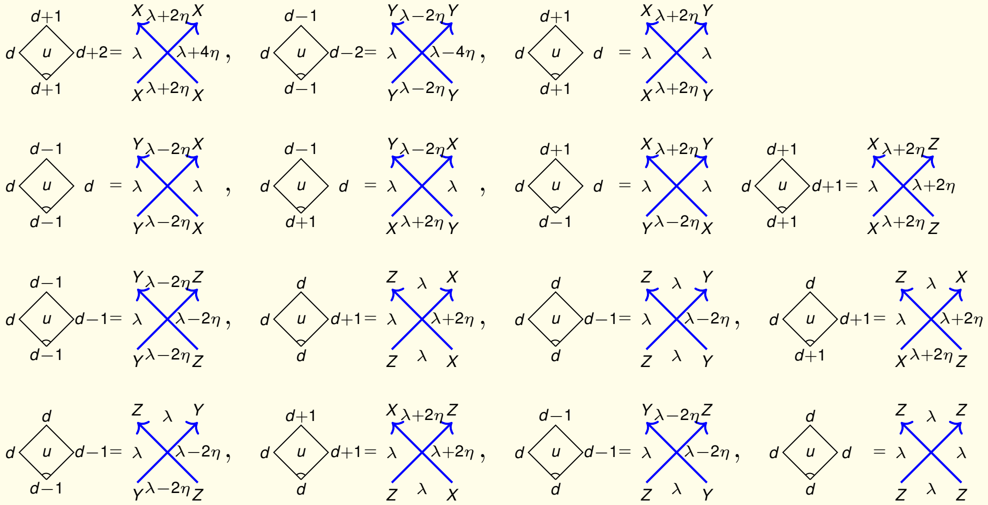



- See also [Yagi '15,17] for other applications to quiver theories

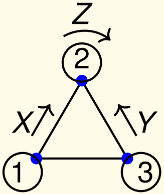
# Relation to the $Z_2$ quiver theory

- Our claim: The $\mathrm{SU}(3)$ sector of the interpolating theory is the vertex model corresponding to a dilute CSOS model
- The dynamical parameter/left height tracks the gauge coupling
- Crossing $X, Y$: $\lambda \to \lambda \pm 2\eta$
- Crossing $Z$: $\lambda$ unchanged (dilute)
- $\lambda \sim \lambda \pm 4\eta$ (cyclic)
- In the RSOS model the cyclicity means that the weights are invariant under $d \to d \pm 2$

# Dynamical 15-vertex model



# XXX spin chain as a dilute CSOS model



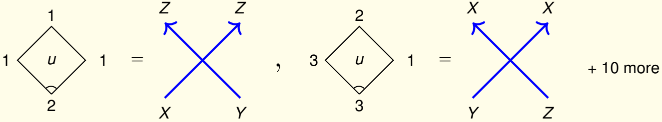

- Need 3 heights $d = 1, 2, 3$, identify $0 \sim 3, 4 \sim 1$
- Use standard $\mathrm{SU}(3)$ Heisenberg $R$-matrix

$$R(u) = \frac{1}{u + i}\,(uI + iP)$$

- Non-dynamical: $R(u)$ doesn't depend on $d$.
- Star-triangle relation is satisfied
- We have a 3-1 map from the XXX spin chain to a face model

# Extension to a 27-vertex model

- The construction naturally allows 12 more vertices

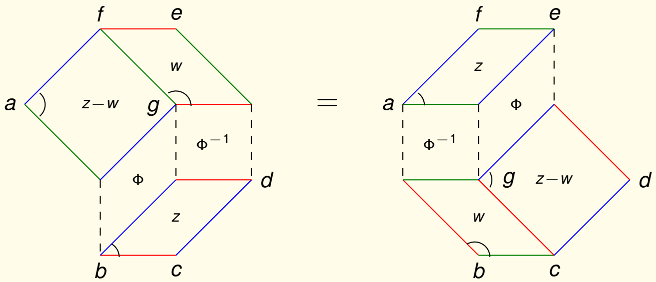



- These interactions are not allowed by $\mathcal{N}=2,4$ SUSY
- Present in $\mathcal{N}=1$ SCFT's (e.g. Leigh-Strassler)

$$\mathcal{W}_{LS} = \kappa \mathrm{Tr}\left( X[Y,Z]_q + \frac{h}{3}\left((X)^3 + (Y)^3 + (Z)^3\right)\right)$$

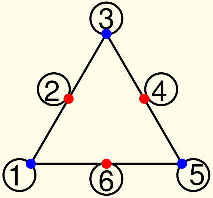
- Quasi-Hopf symmetry [Dlamini-KZ '19]: Expect a quasi-Hopf version of star-triangle relation

# Quasi-Hopf star-triangle relation




(Last Φ on left and first Φ on right not shown)

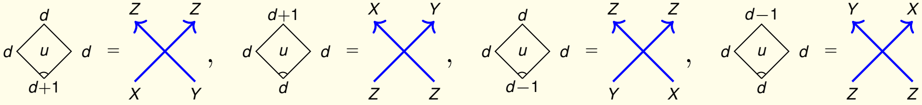
# $\mathbb{Z}_2$ orbifold as a dilute CSOS model




- We define an RSOS/dynamical 15-vertex model as above
- Now the allowed heights are $d = 1, \ldots, 6$
- Dynamical: $R(u, \kappa)$ for $d$ odd, $R(u, 1/\kappa)$ for $d$ even
- Expect elliptic dependence on $u$
- Can the star-triangle relation be satisfied with this input?

# Generalisations

- For $Z_k$ orbifolds, we just need $3k$ heights
- Every $k$ heights have equal Boltzmann weights
- For $\mathcal{N} = 1$ theories, we can extend to a 19-vertex model




- Vertices such as $XZ \to YY$ are not possible for $k > 1$
- Conjecture: Spin chains for a large class of $\mathcal{N} = 1$ quiver theories are described by dynamical 19-vertex models, while those for $\mathcal{N} = 2$ quiver theories are described by dynamical 15-vertex models

# Summary

- Spin chains for $\mathcal{N}=2$ orbifold theories are dynamical
- Magnon scattering can get complicated and intricate
- Showed the 2-magnon CBA for the X and Z vacuum
- The 3-magnon CBA for the Z vacuum is tractable
- The $\hat{D}_4$ orbifold theory brings in even more unusual features
- The naively broken $\mathrm{SU}(4)_R$ generators are not lost but can be upgraded to generators of a quantum groupoid
- We have been studying short chains with the goal of better understanding the implications of this quantum symmetry
- There is a link between supersymmetric gauge theories and dilute RSOS models which we are starting to uncover

# Outlook

- Unrestricted three-magnon problem, also for X vacuum
- Integrability? Solvability?
- Fully understand the quantum group symmetries, extend to full $\mathrm{PSU}(2,2|4)$
- Clarify links to elliptic quantum groups (Felder)
- Find Boltzmann weights realising the required adjacency graphs
- Generalise to $Z_k$ and more general ADE quivers (expect higher-genus theta functions to appear)
- Higher loops? Supergravity side?
- Lessons from link to RSOS models? How general is the mapping of gauge theories to statistical models?

Thanks for your attention!