Gravitational Scattering at 4th Post-Minkowskian Order.

Gregor Kälin

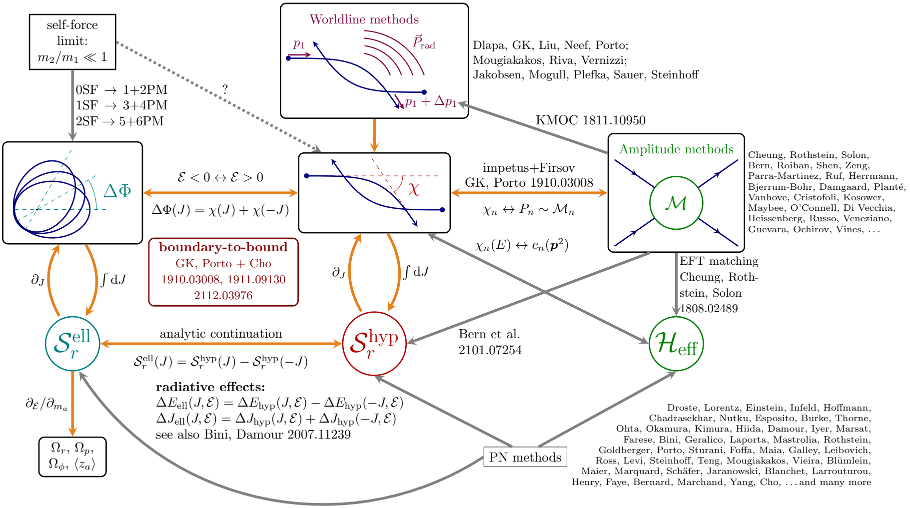
based on
[2006.01184] [2207.00580] [2209.01091] [2210.05541]
with C. Dlapa, R. Jinno, Z. Liu, J. Neef, R.A. Porto, H. Rubira

15.12.2022

European Research Council
Established by the European Commission

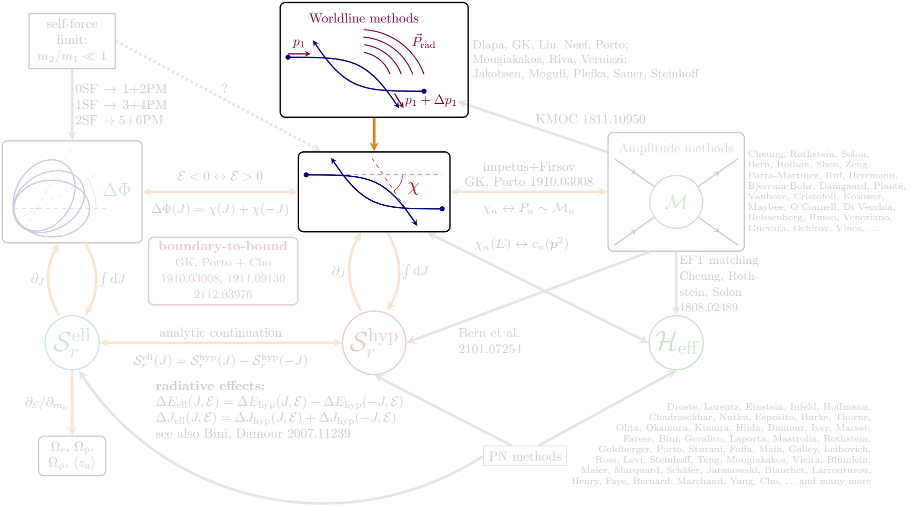
DESY.

Overview.

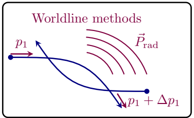


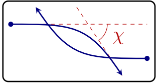
self-force limit: $m_2/m_1 \ll 1$
0SF → 1+2PM
1SF → 3+4PM
2SF → 5+6PM
?
Worldline methods
p_1
$\vec{P}_{\rm rad}$
$p_1 + \Delta p_1$
Dlapa, GK, Liu, Neef, Porto;
Mougiakakos, Riva, Vernizzi;
Jakobsen, Mogull, Plefka, Sauer, Steinhoff
KMOC 1811.10950
$\Delta\Phi$
$\mathcal{E} < 0 \leftrightarrow \mathcal{E} > 0$
$\Delta\Phi(J) = \chi(J) + \chi(-J)$
χ
impetus+Firsov
GK, Porto 1910.03008
$\chi_n \leftrightarrow P_n \sim \mathcal{M}_n$
Amplitude methods
$\mathcal{M}$
Cheung, Rothstein, Solon,
Bern, Roiban, Shen, Zeng,
Parra-Martinez, Ruf, Herrmann,
Bjerrum-Bohr, Damgaard, Planté,
Vanhove, Cristofoli, Kosower,
Maybee, O'Connell, Di Vecchia,
Heissenberg, Russo, Veneziano,
Guevara, Ochirov, Vines, ...
boundary-to-bound
GK, Porto + Cho
1910.03008, 1911.09130
2112.03976
∂_J
$\int \mathrm{d}J$
$\chi_n(E) \leftrightarrow c_n(\boldsymbol{p}^2)$
EFT matching
Cheung, Rothstein, Solon
1808.02489
$\mathcal{S}_r^{\rm ell}$
analytic continuation
$\mathcal{S}_r^{\rm ell}(J) = \mathcal{S}_r^{\rm hyp}(J) - \mathcal{S}_r^{\rm hyp}(-J)$
$\mathcal{S}_r^{\rm hyp}$
Bern et al.
2101.07254
$\mathcal{H}_{\rm eff}$
$\partial_{\mathcal{E}}/\partial_{m_a}$
Ω_r, Ω_p, Ω_ϕ, $\langle z_a \rangle$
radiative effects:
$\Delta E_{\rm ell}(J,\mathcal{E}) = \Delta E_{\rm hyp}(J,\mathcal{E}) - \Delta E_{\rm hyp}(-J,\mathcal{E})$
$\Delta J_{\rm ell}(J,\mathcal{E}) = \Delta J_{\rm hyp}(J,\mathcal{E}) + \Delta J_{\rm hyp}(-J,\mathcal{E})$
see also Bini, Damour 2007.11239
PN methods
Droste, Lorentz, Einstein, Infeld, Hoffmann,
Chadrasekhar, Nutku, Esposito, Burke, Thorne,
Ohta, Okamura, Kimura, Hiida, Damour, Iyer, Marsat,
Farese, Bini, Geralico, Laporta, Mastrolia, Rothstein,
Goldberger, Porto, Sturani, Foffa, Maia, Galley, Leibovich,
Ross, Levi, Steinhoff, Teng, Mougiakakos, Vieira, Blümlein,
Maier, Marquard, Schäfer, Jaranowski, Blanchet, Larrouturou,
Henry, Faye, Bernard, Marchand, Yang, Cho, ... and many more

Overview.


Worldline methods
p_1
$\vec{P}_{\rm rad}$
$p_1 + \Delta p_1$
Dlapa, GK, Liu, Neef, Porto;
Mougiakakos, Riva, Vernizzi;
Jakobsen, Mogull, Plefka, Sauer, Steinhoff
self-force limit: $m_2/m_1 \ll 1$
0SF → 1+2PM
1SF → 3+4PM
2SF → 5+6PM
?
KMOC 1811.10950
$\Delta\Phi$
χ
$\mathcal{E} < 0 \leftrightarrow \mathcal{E} > 0$
$\Delta\Phi(J) = \chi(J) + \chi(-J)$
impetus+Firsov
GK, Porto 1910.03008
$\chi_n \leftrightarrow P_n \sim \mathcal{M}_n$
Amplitude methods
$\mathcal{M}$
Cheung, Rothstein, Solon,
Bern, Roiban, Shen, Zeng,
Parra-Martinez, Ruf, Herrmann,
Bjerrum-Bohr, Damgaard, Planté,
Vanhove, Cristofoli, Kosower,
Maybee, O'Connell, Di Vecchia,
Heissenberg, Russo, Veneziano,
Guevara, Ochirov, Vines, . . .
boundary-to-bound
GK, Porto + Cho
1910.03008, 1911.09130
2112.03976
$\chi_n(E) \leftrightarrow c_n(\boldsymbol{p}^2)$
∂_J
$\int dJ$
EFT matching
Cheung, Rothstein, Solon
1808.02489
$\mathcal{S}_r^{\rm ell}$
analytic continuation
$\mathcal{S}_r^{\rm ell}(J) = \mathcal{S}_r^{\rm hyp}(J) - \mathcal{S}_r^{\rm hyp}(-J)$
$\mathcal{S}_r^{\rm hyp}$
Bern et al.
2101.07254
$\mathcal{H}_{\rm eff}$
$\partial_{\mathcal{E}}/\partial_{m_a}$
radiative effects:
$\Delta E_{\rm ell}(J,\mathcal{E}) = \Delta E_{\rm hyp}(J,\mathcal{E}) - \Delta E_{\rm hyp}(-J,\mathcal{E})$
$\Delta J_{\rm ell}(J,\mathcal{E}) = \Delta J_{\rm hyp}(J,\mathcal{E}) + \Delta J_{\rm hyp}(-J,\mathcal{E})$
see also Bini, Damour 2007.11239
Ω_r, Ω_p, Ω_ϕ, $\langle z_a \rangle$
PN methods
Droste, Lorentz, Einstein, Infeld, Hoffmann,
Chadrasekhar, Nutku, Esposito, Burke, Thorne,
Ohta, Okamura, Kimura, Hiida, Damour, Iyer, Marsat,
Farese, Bini, Geralico, Laporta, Mastrolia, Rothstein,
Goldberger, Porto, Sturani, Foffa, Maia, Galley, Leibovich,
Ross, Levi, Steinhoff, Teng, Mougiakakos, Vieira, Blümlein,
Maier, Marquard, Schäfer, Jaranowski, Blanchet, Larrouturou,
Henry, Faye, Bernard, Marchand, Yang, Cho, . . . and many more

Overview.





[GK, Porto 2006.01184]

PM-EFT: Coupling gravity to massive worldlines

- Purely classical
- Perturbative expansion in G: QFT toolbox
- EFT methodology: Full action $\to$ effective action $\to$ deflection/fluxes/waveform/...
- Complete: radiation, finite size, spin, n-body
- Spin: [Liu, Porto, Yang 2102.10059]

Full theory

$$\mathcal{S}_{\rm EH} = -2M_{\rm Pl}^2 \int {\rm d}^D x \sqrt{-g} R[g]$$

$$\mathcal{S}_{\rm pp} = -\sum_a \frac{m_a}{2} \int {\rm d}\tau_a g_{\mu\nu}(x_a(\tau_a)) \dot{x}_a^\mu(\tau_a) \dot{x}_a^\nu(\tau_a) + \dots$$

$$\mathcal{S}_{\rm GF} + \mathcal{S}_{\rm TD} = \dots$$

- 2-point Lagrangian: 2 terms
- 3-point Lagrangian: 6 terms
- 4-point Lagrangian: 18 terms
- 5-point Lagrangian: 36 terms

Feynman diagrams

(Classical) Path integral for the eff. action

$$e^{i\mathcal{S}_{\rm eff}[x_a]} = \int \mathcal{D}h_{\mu\nu} e^{i\mathcal{S}_{\rm EH}[h] + i\mathcal{S}_{\rm GF}[h] + i\mathcal{S}_{\rm TD}[h] + i\mathcal{S}_{\rm pp}[x_a, h]}$$

Radiation reaction.

- Causal boundary conditions: in-in formalism = doubling of fields/WLs

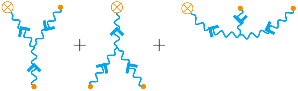
$$\begin{aligned}\mathcal{S}[h_1, h_2] = \mathcal{S}_{\mathrm{EH}}[h_1] - \mathcal{S}_{\mathrm{EH}}[h_2] \\ - \sum_{A=1}^{2} \frac{\kappa m_A}{2} \int \mathrm{d}\tau_A \Big[h_{1,\mu\nu}(x_{1,A}(\tau_A)) \dot{x}^{\mu}_{1,A}(\tau_A) \dot{x}^{\nu}_{1,A}(\tau_A) \\ - h_{2,\mu\nu}(x_{2,A}(\tau_A)) \dot{x}^{\mu}_{2,A}(\tau_A) \dot{x}^{\nu}_{2,A}(\tau_A) \Big]\end{aligned}$$

- BUT: when computing the *variation* of the action it simplifies almost to the usual in-out rules with the difference Feynman → ret/adv propagator.
- Remember also Stefano's (PN-EFT) and Gustav's (WQFT) talk.

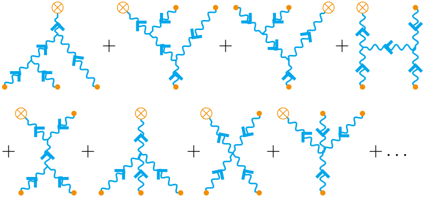
Diagrammatically:

$$\left.\frac{\delta \mathcal{S}_{\mathrm{eff}}[x_+, x_-]}{\delta x_{1-}^{\alpha}}\right|^{\mathrm{1PM}}_{\substack{x_- \to 0 \\ x_+ \to x}} =$$

$$\left.\frac{\delta \mathcal{S}_{\mathrm{eff}}[x_+, x_-]}{\delta x_{1-}^{\alpha}}\right|^{\mathrm{2PM}}_{\substack{x_- \to 0 \\ x_+ \to x}} =$$



$$\left.\frac{\delta \mathcal{S}_{\mathrm{eff}}[x_+, x_-]}{\delta x_{1-}^{\alpha}}\right|^{\mathrm{3PM}}_{\substack{x_- \to 0 \\ x_+ \to x}} =$$



Trajectories.

- Variation = equations of motion:

$$\left.\frac{\delta \mathcal{S}_{\rm eff}^{\rm in\text{-}in}}{\delta x_{a-}^{\mu}}\right|_{\substack{x_- \to 0 \\ x_+ \to x}} = 0$$

- Compute trajectories in a Post-Minkowskian expansion:

$$x_a^{\mu}(\tau_1) = b_a^{\mu} + u_a^{\mu}\tau_a + \sum_n G^n \delta^{(n)} x_a^{\mu}(\tau_a)$$

  - $b = b_1 - b_2$ impact parameter
  - u_a incoming velocities at past infinity

$$u_1 \cdot u_2 = \gamma\,, \quad u_a \cdot b = 0\,.$$

  - Single scale γ

Observables.

Using the above trajectories we can directly compute observables:

Impulse

$$\Delta p_1^\mu = m_1 \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \ddot{x}_1^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \frac{\delta \mathcal{L}_n^{\text{in-in}}}{\delta x_{1-}^\nu} \Bigg|_{\substack{x_- \to 0 \\ x_+ \to x}} ,$$

Generic "boundary" observable

$$\Delta \mathcal{O}^{\mu_1 \cdots \mu_n} = \int_{-\infty}^{+\infty} \mathrm{d}\tau \mathcal{I}_{\mathcal{O}}^{\mu_1 \cdots \mu_n}[x_{1\pm}(\tau), x_{2\pm}(\tau)]$$

The whole procedure is fully automatized in the GiNaC-based C++ library WoLF (developed with J. Neef)

(to be published soon)

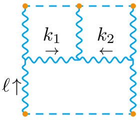
Integration procedure at 4PM.

Short summary

Most of it was discussed in Christoph's talk this morning.

- PaVe-style tensor reduction
- Map to integral families
- IBP reduction (careful about ret/adv vs Feynman): 1094 MIs
- Compute MIs by method of differential equation
- Compute (independent) boundary integrals using the method of regions
  - cons: $\mathrm{pot} + \mathrm{Re}(\mathrm{rad}^2)$ (with Feynman $i0$)
  - full: $\mathrm{pot} + \mathrm{rad}^1 + \mathrm{rad}^2$ (with ret/adv $i0$)
  - pot: Direct via parametrization
  - rad: PN-style factorization in momentum space
    4PM rad^2 example: $\ell \sim (v_\infty, 1)\,, k_i \sim (v_\infty, v_\infty)$

$$\int_{\ell,k_1,k_2} \frac{\delta(k_1-\ell)\delta(k_2+\ell)\delta(\ell)}{\ell^2(\ell-k_1)^2(k_2+\ell-q)^2(l-q)^2(k_1+k_2)^2k_2^2k_1^2}$$



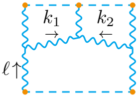
Integration procedure at 4PM.

Short summary

Most of it was discussed in Christoph's talk this morning.

- PaVe-style tensor reduction
- Map to integral families
- IBP reduction (careful about ret/adv vs Feynman): 1094 MIs
- Compute MIs by method of differential equation
- Compute (independent) boundary integrals using the method of regions
  - cons: $\mathrm{pot} + \mathrm{Re}(\mathrm{rad}^2)$ (with Feynman $i0$)
  - full: $\mathrm{pot} + \mathrm{rad}^1 + \mathrm{rad}^2$ (with ret/adv $i0$)
  - pot: Direct via parametrization
  - rad: PN-style factorization in momentum space
    4PM rad^2 example: $\ell \sim (v_\infty, 1)\,, k_i \sim (v_\infty, v_\infty)$

$$\int_{\ell,k_1,k_2} \frac{\delta(k_1-\ell)\delta(k_2+\ell)\delta(\ell)}{\ell^2(\ell-k_1)^2(k_2+\ell-q)^2(l-q)^2(k_1+k_2)^2k_2^2k_1^2}$$



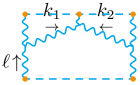
Integration procedure at 4PM.

Short summary

Most of it was discussed in Christoph's talk this morning.

- PaVe-style tensor reduction
- Map to integral families
- IBP reduction (careful about ret/adv vs Feynman): 1094 MIs
- Compute MIs by method of differential equation
- Compute (independent) boundary integrals using the method of regions
  - cons: $\mathrm{pot} + \mathrm{Re}(\mathrm{rad}^2)$ (with Feynman $i0$)
  - full: $\mathrm{pot} + \mathrm{rad}^1 + \mathrm{rad}^2$ (with ret/adv $i0$)
  - pot: Direct via parametrization
  - rad: PN-style factorization in momentum space
    4PM rad^2 example: $\ell \sim (v_\infty, 1)\,, k_i \sim (v_\infty, v_\infty)$

$$\int_{\ell,k_1,k_2} \frac{\delta(k_1-\ell)\delta(k_2+\ell)\delta(\ell)}{\ell^2(\ell-k_1)^2(k_2+\ell-q)^2(l-q)^2(k_1+k_2)^2 k_2^2 k_1^2}$$



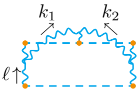
Integration procedure at 4PM.

Short summary

Most of it was discussed in Christoph's talk this morning.

- PaVe-style tensor reduction
- Map to integral families
- IBP reduction (careful about ret/adv vs Feynman): 1094 MIs
- Compute MIs by method of differential equation
- Compute (independent) boundary integrals using the method of regions
  - cons: $\mathrm{pot} + \mathrm{Re}(\mathrm{rad}^2)$ (with Feynman $i0$)
  - full: $\mathrm{pot} + \mathrm{rad}^1 + \mathrm{rad}^2$ (with ret/adv $i0$)
  - pot: Direct via parametrization
  - rad: PN-style factorization in momentum space
    4PM rad^2 example: $\ell \sim (v_\infty, 1)\,, k_i \sim (v_\infty, v_\infty)$

$$\int_{\ell,k_1,k_2} \frac{\delta(k_1-\ell)\delta(k_2+\ell)\delta(\ell)}{\ell^2(\ell-k_1)^2(k_2+\ell-q)^2(l-q)^2(k_1+k_2)^2k_2^2k_1^2}$$



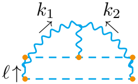
Integration procedure at 4PM.

Short summary

Most of it was discussed in Christoph's talk this morning.

- PaVe-style tensor reduction
- Map to integral families
- IBP reduction (careful about ret/adv vs Feynman): 1094 MIs
- Compute MIs by method of differential equation
- Compute (independent) boundary integrals using the method of regions
  - cons: $\mathrm{pot} + \mathrm{Re}(\mathrm{rad}^2)$ (with Feynman $i0$)
  - full: $\mathrm{pot} + \mathrm{rad}^1 + \mathrm{rad}^2$ (with ret/adv $i0$)
  - pot: Direct via parametrization
  - rad: PN-style factorization in momentum space
    4PM rad^2 example: $\ell \sim (v_\infty, 1)\,, k_i \sim (v_\infty, v_\infty)$

$$\int_{\ell,k_1,k_2} \frac{\delta(k_1-\ell)\delta(k_2+\ell)\delta(\ell)}{\ell^2(\ell-k_1)^2(k_2+\ell-q)^2(l-q)^2(k_1+k_2)^2k_2^2k_1^2}$$



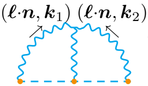
Integration procedure at 4PM.

Short summary

Most of it was discussed in Christoph's talk this morning.

- PaVe-style tensor reduction
- Map to integral families
- IBP reduction (careful about ret/adv vs Feynman): 1094 MIs
- Compute MIs by method of differential equation
- Compute (independent) boundary integrals using the method of regions
  - cons: $\mathrm{pot} + \mathrm{Re}(\mathrm{rad}^2)$ (with Feynman $i0$)
  - full: $\mathrm{pot} + \mathrm{rad}^1 + \mathrm{rad}^2$ (with ret/adv $i0$)
  - pot: Direct via parametrization
  - rad: PN-style factorization in momentum space
    4PM rad^2 example: $\ell \sim (v_\infty, 1)\,, k_i \sim (v_\infty, v_\infty)$

$$\int_{\ell,k_1,k_2} \frac{\delta(k_1-\ell)\delta(k_2+\ell)\delta(\ell)}{\ell^2(\ell-k_1)^2(k_2+\ell-q)^2(l-q)^2(k_1+k_2)^2k_2^2k_1^2}$$



$$\int_{\boldsymbol{\ell}} \frac{1}{[\boldsymbol{\ell}^2]^2[(\boldsymbol{\ell}-\boldsymbol{q})^2]^2} \int_{\boldsymbol{k}_1,\boldsymbol{k}_2} \frac{v_\infty^{2d-6}}{(\boldsymbol{k}_1-\boldsymbol{k}_2)^2(\boldsymbol{k}_2^2-(\boldsymbol{\ell}\cdot\boldsymbol{n})^2)(\boldsymbol{k}_1^2-(\boldsymbol{\ell}\cdot\boldsymbol{n})^2)}$$

[Galley, Leibovich, Porto, Ross 1511.07379]

Radiation-reacted impulse at 4PM.

$$\Delta^{(n)} p_1^\mu = c_{1b}^{(n)}\, \hat{b}^\mu + \sum_a c_{1\check{u}_a}^{(n)}\, \check{u}_a^\mu$$

$$\begin{aligned}
\frac{c_{1b}^{(4)\mathrm{tot}}}{\pi} = &-\frac{3h_1 m_1 m_2\left(m_1^3+m_2^3\right)}{64\left(\gamma^2-1\right)^{5/2}} + m_1^2 m_2^2 (m_1+m_2)\Bigg[\frac{21\sqrt{\gamma^2-1}h_2 E\left(\frac{\gamma-1}{\gamma+1}\right)^2}{32(\gamma-1)^2(\gamma+1)} + \frac{3h_3 K\left(\frac{\gamma-1}{\gamma+1}\right)^2}{16\left(\gamma^2-1\right)^{3/2}} - \frac{3h_4 E\left(\frac{\gamma-1}{\gamma+1}\right)K\left(\frac{\gamma-1}{\gamma+1}\right)}{16\left(\gamma^2-1\right)^{3/2}} + \frac{\pi^2 h_5}{8\sqrt{\gamma^2-1}} + \frac{h_6\log\left(\frac{\gamma-1}{2}\right)}{16\left(\gamma^2-1\right)^{3/2}} \\
&+ \frac{3h_7 \mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{(\gamma-1)(\gamma+1)^2} - \frac{3h_7\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)}{4(\gamma-1)(\gamma+1)^2}\Bigg] + m_1^3 m_2^2\Bigg[\frac{h_8}{48\left(\gamma^2-1\right)^3} + \frac{\sqrt{\gamma^2-1}h_9}{768(\gamma-1)^3\gamma^9(\gamma+1)^4} + \frac{h_{10}\log\left(\frac{\gamma+1}{2}\right)}{8\left(\gamma^2-1\right)^2} - \frac{h_{11}\log\left(\frac{\gamma+1}{2}\right)}{32\left(\gamma^2-1\right)^{5/2}} + \frac{h_{12}\log(\gamma)}{16\left(\gamma^2-1\right)^{5/2}} \\
&- \frac{h_{13}\cosh^{-1}(\gamma)}{8(\gamma-1)(\gamma+1)^4} + \frac{h_{14}\cosh^{-1}(\gamma)}{16\left(\gamma^2-1\right)^{7/2}} + \frac{3h_{15}\log\left(\frac{\gamma+1}{2}\right)\coth^{-1}(\gamma)}{4\sqrt{\gamma^2-1}} - \frac{3h_{16}\cosh^{-1}(\gamma)\coth^{-1}(\gamma)}{8\left(\gamma^2-1\right)^2} - \frac{3h_{17}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)}{64\sqrt{\gamma^2-1}} - \frac{3}{32}\sqrt{\gamma^2-1}h_{18}\mathrm{Li}_2\left(\frac{1-\gamma}{\gamma+1}\right)\Bigg] \\
&+ m_1^2 m_2^3\Bigg[\frac{3h_{15}\log\left(\frac{2}{\gamma-1}\right)\log\left(\frac{\gamma+1}{2}\right)}{8\sqrt{\gamma^2-1}} + \frac{3h_{16}\log\left(\frac{\gamma-1}{2}\right)\cosh^{-1}(\gamma)}{16\left(\gamma^2-1\right)^2} + \frac{h_{19}}{48\left(\gamma^2-1\right)^3} + \frac{h_{20}}{192\gamma^7\left(\gamma^2-1\right)^{5/2}} + \frac{h_{21}\log\left(\frac{\gamma+1}{2}\right)}{8\left(\gamma^2-1\right)^2} + \frac{h_{22}\log\left(\frac{\gamma+1}{2}\right)}{16\left(\gamma^2-1\right)^{3/2}} + \frac{h_{23}\log(\gamma)}{2\left(\gamma^2-1\right)^{3/2}} \\
&- \frac{h_{24}\cosh^{-1}(\gamma)}{16\left(\gamma^2-1\right)^3} + \frac{h_{25}\cosh^{-1}(\gamma)}{16\left(\gamma^2-1\right)^{7/2}} + \frac{3h_{26}\cos^{-1}(\gamma)^2}{32\left(\gamma^2-1\right)^{7/2}} + \frac{3h_{27}\log^2\left(\frac{\gamma+1}{2}\right)}{2\sqrt{\gamma^2-1}} + \frac{3h_{28}\log\left(\frac{\gamma+1}{2}\right)\cosh^{-1}(\gamma)}{16\left(\gamma^2-1\right)^2} + \frac{h_{29}\mathrm{Li}_2\left(\frac{1-\gamma}{\gamma+1}\right)}{4\sqrt{\gamma^2-1}} + \frac{3h_{30}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)}{8\sqrt{\gamma^2-1}}\Bigg],
\end{aligned}$$

$$c_{1\check{u}_1}^{(4)\mathrm{tot}} = \frac{9\pi^2 h_{31} m_1 m_2^2 (m_1+m_2)^2}{32\left(\gamma^2-1\right)} + \frac{2h_{32} m_1 m_2^2\left(m_1^2+m_2^2\right)}{\left(\gamma^2-1\right)^3} + m_1^2 m_2^3\Bigg[\frac{4h_{33}}{3\left(\gamma^2-1\right)^3} - \frac{8h_{34}}{3\left(\gamma^2-1\right)^{5/2}} + \frac{8h_{35}\cosh^{-1}(\gamma)}{\left(\gamma^2-1\right)^3} - \frac{16h_{36}\cosh^{-1}(\gamma)}{\left(\gamma^2-1\right)^{3/2}}\Bigg],$$

$$\begin{aligned}
c_{1\check{u}_2}^{(4)\mathrm{tot}} = &-m_1^4 m_2\left(\frac{9\pi^2 h_{31}}{32\left(\gamma^2-1\right)} + \frac{2h_{32}}{\left(\gamma^2-1\right)^3}\right) + m_1^3 m_2^2\Bigg[-\frac{4h_{37}}{3\left(\gamma^2-1\right)^3} + \frac{h_{38}}{705600\gamma^8\left(\gamma^2-1\right)^{5/2}} + \frac{\pi^2 h_{39}}{192\left(\gamma^2-1\right)^2} + \frac{h_{40}\cosh^{-1}(\gamma)}{6720\gamma^9\left(\gamma^2-1\right)^3} + \frac{32h_{41}\cosh^{-1}(\gamma)}{3\left(\gamma^2-1\right)^{3/2}} \\
&+ \frac{8h_{42}\cos^{-1}(\gamma)^2}{\left(\gamma^2-1\right)^2} + \frac{32h_{43}\cosh^{-1}(\gamma)^2}{\left(\gamma^2-1\right)^{7/2}} + \frac{h_{44}\log(2)\cosh^{-1}(\gamma)}{8\left(\gamma^2-1\right)^2} + \frac{3h_{45}\left(\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right) - 4\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)\right)}{16\left(\gamma^2-1\right)^2} \\
&+ \frac{3h_{46}\left(\log\left(\frac{\gamma+1}{2}\right)\cosh^{-1}(\gamma) - 2\mathrm{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)\right)}{8\left(\gamma^2-1\right)^2} + -\frac{h_{47}\left(\mathrm{Li}_2\left(-\left(\gamma-\sqrt{\gamma^2-1}\right)^2\right) - 2\log(\gamma)\cosh^{-1}(\gamma)\right)}{16\left(\gamma^2-1\right)^2}\Bigg] + m_1^2 m_2^3\Bigg[-\frac{2h_{48}}{45\left(\gamma^2-1\right)^3} \\
&+ \frac{h_{49}}{1440\gamma^7\left(\gamma^2-1\right)^{5/2}} + \frac{\pi^2 h_{50}}{48\left(\gamma^2-1\right)^2} + \frac{h_{51}\cosh^{-1}(\gamma)}{480\gamma^8\left(\gamma^2-1\right)^3} - \frac{16h_{52}\cosh^{-1}(\gamma)}{5\left(\gamma^2-1\right)^{3/2}} + \frac{16h_{53}\cos^{-1}(\gamma)^2}{\left(\gamma^2-1\right)^2} + \frac{32h_{54}\cos^{-1}(\gamma)^2}{\left(\gamma^2-1\right)^{7/2}} - \frac{h_{55}\log(2)\cosh^{-1}(\gamma)}{4\left(\gamma^2-1\right)^2} \\
&+ \frac{h_{56}\left(\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right) - 4\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)\right)}{32\left(\gamma^2-1\right)^2} + \frac{h_{57}\left(\log\left(\frac{2}{\gamma+1}\right)\cosh^{-1}(\gamma) + 2\mathrm{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)\right)}{4\left(\gamma^2-1\right)^2} + \frac{h_{58}\left(\mathrm{Li}_2\left(-\left(\gamma-\sqrt{\gamma^2-1}\right)^2\right) - 2\log(\gamma)\cosh^{-1}(\gamma)\right)}{8\left(\gamma^2-1\right)^2}\Bigg].
\end{aligned}$$

Checks on the impulse.

- Recovered conservative (potential+Feynman rad^2) contributions.
  - Previously obtained by [Bern et al.; Dlapa et al.]
- New radiative pieces in agreement with partial results in PN [Bini, Damour, Geralico; Cho, Dandapat, Gopakumar] and PM [Manohar, Ridgway, Shen].
  - e.g. linear response: b-direction, rad^1 pieces.
- Total mechanical impulse checks with PN data.
- Fulfills many consistency conditions: pole cancellation, IR div cancellation, mass-polynomiality, on-shellness, . . .
  (see also [Bini, Damour, Geralico 2210.07165])
- Numerical checks of boundary integrals (also using machine learning techniques) [Jinno, GK, Liu, Rubira 2209.01091]

Massless limit of the impulse.

- No smooth massless limit, keeping s fixed ($\gamma = (s - m_1^2 - m_2^2)/(2m_1 m_2)$):

$$\Delta^{(4)} p_1^\mu \xrightarrow{m_a \to 0} \frac{35\pi s^3}{64 m_1 b^4}\left(7 + 8\log(2) - 12\log(2)^2\right)\hat{b}^\mu + \mathcal{O}(m_a)$$

  Note the absence of $\mathcal{O}(m_a^0)$ terms!
- Comes from conservative + rad[1] (= linear response) pieces.
- Could only be "fixed" by inconsistent conservative-looking contributions (note the mass structure!).
  See also [Damour, Bini, Geralico 2210.07165] using a symmetry & mass-polynomiality argument.

Radiated energy.

- $\Delta E_{\text{hyp}} = P_{\text{rad}} \cdot \frac{m_1 u_1 + m_2 u_2}{M^2}$

$$\begin{aligned}\Delta E^{\text{4PM}}_{\text{hyp}} = -\frac{G^4 M^5 \nu^2}{b^4 \Gamma} \Bigg\{ & \frac{15\pi^2 (\gamma^2-1)\left(27(\gamma^2-1)h_{31} + 2h_{50}\right) + 64(45h_{32} - h_{48})}{1440(\gamma^2-1)^3} + \frac{h_{49}}{1440\gamma^7(\gamma^2-1)^{5/2}} - \operatorname{arccosh}^2(\gamma)\left(\frac{16h_{53}}{(\gamma^2-1)^2} + \frac{32h_{54}}{(\gamma^2-1)^{7/2}}\right) \\
& - \frac{h_{55}\log(2)\operatorname{arccosh}(\gamma)}{4(\gamma^2-1)^2} + \frac{h_{57}\log\left(\frac{2}{\gamma+1}\right)\operatorname{arccosh}(\gamma)}{4(\gamma^2-1)^2} - \frac{h_{58}\log(\gamma)\operatorname{arccosh}(\gamma)}{4(\gamma^2-1)^2} + \operatorname{arccosh}(\gamma)\left(\frac{h_{51}}{480\gamma^8(\gamma^2-1)^3} - \frac{16h_{52}}{5(\gamma^2-1)^{3/2}}\right) \\
& - \frac{h_{56}\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{8(\gamma^2-1)^2} + \frac{h_{56}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)}{32(\gamma^2-1)^2} + \frac{h_{57}\mathrm{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)}{2(\gamma^2-1)^2} + \frac{h_{58}\mathrm{Li}_2\left(-\left(\gamma-\sqrt{\gamma^2-1}\right)^2\right)}{8(\gamma^2-1)^2} \\
& + \nu\Bigg[\frac{4(-45h_{32} + 30h_{33} - 30h_{37} + h_{48})}{45(\gamma^2-1)^3} + \frac{\pi^2\left(54(\gamma^2-1)h_{31} + h_{39} - 4h_{50}\right)}{96(\gamma^2-1)^2} - \operatorname{arccosh}^2(\gamma)\left(\frac{16(h_{42} - 2h_{53})}{(\gamma^2-1)^2} - \frac{64(h_{43} + h_{54})}{(\gamma^2-1)^{7/2}}\right) \\
& + \frac{h_{38} - 490\gamma\left(3840\gamma^7 h_{34} + h_{49}\right)}{352800\gamma^8(\gamma^2-1)^{5/2}} + \frac{(3h_{46} + 2h_{57})\log\left(\frac{\gamma+1}{2}\right)\operatorname{arccosh}(\gamma)}{4(\gamma^2-1)^2} + \frac{(h_{44} + 2h_{55})\log(2)\operatorname{arccosh}(\gamma)}{4(\gamma^2-1)^2} + \frac{(h_{47} + 2h_{58})\log(\gamma)\operatorname{arccosh}(\gamma)}{4(\gamma^2-1)^2} \\
& + \operatorname{arccosh}(\gamma)\left(\frac{53760\gamma^9 h_{35} - 14\gamma h_{51} + h_{40}}{3360\gamma^9(\gamma^2-1)^3} - \frac{32(15h_{36} - 10h_{41} - 3h_{52})}{15(\gamma^2-1)^{3/2}}\right) + \frac{(h_{56} - 6h_{45})\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{4(\gamma^2-1)^2} - \frac{(h_{56} - 6h_{45})\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)}{16(\gamma^2-1)^2} \\
& - \frac{(3h_{46} + 2h_{57})\mathrm{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)}{2(\gamma^2-1)^2} - \frac{(h_{47} + 2h_{58})\mathrm{Li}_2\left(-\left(\gamma-\sqrt{\gamma^2-1}\right)^2\right)}{8(\gamma^2-1)^2}\Bigg]\Bigg\}.\end{aligned}$$

- Agrees with state-of-the art PN results.

$$\begin{aligned}\frac{b^4 \Delta E^{\text{4PM}}_{\text{hyp}}}{G^4 M^5 \nu^2} = & \frac{1568}{45 v_\infty} + \left(\frac{18608}{525} - \frac{1136\nu}{45}\right) v_\infty + \frac{3136 v_\infty^2}{45} \\
& + \left(\frac{764\nu^2}{45} - \frac{356\nu}{63} + \frac{220348}{11025}\right) v_\infty^3 + \left(\frac{1216}{105} - \frac{2272\nu}{45}\right) v_\infty^4 \\
& + \left(-\frac{622\nu^3}{45} + \frac{3028\nu^2}{1575} - \frac{199538\nu}{33075} - \frac{151854}{13475}\right) v_\infty^5 \\
& + \left(\frac{1528\nu^2}{45} - \frac{8056\nu}{1575} + \frac{117248}{1575}\right) v_\infty^6 + O\left(v_\infty^7\right).\end{aligned}$$

- Extract the energy flux in an adiabatic expansion: can be used for generic bound and unbound orbits (B2B!).

High-energy (massless) limit of the radiated energy.

The second energy crisis

- Similar high-energy/massless limit problem for the energy

$$\frac{b^4 \Gamma \Delta E^{\mathrm{4PM}}_{\mathrm{hyp}}}{G^4 M^5 \nu^2} \xrightarrow{\gamma\to\infty} \frac{13696}{105} \gamma^3 \nu \log(2\gamma)\,.$$

- Remember, already at 3PM there is a problem

$$\frac{b^3 \Gamma \Delta E^{\mathrm{3PM}}_{\mathrm{hyp}}}{G^3 M^4 \nu^2} \xrightarrow{\gamma\to\infty} \frac{35}{8} \pi \gamma^3 (1 + \log(4))$$

- More energy emitted than available. Break-down of perturbation theory for very large γ! $\Rightarrow$ non-analyticity in G. [Kovacs, Thorne; Gruzinov, Veneziano; Ciafaloni, Colferai, Coradeschi, Veneziano; Di Vecchia, Heissenberg, Russo, Veneziano]
- Again: $\mathcal{O}(m_a^0)$ terms are absent.

Relative scattering angle.

- Computed relative scattering angle (see e.g. [Bini, Damour, Geralico 2107.08896])

$$
\begin{aligned}
\frac{\chi^{(4)\mathrm{cons}}_{b,\mathrm{rel}}(\gamma)}{\pi\Gamma} = {} & \frac{3h_{61}}{128\left(\gamma^2-1\right)^3} + \nu\Bigg[-\frac{3h_3 \mathrm{K}^2\left(\frac{\gamma-1}{\gamma+1}\right)}{32\left(\gamma^2-1\right)^2} + \frac{3h_4 \mathrm{E}\left(\frac{\gamma-1}{\gamma+1}\right)\mathrm{K}\left(\frac{\gamma-1}{\gamma+1}\right)}{32\left(\gamma^2-1\right)^2} + \frac{\pi^2 h_5}{16(1-\gamma^2)} + \frac{3h_{27}\log^2\left(\frac{\gamma+1}{2}\right)}{4(1-\gamma^2)} - \frac{h_6\log\left(\frac{\gamma-1}{2}\right)}{32\left(\gamma^2-1\right)^2} + \frac{3h_{15}\log\left(\frac{\gamma-1}{2}\right)\log\left(\frac{\gamma+1}{2}\right)}{16\left(\gamma^2-1\right)} \\
& - \frac{h_{22}\log\left(\frac{\gamma+1}{2}\right)}{32\left(\gamma^2-1\right)^2} - \frac{h_{23}\log(\gamma)}{4\left(\gamma^2-1\right)^2} + \frac{3h_{26}\operatorname{arccosh}^2(\gamma)}{64\left(\gamma^2-1\right)^4} + \frac{h_{24}\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^{7/2}} - \frac{3h_{16}\log\left(\frac{\gamma-1}{2}\right)\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^{5/2}} - \frac{3h_{28}\log\left(\frac{\gamma+1}{2}\right)\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^{5/2}} \\
& - \frac{h_{62}}{384\gamma^7\left(\gamma^2-1\right)^3} - \frac{21h_2\mathrm{E}^2\left(\frac{\gamma-1}{\gamma+1}\right)}{64(\gamma-1)^2(\gamma+1)} - \frac{3\sqrt{\gamma^2-1}h_7\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{2(\gamma-1)^2(\gamma+1)^3} + \frac{h_{29}\mathrm{Li}_2\left(\frac{1-\gamma}{\gamma+1}\right)}{8(1-\gamma^2)} + \left(\frac{3\sqrt{\gamma^2-1}h_7}{8(\gamma-1)^2(\gamma+1)^3} + \frac{3h_{30}}{16-16\gamma^2}\right)\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Gamma\chi^{(4)\mathrm{1rad}}_{b,\mathrm{rel}}(\gamma)}{\pi\nu} = {} & \frac{h_{64}}{96\left(\gamma^2-1\right)^{7/2}} + \frac{h_{65}\log\left(\frac{\gamma+1}{2}\right)}{16\left(\gamma^2-1\right)^{5/2}} + \frac{h_{63}\operatorname{arcsinh}\left(\frac{\sqrt{\gamma-1}}{\sqrt{2}}\right)}{8\left(\gamma^2-1\right)^4} - \frac{h_{25}\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^4} \\
& + \nu\left[\frac{h_{67}}{96\left(\gamma^2-1\right)^{7/2}} + \frac{h_{68}\log\left(\frac{\gamma+1}{2}\right)}{16\left(\gamma^2-1\right)^{5/2}} - \frac{\operatorname{arccosh}(\gamma)\left((\gamma+1)h_{14}+(\gamma-3)h_{25}\right)}{32\left(\gamma^2-1\right)^4} + \frac{h_{66}\operatorname{arcsinh}\left(\frac{\sqrt{\gamma-1}}{\sqrt{2}}\right)}{8(\gamma-1)^2(\gamma+1)^4}\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Gamma\chi^{(4)\mathrm{2rad}}_{b,\mathrm{rel}}(\gamma)}{\pi\nu^2} = {} & \frac{\log\left(\frac{\gamma+1}{2}\right)\left(2\left(\gamma^2-1\right)h_{22}+h_{11}\right)}{64(\gamma-1)^3(\gamma+1)^2} - \frac{\log(\gamma)\left(h_{12}-8\left(\gamma^2-1\right)h_{23}\right)}{32(\gamma-1)^3(\gamma+1)^2} + \frac{\operatorname{arccosh}(\gamma)\left(2(\gamma-1)^2h_{13}-(\gamma+1)h_{24}\right)}{32\left(\gamma^2-1\right)^{7/2}} \\
& + \frac{3\sqrt{\gamma^2-1}\left(h_{16}+h_{28}\right)\log\left(\frac{\gamma+1}{2}\right)\operatorname{arccosh}(\gamma)}{32(\gamma-1)^3(\gamma+1)^2} - \frac{h_9-4\gamma^2(\gamma+1)h_{20}}{1536\gamma^9\left(\gamma^2-1\right)^3} - \frac{3\left(h_{15}-4h_{27}\right)\log^2\left(\frac{\gamma+1}{2}\right)}{16(\gamma-1)} \\
& - \frac{3h_{26}\operatorname{arccosh}^2(\gamma)}{64(\gamma-1)^4(\gamma+1)^3} + \left(\frac{3}{64}(\gamma+1)h_{18} + \frac{h_{29}}{8(\gamma-1)}\right)\mathrm{Li}_2\left(\frac{1-\gamma}{\gamma+1}\right) + \frac{3\left(h_{17}+8h_{30}\right)\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)}{128(\gamma-1)},
\end{aligned}
$$

- Agrees with state-of-the art PN results [Bini, Damour, Geralico 2107.08896].
- Remember also Stefano's talk: Relative frame with recoil.
- Includes new conservative-looking terms $\propto \nu^2$.

Partial resummation via Firsov.

Used extensively in the context of the *boundary-to-bound (B2B)* map, the Firsov formula can be used to resum certain contributions to all order in G.

$$\chi(b) = -\pi + 2b \int_{r_{\min}}^{\infty} \frac{\mathrm{d}r}{r\sqrt{r^2\mathbf{p}^2(r)/p_\infty^2 - b^2}}$$

$$\Leftrightarrow$$

$$\mathbf{p}^2(r)/p_\infty^2 = \exp\left[\frac{2}{\pi}\int_{r|\mathbf{p}(r)/p_\infty|}^{\infty} \frac{\chi(b)\mathrm{d}b}{\sqrt{b^2 - r^2\mathbf{p}^2(r)/p_\infty^2}}\right]$$

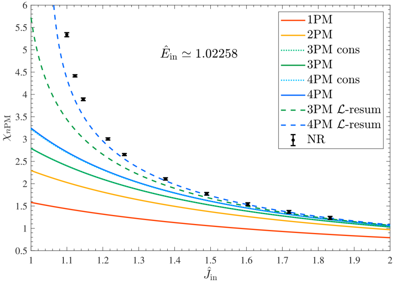
1. Compute $\mathbf{p}^2(r) = p_\infty^2(1 + \sum_{n=1}^{m} f_n(GM/r)^n)$ from scattering angle up to fixed order m in G via Firsov.
2. Use the truncated $\mathbf{p}$ to compute an all-order expression for the angle $\chi[f_m]$ using the former relation. We call this the "f_m-theory" resummation.
3. Profit!

(Note: A partial resummation of this kind needs to be performed in B2B to obtain quasi-circular bound from unbound observables at a fixed PN order.)

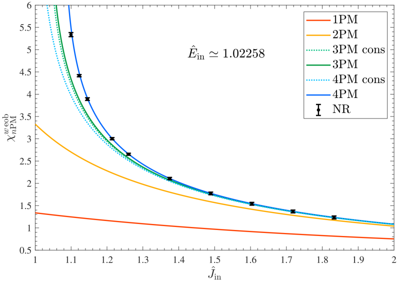
Partial resummation via Firsov.

Thibault Damour and Piero Rettegno performed this resummation and compared to NR!

Plot from [Damour, Rettegno 2211.01399]



Non-resummed vs $\mathcal{L}$-resummed



Firsov-resummed

Summary.

- Our PM-EFT approach is a complete, systematic, and efficient framework for the description of the gravitational n-body problem, including spin and tidal effects.
- Completed 4PM impulse, energy loss, and energy flux by adding radiation-reaction effects.
- The (resummed) full 4PM kinematics agrees nicely with NR data.
- Ultra-relativistic & massless limit requires deeper analysis.
- Via B2B: eccentric bound orbits.

Challenges.

"Shut up and calculate?!"

- Spin & tidal effects at 4PM: book-keeping
- 5PM: Integration (IBPs, DEs, BCs)
- Ultra-relativistic & massless limit
- B2B: Non-local terms
- PM waveforms
- Resummation strategies

*This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 817791)

Backup slides

Numerical integration using machine learning.

- Why?
  - Cross-checks are important and incredibly useful
  - Might be the only available method at higher loops
  - Analytical bootstrap
- Need fast algorithms for high precision!
- Our idea: teach a neural network to optimize the Monte-Carlo integration making use of the *normalizing flows* technology.
  - *Importance sampling:* Pick points for sampling such that regions of large integrand f gain more weight.
  - i.e. take a distribution $x(G)$, $\mathrm{d}G = g(x)\mathrm{d}x$

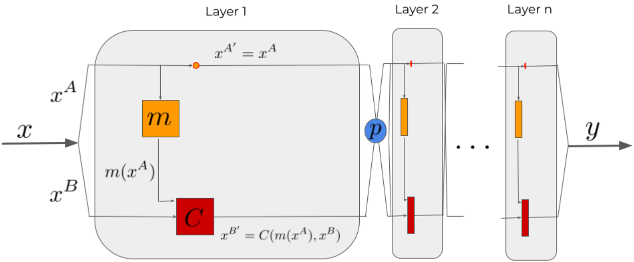
$$I = \int_\Omega \mathrm{d}x f(x) = \int_{\tilde{\Omega}} \mathrm{d}G \frac{f(x(G))}{g(x(G))}$$

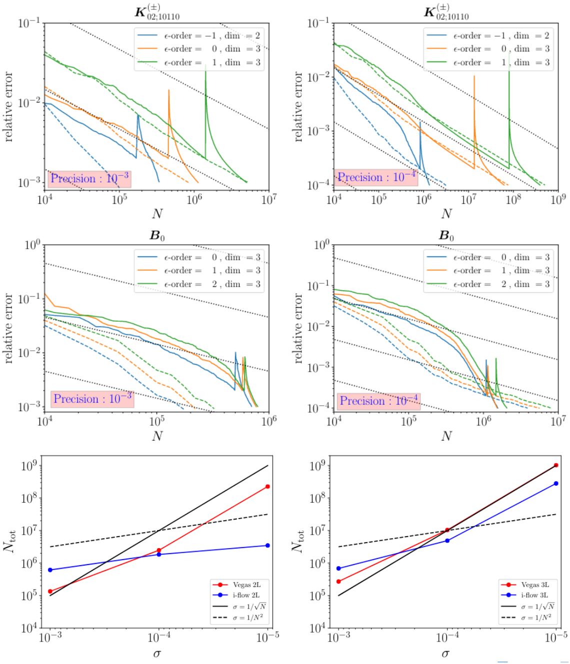
that minimizes the variance

$$\sigma^2 = \frac{1}{N-1}\left[\frac{1}{N}\sum_i \left(\frac{f(G_i)}{g(G_i)}\right)^2 - \left(\frac{1}{N}\sum_i \frac{f(G_i)}{g(G_i)}\right)^2\right]$$

A machine learning setup using neural networks is able to learn such distribution s.t.

- it is fast to evaluate,
- it is fast to invert.





$K^{(\pm)}_{02;10110}$
relative error
N
ε-order = −1 , dim = 2
ε-order = 0 , dim = 3
ε-order = 1 , dim = 3
Precision : 10^{-3}
Precision : 10^{-4}
B_0
ε-order = 0 , dim = 3
ε-order = 1 , dim = 3
ε-order = 2 , dim = 3
N_{tot}
σ
Vegas 2L
i-flow 2L
Vegas 3L
i-flow 3L
$\sigma = 1/\sqrt{N}$
$\sigma = 1/N^2$

Even terms of PN expanded relative angle have ν^2 term at 5PN:

$$\left.\frac{\Gamma^3\chi^{(4)}_{j,\mathrm{rel}}(\gamma)-\chi^{(4)}_{j,\mathrm{Sch}}(\gamma)}{\pi}\right|_{\mathrm{even}} = -\frac{15\nu}{4}+\left(\frac{123\pi^2\nu}{256}-\frac{557\nu}{16}\right)v_\infty^2+\left(\frac{33601\pi^2\nu}{16384}-\frac{6113\nu}{96}-\frac{37}{5}\nu\log\left(v_\infty/2\right)\right)v_\infty^4$$
$$+\left(\frac{1491\nu^2}{400}+\frac{93031\pi^2\nu}{32768}-\frac{615581\nu}{19200}-\frac{1357}{280}\nu\log\left(v_\infty/2\right)\right)v_\infty^6+\mathcal{O}\left(v_\infty^8\right)\,,$$