Strong field amplitudes and classical physics

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Based on

1. Eikonal amplitudes from curved backgrounds
   Tim Adamo, A.C. and Piotr Tourkine
   SciPost Phys. 13 (2022)
2. Classical physics from amplitudes on curved backgrounds
   Tim Adamo, A.C. and Anton Ilderton
   JHEP 08 (2022) 281
3. All orders waveforms from amplitudes
   Tim Adamo, A.C., Anton Ilderton and Sonja Klisch
   arXiv:2210.04696
4. Large gauge transformations and the structure of amplitudes
   A.C., Asaad Elkhidir, Anton Ilderton and Donal O'Connell
   arXiv:2211.16438

Outline

Motivation

- The post-Minkowskian approximation in general relativity
- On-shell data as natural building blocks (KMOC)

Classical observables on curved background

- The post-background approximation
- Strong field amplitudes as natural building blocks

Main results

- Recovering memory effects neglected perturbatively
- Relation between 3-points and large gauge transformations
- Self-force results on plane wave backgrounds

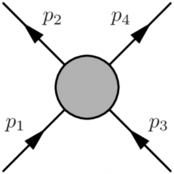
The two-body problem in GR

- Gravitational waves carry fingerprints of a two-body dynamics

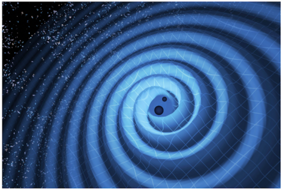
$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad , \quad \ddot{x}_a^\mu = -\Gamma^\mu_{\alpha\beta} \dot{x}_a^\alpha \dot{x}_a^\beta$$

... however, no exact solution is known!

- The post-Minkowskian approximation (PM) has gained a renewed attention after a remarkable state of the art calculation from scattering amplitudes (Zvi Bern et al.)



$\Rightarrow$



Credit: Tim Pyle

The post-Minkowskian approximation

- The change in momentum due to a scattering is

$$\Delta p_1^\mu = -\frac{1}{2}\int_{-\infty}^{+\infty} d\sigma\, \partial^\mu g_{\alpha\beta}(x_1(\sigma)) p_1^\alpha(\sigma) p_1^\beta(\sigma)$$

Expanding around straight trajectories in the weak field limit

$$x_a^\mu(\sigma) = x_{a,0}^\mu + \sigma p_a^\mu + \ldots \quad ; \quad h^{\mu\nu}(x(\sigma)) = -16\pi G \mathcal{P}^{\mu\nu\alpha\beta} T_{\alpha\beta} + \ldots$$

Classical result at 1PM

The Fourier domain computation contains a scattering amplitude

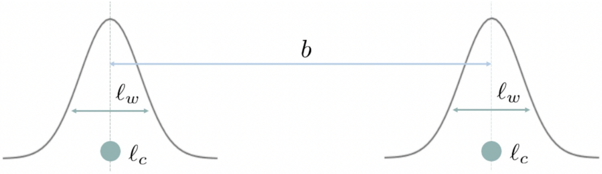
$$\Delta p_1^\mu = \int_q e^{iq\cdot b} \hat{\delta}\,(q\cdot p_1)\,\hat{\delta}\,(q\cdot p_2)\; iq^\mu\, \underbrace{8\pi G\; p_1^\alpha p_1^\beta \frac{\mathcal{P}_{\alpha\beta;\alpha'\beta'}}{q^2} p_2^{\alpha'} p_2^{\beta'}}_{\mathcal{A}_4^{tree}}$$

The KMOC formalism

- Binary system as superposition of single particle states

$$|\psi\rangle = \int d\Phi\left(p_1\right) d\Phi\left(p_2\right) \phi_1\left(p_1\right) \phi_2\left(p_2\right) e^{\frac{ib\cdot p_1}{\hbar}} \left|p_1 p_2\right\rangle$$

- Classical limit $\leftrightarrow$ Goldilocks relations $\ell_c \ll \ell_w \ll \ell_s$



Credit: Ben Maybee, 2105.10268

Main idea (Kosower, Maybee, O'Connell)

Classical observables from on-shell amplitudes to all PM orders

$$\langle\psi| S^{\dagger}\mathbb{P}_1^{\mu} S |\psi\rangle = p_1^{\mu} + \int_q e^{iq\cdot b}\hat{\delta}\left(q\cdot p_1\right)\hat{\delta}\left(q\cdot p_2\right) iq^{\mu}\mathcal{A}_4^{tree} + \ldots$$

The post-background approximation

- We have defined a classical observable around Minkowski

$$\Delta p^\mu = \int_{-\infty}^{+\infty} d\sigma \partial^\mu g_{\alpha\beta}(x(\sigma)) p^\alpha(\sigma) p^\beta(\sigma) \quad , \quad g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

...however, we could have chosen any curved spacetime

$$g_{\alpha\beta} = g^0_{\alpha\beta} + h_{\alpha\beta}$$

where $g^0_{\alpha\beta}$ is an exact solution to Einstein field equations

Question

Can we recover these expansions around g^0 from KMOC?

QFT on curved backgrounds

- Consider QFT in presence of a non trivial background, where the $\mathcal{S}$-matrix carries information on the background g^0

$$\mathcal{S}|\psi\rangle = \int d\Phi(p', p)\phi(p) \underbrace{\langle p'|\mathcal{S}|p\rangle}_{\text{2-point}} |p'\rangle + \ldots$$

We call the building blocks "strong field amplitudes"

$$\langle p'|\mathcal{S}|p\rangle \quad , \quad \langle p'|\mathcal{S}|p, k^{\eta}\rangle \quad , \quad \langle p'|\mathcal{S}|p, k_1^{\eta_1}, k_2^{\eta_2}\rangle \quad \ldots$$

KMOC on curved backgrounds

Observables on g^0 can be computed from strong field amplitudes

$$\langle \Delta \mathcal{O}\rangle = \lim_{\hbar \to 0} \left[\langle \psi|\mathcal{S}^{\dagger}\hat{\mathcal{O}}\mathcal{S}|\psi\rangle - \langle \psi|\hat{\mathcal{O}}|\psi\rangle \right]$$

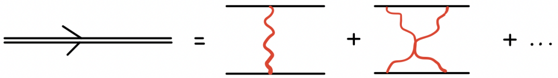
Strong field amplitudes

- The simplest strong field amplitude is a scalar 2-point given by the quadratic part of the action $S[\Phi]$ on $\Phi = \epsilon_1 \Phi_{in} + \epsilon_2 \Phi_{out}$

$$S[\Phi] = \int d^4x\sqrt{-g}\left(g^{\mu\nu}(x)\partial_\mu\Phi(x)\partial_\nu\Phi^*(x) + m^2|\Phi(x)|^2\right)$$

$$\langle p'|\,\mathcal{S}\,|p\rangle := \left.\frac{\partial^2 S[\Phi]}{\partial\epsilon_1\partial\epsilon_2}\right|_{\epsilon_1=\epsilon_2=0}$$

- N-points are defined by the multilinear part of the action. Hard to compute as they resum infinite amplitudes on η



2-points on stationary backgrounds

- Consider the KG equation on a stationary background

$$(\Box + m^2)\Phi(x) = h^{\mu\nu}(x)\partial_\mu\partial_\nu\Phi(x) + \dots$$

We can apply a WKB approximation (Kol, O'Connell, Telem)

$$\chi(x_\perp) := 2M \int_q \hat{\delta}(2P\cdot q)\hat{\delta}(2p\cdot q)\mathrm{e}^{-\mathrm{i}q_\perp\cdot x_\perp/\hbar}\tilde{h}^{\mu\nu}(q)p_\mu p_\nu$$

Relation with traditional amplitudes (Adamo, C., Tourkine)

2-points on g^0 as eikonal amplitudes. The quadratic part of the action resums an infinite number of amplitudes in Minkowski

$$\langle p'|\,\mathcal{S}\,|p\rangle = N\,\hat{\delta}(p_0' - p_0)\int_{x_\perp} e^{-iq_\perp\cdot x^\perp}\left(e^{i\chi(x_\perp)/\hbar} - 1\right)$$

2-points on Kerr

- $\langle p'| \mathcal{S} |p\rangle$ on Kerr depends on the following eikonal amplitude

$$I_a(q_\perp) = \int \mathrm{d}^2 x_\perp \mathrm{e}^{-\mathrm{i} q_\perp \cdot x_\perp/\hbar} |x_\perp - a_\perp|^{\boldsymbol{\alpha}} |x_\perp + a_\perp|^{2\boldsymbol{\beta}}$$

- Analytic continuation provides a KLT-like factorization

$$I_a(q_\perp) = -\left(\tilde{I}_1, \tilde{I}_2\right) \mathcal{K} (I_1, I_2)^{\mathrm{T}}$$

where

$$\mathcal{K} := \frac{\mathrm{i}}{2} \begin{pmatrix} 1 - e^{2\mathrm{i}\pi\boldsymbol{\beta}} & -e^{\mathrm{i}\pi\boldsymbol{\alpha}}\left(-1 + e^{2i\pi\boldsymbol{\beta}}\right) \\ -e^{\mathrm{i}\pi\boldsymbol{\alpha}}\left(-1 + e^{2\mathrm{i}\pi\boldsymbol{\beta}}\right) & 1 - e^{2\mathrm{i}\pi(\boldsymbol{\alpha}+\boldsymbol{\beta})} \end{pmatrix}$$

- Complex poles at $i\alpha_\pm(s) = n$, $n \in N$ (Adamo, C., Tourkine)

2-point on plane wave backgrounds

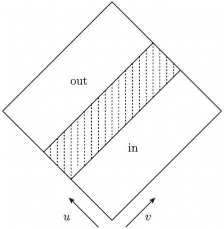
- We can also consider "strong field amplitudes" on non stationary backgrounds like gravitational plane waves

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\mathrm{d}v - H_{ab}(u)\,x^a x^b\,(\,\mathrm{d}u)^2 - \mathrm{d}x^{\perp}\mathrm{d}x^{\perp}$$

- A scalar 2-point is given by

$$\langle p'|\mathcal{S}|p\rangle = \frac{4\pi\,\hat{\delta}\left(p'_+ - p_+\right)}{\sqrt{|\det(c)|\hbar}}\,e^{-\frac{\mathrm{i}}{2p_+\hbar}\mathrm{q}_\perp\cdot c^{-1}\cdot\mathrm{q}_\perp}$$

where c is a 2×2 matrix encoding classical memory effects



Observables from 2-points

- 2-points can be used to construct a semiclassical final state

$$\mathcal{S}|\psi\rangle = \int d\Phi\left(p', p\right)\phi_b(p)\left\langle p'\left|\mathcal{S}\right| p\right\rangle\left|p'\right\rangle$$

- For Schwarzschild and Kerr, stationary phase arguments gives

$$\mathcal{S}|\psi\rangle = \int d^4p\,\phi_b(p - \partial_b\chi(b))\,|p\rangle \quad \Rightarrow \quad \Delta p^\mu = \partial_b^\mu\chi(b) + \dots$$

- For plane waves, memory effects appear (Adamo, C., Ilderton)

$$\Delta p^i = \partial_{x^-}E^i_a(x_f)z^a + \dots \quad , \quad E^i_a = b^i_a + \sqrt{G}c^i_a x^-$$

Comparison with on-shell amplitudes

- The 2-point on a plane wave background shows that the impulse has a linear term in $\kappa \sim \sqrt{G}$ (Adamo, C., Ilderton)

$$\Delta p^{\mu} = \sqrt{G} c^{\mu}_{a} z^{a} + \dots$$

- From a perturbative approach, the leading term should be a 4-point Compton amplitude... but this scale as $\kappa^2 \sim G$

$$\Delta p^{\mu} = \int_{q} d\Phi(k)\, \hat{\delta}(2q \cdot p_1)\hat{\delta}^{+}(2q \cdot k - q^2)$$

$$\times \alpha(k-q)\alpha(k) e^{-iz\cdot q} i q^{\mu} \mathcal{A}_4^{tree} \sim G$$

Solution (C., Ilderton, Elkhidir, O'Connell)

3-point amplitudes on Minkowski are actually non vanishing when large gauge transformations are included in the LSZ reduction

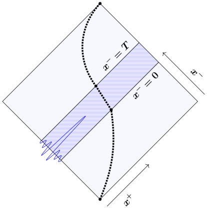
Beyond geodesics motion

- Consider a 3-point amplitude for a scalar particle emitting a graviton on a plane wave background (Adamo, Ilderton)

$$\langle p', k^\eta | \, \mathcal{S} \, | p \rangle \sim \frac{2i\kappa}{\hbar^{3/2}} \int_x \frac{e^{i\mathcal{V}(x)}}{\sqrt{|E(x)|}} \mathcal{E}^\eta_{\mu\nu}(k;x) P^\mu(x) P'^\nu(x)$$

$$\mathcal{V}(x) := \int_y \frac{\theta(x-y) P(y) \cdot \bar{K}(y)}{p_+ - k_+}$$

- If we use this strong field amplitude with KMOC we obtain observables containing a series of all order PM contributions.



Strong field waveform

- Consider this operator for the radiation emitted on $\mathcal{I}^+$

$$\mathbb{O}_{\vec{\mu}}(u, r, \hat{x}) = -\frac{i\hbar^2}{4\pi r}\int_0^\infty \hat{\mathrm{d}}\omega\, \mathrm{e}^{-i\omega u}\, \underbrace{C^\eta_{\vec{\mu}}(k)}_{helicity}\, a_\eta(k)\Big|_{k=\hbar\omega\hat{x}} + \text{ c.c.}$$

- The waveform $W_{\vec{\mu}}$ is the leading coefficient in $1/r$

$$W_{\mu\nu\sigma\rho}(u, \hat{x}) = \frac{i\kappa}{2\pi\hbar^{\frac{1}{2}}}\int_0^\infty \hat{\mathrm{d}}\omega \mathrm{e}^{-i\omega u} k_{[\mu}\varepsilon^{-\eta}_{\nu]} k_{[\sigma}\varepsilon^{-\eta}_{\rho]}$$

$$\times \int \mathrm{d}\Phi(p') \underbrace{\langle\psi|\, \mathcal{S}^\dagger\, |p'\rangle\, \langle p', k^\eta|\, \mathcal{S}\, |\psi\rangle}_{LO}\, |_{k=\hbar\omega\hat{x}} + \text{ c.c.}$$

Strong field waveform

General result - 1PB (post-background)

$$W_{\mu\nu\sigma\rho} = -\frac{\kappa^2}{\pi}\hat{x}_{[\mu}\hat{x}_{[\sigma}\int_y \delta(u-\overline{\mathcal{V}}(y))\left[\mathcal{D}^2 T^0_{\rho]\nu]}(\hat{x},y) - \mathcal{D}T^1_{\rho]\nu]}(\hat{x},y)\right]$$

$$T^0_{\nu\rho}(\hat{x},y) := \frac{\mathbb{P}_{\nu\alpha}(\hat{x},y)\mathbb{P}_{\rho\beta}(\hat{x},y)P^\alpha(y)P^\beta(y) - \frac{1}{2}\eta_{\nu\rho}m^2}{\sqrt{|E(y)|}}$$

$$T^1_{\nu\rho}(\hat{x},y) := \frac{\sigma_{\nu\rho}(y)}{\hat{x}_+\sqrt{|E(y)|}}p_+^2$$

Impulsive wave for $\nu \sim \sqrt{G}\lambda|u|$ (Adamo, C., Ilderton, Klisch)

$$W_{\mu\nu\sigma\rho} = -\frac{\kappa^2 p_+}{\pi^2\sqrt{8}}\delta^+_{[\mu}\delta^+_{[\sigma}(-1)^{(a)}\delta^a_{\rho]}\delta^a_{\nu]}\frac{\partial^2}{\partial u^2}\left(\frac{\nu\log\left(\nu+\sqrt{\nu^2-1}\right)}{\sqrt{\nu^2-1}}\right)$$

Conclusion

Summary

- Strong field amplitudes are the natural building blocks to study perturbation theory around non trivial backgrounds
- Observables from the classical limit of strong field amplitudes

Main results

- Recovering memory effects which were neglected
- 3-point amplitudes are non vanishing and related to memory
- Self-force results from strong field amplitudes

Main message

We can gain a deeper understanding of perturbation theory on a flat spacetime by studying amplitudes on curved backgrounds