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Equivariant iterated Eisenstein integrals and modular graph forms

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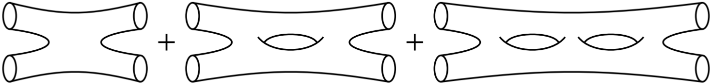
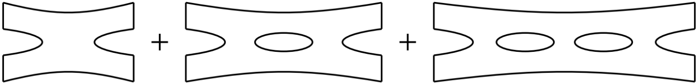
Introduction

- Feynman integrals and string amplitudes are fruitful settings for studying special functions.
- We obtain special types of **iterated integrals** by working order-by-order in the dimensional regulator ϵ. These include:
  - Multiple polylogarithms (MPLs).
  - Elliptic multiple polylogarithms (eMPLs).
  - **Iterated integrals of modular forms**.
- In this talk, we focus on Modular Graph Forms (MGFs).
  - Show up in **genus one closed-string** amplitudes.
  - Conjecturally evaluate to single-valued MZV's at the cusp $\tau \to i\infty$.
  - Can be thought of as versions of **single-valued eMZV's**.
  - MGFs are **non-holomorphic modular forms.**
  - Can be written in terms of non-holomorphic combinations of **iterated integrals of Eisenstein series**.

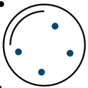
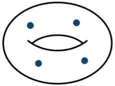
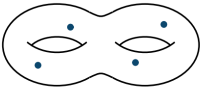
String amplitudes and special functions

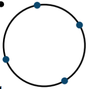
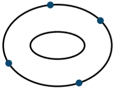
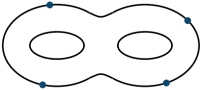
- String amplitudes admit an expansion in **genus**: [Figures taken from PhD thesis of J. Gerken]

$\mathcal{A}_{\text{closed}} =$  $+ \dots$

$\mathcal{A}_{\text{open}} =$  $+ \dots$

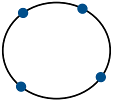
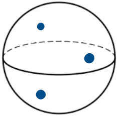
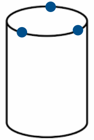
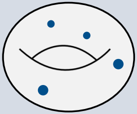
- The boundaries may be conformally mapped to punctures, leading to:

$\mathcal{A}_{\text{closed}} = g_s^{-2} \int_{\mathcal{M}_{0,4}}$  $+ \int_{\mathcal{M}_{1,4}}$  $+ g_s^{2} \int_{\mathcal{M}_{2,4}}$  $+ \dots$

$\mathcal{A}_{\text{open}} = g_s^{-1} \int_{\mathcal{M}_{0,4}}$  $+ \int_{\mathcal{M}_{1,4}}$  $+ g_s \int_{\mathcal{M}_{2,4}}$  $+ \dots$

String amplitudes and special functions

- Various types of special functions show up depending on whether we have **open/closed** strings, and depending on the **genus**:

	Open string	Closed string
g = 0	 Disk: (MZV's)	 Riemann sphere: (sv. MZV's)
g = 1	 Cylinder: (eMZV's)	 **Torus:** **MGF's** **(≈ sv. eMZV's)**

- In this talk we consider the **MGF's**, which can be expressed in terms of non-holomorphic combinations of iterated integrals of Eisenstein series.

Introduction: Connection to Feynman integrals

- Various Feynman integrals can be solved in terms of **iterated integrals of modular forms**: e.g.: [Adams, Weinzierl, 1704.08895], [Adams, Weinzierl, arXiv:1802.05020]

$$I\left(f_1, f_2, \ldots, f_n; q\right) = (2\pi i)^n \int_{\tau_0}^{\tau} d\tau_1\, f_1\left(\tau_1\right) \int_{\tau_0}^{\tau_1} d\tau_2\, f_2\left(\tau_2\right) \ldots \int_{\tau_0}^{\tau_{n-1}} d\tau_n\, f_n\left(\tau_n\right)$$

(In this talk we do not consider z-dependence, in which case we would consider kernels $f^{(k)}(z \mid \tau)$ from the Kronecker-Eisenstein series.)

- Such representations can sometimes be obtain from **ϵ-factorized differential equations** of the form $(d + \epsilon A)I = 0$.
- Integrating a modular form **does not** usually result in another modular form.

$$\int_{\tau}^{i\infty} \mathrm{d}\tau_1 \left(\tau_1\right)^j \mathrm{G}_k\left(\tau_1\right) \overset{\tau \to -1/\tau}{\longrightarrow} (-1)^j \left(\int_{\tau}^{i\infty} - \int_{0}^{i\infty} \right) \mathrm{d}\tau_1 \left(\tau_1\right)^{k-j-2} \mathrm{G}_k\left(\tau_1\right)$$

- The contributions from $\int_0^{i\infty}$ are known as **multiple modular values** (MMV's.)
- We can construct **non-holomorphic** combinations of iterated Eisenstein integrals that **do** yield modular forms. We study these special combinations in this talk!

Multiple Modular Values (MMV's)

- MMV's are numbers that **extend beyond the realm of Multiple Zeta Values (MZV's)**. For example, we have:

$$\mathfrak{m}\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix}\right] = \int_0^{i\infty} \mathrm{d}\tau_1\, \tau_1^{j_1} \mathrm{G}_{k_1}(\tau_1)$$

$$\mathfrak{m}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] = \int_0^{i\infty} \mathrm{d}\tau_2\, \tau_2^{j_2} \mathrm{G}_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} \mathrm{d}\tau_1\, \tau_1^{j_1} \mathrm{G}_{k_1}(\tau_1)$$

- The following examples at **weight ≥ 14** contain new numbers: [Brown, 1904.00179]

$$\mathfrak{m}\left[\begin{smallmatrix} 0 & 0 \\ 4 & 10 \end{smallmatrix}\right] = \frac{7613\pi^{14}}{1361455395300} - \frac{4}{27}\pi^2\rho^{-1}(f_3 f_9) - \frac{1024\pi^{14} c(\Delta_{12}, 12)}{652995}$$

$$\mathfrak{m}\left[\begin{smallmatrix} 1 & 0 \\ 4 & 10 \end{smallmatrix}\right] = -\frac{4i\pi^{11}\zeta_3}{2525985} - \frac{i\pi^5}{243}\zeta_9 + \frac{11i\pi^3}{270}\zeta_{11} - \frac{128i\pi^{13}\Lambda(\Delta_{12}, 12)}{1913625}$$

(The completed L-function of a holomorphic cusp form $\Delta(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ is $\Lambda(\Delta, t) = (2\pi)^{-t}\Gamma(t)\sum_{n=1}^{\infty} a(n)n^{-t}$, which converges absolutely for $\mathrm{Re}(t) > s + \frac{1}{2}$ and can be extended to a meromorphic function.)

Modular Forms

- MGFs can be thought of as **generalizations of Eisenstein series**. Let us briefly review these.
- The **holomorphic** Eisenstein series $\mathrm{G}_k(\tau)$ is given by:

$$\mathrm{G}_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^k} = \sum_{p\in\Lambda'} \frac{1}{p^k}, \quad k \geq 4\,,$$

  where the **discrete momentum** $p = m\tau + n \in \Lambda'$ and $\Lambda' = (\mathbb{Z}\tau + \mathbb{Z})\setminus\{0\}$.
- The Eisenstein series $\mathrm{G}_k(\tau)$ is a **modular form** of weight k:

$$\mathrm{G}_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \mathrm{G}_k(\tau) \quad \text{for} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

- Modular forms admit **q-series**, where $q = e^{2\pi i\tau}$, due to T-invariance ($\tau \to \tau+1$), e.g:

$$G_4(\tau) = 2\zeta_4\left(1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5)\right)$$

- If the zeroth power in q has coefficient zero, we call it a **cusp form**.

Non-Holomorphic Modular Forms

- The **non-holomorphic** Eisenstein series $\mathrm{E}_k(\tau)$ is given by:

$$\mathrm{E}_k(\tau) = \left(\frac{\operatorname{Im}\tau}{\pi}\right)^k \sum_{p\in\Lambda'} \frac{1}{|p|^{2k}}\,, \quad k \geq 2$$

- It is modular invariant, such that:

$$\mathrm{E}_k\left(\frac{a\tau+b}{c\tau+d}\right) = \mathrm{E}_k(\tau) \quad \text{for } \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

- More generally, a **non-holomorphic modular form** $h(\tau)$ of weight (a,b) satisfies:

$$h\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^a(\gamma\bar{\tau}+\delta)^b h(\tau)$$

- The simplest example is $\operatorname{Im}(\tau') = \frac{\operatorname{Im}(\tau)}{|\gamma\tau+\delta|^2}$ which is a non-holomorphic modular form of weight $(-1,-1)$.
- Non-holomorphic modular forms admit expansions in $q, \bar{q}$ and $\operatorname{Im}(\tau)$:

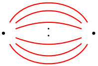
$$h(\tau) = \sum_{n,m\geq 0}\sum_{r\in\mathbb{Z}} c_{n,m,r} \operatorname{Im}(\tau)^{\mathrm{r}}\, q^n \bar{q}^m\,.$$

- The coefficients $c_{n,m,r}$ contain odd zeta's for E_k and **MZV's in general**.

Modular Graph Forms

[D'Hoker, Gürdogan, Green, Vanhove 1512.06779], [D'Hoker, Green 1603.00839]

- **Modular Graph Forms** (MGFs) arise in the **low-energy** (α'-expansion) of **genus-one closed string** amplitudes. (In type II or the Heterotic string.)
- For **dihedral graphs** the definition of MGFs reduces to the following nested sums over discrete torus momenta:



$$\mathcal{C}^+\left[\begin{smallmatrix} a_1 & \dots & a_R \\ b_1 & \dots & b_R \end{smallmatrix}\right](\tau) = \left(\prod_{j=1}^{R} \frac{(\mathrm{Im}\,\tau)^{a_j}}{\pi^{b_j}}\right) \sum_{p_1,\dots,p_R \in \Lambda'} \frac{\delta(p_1 + \ldots + p_R)}{p_1^{a_1}\bar{p}_1^{b_1} \ldots p_R^{a_R}\bar{p}_R^{b_R}} \,.$$

- In general MGF's can be represented by a connected graph of discrete momenta, with a momentum conserving delta-function for each vertex.
- We have the special cases:

$$\mathrm{G}_k(\tau) = \mathrm{Im}(\tau)^{-k}\mathcal{C}^+\left[\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right](\tau)\,, \qquad \mathrm{E}_k(\tau) = \mathcal{C}^+\left[\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right](\tau)\,.$$

- MGF's are **non-holomorphic modular forms**:

$$\mathcal{C}^+\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\bar{\tau} + d)^{|B|-|A|}\mathcal{C}^+\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right](\tau)\,,$$

where $A = (a_1, \ldots, a_R)$ and $B = (b_1, \ldots, b_R)$ are non-negative integers.

Iterated Eisenstein Integrals

- MGFs satisfy various **non-trivial relations**: e.g. [J. Gerken, PhD thesis]

$$\mathcal{C}^{+}\left[\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}\right](\tau) = \mathrm{E}_3(\tau) + \zeta_3\,, \quad \text{[D. Zagier, Notes on Lattice Sums]}$$
$$\mathcal{C}^{+}\left[\begin{smallmatrix}1&1&1&1\\1&1&1&1\end{smallmatrix}\right](\tau) = 24\mathcal{C}^{+}\left[\begin{smallmatrix}2&1&1\\2&1&1\end{smallmatrix}\right](\tau) - 18\mathrm{E}_4(\tau) + 3\mathrm{E}_2(\tau)^2\,,$$

  which are difficult to obtain from the definition as a lattice-sum.
- Relations between MGF's can be exposed by writing them in terms of **iterated integrals**. Let us define the following kernels:

$$\omega_{+}\left[\begin{smallmatrix}j\\k\end{smallmatrix};\tau,\tau_1\right] = \frac{\mathrm{d}\tau_1}{2\pi i}\left(\frac{\tau-\tau_1}{4\pi\,\mathrm{Im}(\tau)}\right)^{k-2-j}(\bar{\tau}-\tau_1)^j\mathrm{G}_k(\tau_1)\,,$$
$$\omega_{-}\left[\begin{smallmatrix}j\\k\end{smallmatrix};\tau,\tau_1\right] = -\frac{\mathrm{d}\bar{\tau}_1}{2\pi i}\left(\frac{\tau-\bar{\tau}_1}{4\pi\,\mathrm{Im}(\tau)}\right)^{k-2-j}(\bar{\tau}-\bar{\tau}_1)^j\overline{\mathrm{G}_k(\tau_1)}\,,$$

  where $0 \leq j \leq k-2$. These kernels are **modular forms** with **vanishing holomorphic modular** weight. Next, consider iterated integrals of the type:

$$\beta_{+}\left[\begin{smallmatrix}j_1&j_2&\dots&j_\ell\\k_1&k_2&\dots&k_\ell\end{smallmatrix};\tau\right] = \int_{\tau}^{i\infty}\omega_{+}\left[\begin{smallmatrix}j_\ell\\k_\ell\end{smallmatrix};\tau,\tau_\ell\right]\dots\int_{\tau_3}^{i\infty}\omega_{+}\left[\begin{smallmatrix}j_2\\k_2\end{smallmatrix};\tau,\tau_2\right]\int_{\tau_2}^{i\infty}\omega_{+}\left[\begin{smallmatrix}j_1\\k_1\end{smallmatrix};\tau,\tau_1\right]\,,$$
$$\beta_{-}\left[\begin{smallmatrix}j_1&j_2&\dots&j_\ell\\k_1&k_2&\dots&k_\ell\end{smallmatrix};\tau\right] = \int_{\bar{\tau}}^{-i\infty}\omega_{-}\left[\begin{smallmatrix}j_\ell\\k_\ell\end{smallmatrix};\tau,\tau_\ell\right]\dots\int_{\bar{\tau}_3}^{-i\infty}\omega_{-}\left[\begin{smallmatrix}j_2\\k_2\end{smallmatrix};\tau,\tau_2\right]\int_{\bar{\tau}_2}^{-i\infty}\omega_{-}\left[\begin{smallmatrix}j_1\\k_1\end{smallmatrix};\tau,\tau_1\right]\,.$$

Iterated Eisenstein Integrals

- These integrals **fail** to be modular forms by:

$$\beta_\pm\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix};\frac{a\tau+b}{c\tau+d}\right] = \left(\prod_{i=1}^{\ell}(c\bar{\tau}+d)^{k_i-2-2j_i}\right)\beta_\pm\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix};\tau\right] \quad \left(\begin{smallmatrix}\text{mod lower depth} \\ \text{\& MMV's}\end{smallmatrix}\right).$$

- The non-holomorphic Eisenstein series can be written as :

$$\mathrm{E}_k(\tau) = -\frac{(2k-1)!}{(k-1)!^2}\left\{\beta_+\left[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix};\tau\right] + \beta_-\left[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix};\tau\right] - \frac{2\zeta_{2k-1}}{(2k-1)(4\pi\,\mathrm{Im}(\tau))^{k-1}}\right\}.$$

- Because $\mathrm{E}_k(\tau)$ is modular invariant, we identify the **modular invariant combination**:

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix};\tau\right] = \beta_+\left[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix};\tau\right] + \beta_-\left[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix};\tau\right] - \frac{2\zeta_{2k-1}}{(2k-1)(4\pi\,\mathrm{Im}(\tau))^{k-1}}$$

- More generally, we have that:

$$\mathcal{C}^+\left[\begin{smallmatrix} 0 & a \\ 0 & b \end{smallmatrix}\right](\tau) = -\frac{(2i)^{b-a}(a+b-1)!}{(a-1)!(b-1)!}\left(\beta_+\left[\begin{smallmatrix} a-1 \\ a+b \end{smallmatrix};\tau\right] + \beta_-\left[\begin{smallmatrix} a-1 \\ a+b \end{smallmatrix};\tau\right] - \frac{2\zeta_{a+b-1}}{(a+b-1)(4\pi\,\mathrm{Im}(\tau))^{b-1}}\right).$$

and we may identify the combination within the brackets as $\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} a-1 \\ a+b \end{smallmatrix};\tau\right]$.

Iterated Eisenstein Integrals

- We seek to generalize to **higher-depth** $\beta^{\rm eqv}[\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix};\tau]$, which **are modular forms**:

$$\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix};\frac{a\tau+b}{c\tau+d}\right] = \left(\prod_{i=1}^{\ell}(c\bar\tau+d)^{k_i-2-2j_i}\right)\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix};\tau\right] .$$

- A defining property is the **holomorphic differential equation**:

$$2\pi i(\tau-\bar\tau)^2\partial_\tau\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix};\tau\right] = \sum_{i=1}^{\ell}(k_i-j_i-2)\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & \dots & j_i+1 & \dots & j_\ell \\ k_1 & \dots & k_i & \dots & k_\ell \end{smallmatrix};\tau\right]$$
$$-\,\delta_{j_\ell,k_\ell-2}(\tau-\bar\tau)^{k_\ell}\mathrm{G}_{k_\ell}(\tau)\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & \dots & j_{\ell-1} \\ k_1 & \dots & k_{\ell-1} \end{smallmatrix};\tau\right] \ (\mathrm{mod}\ \beta^{\rm sv}_{\Delta})$$

- We may again draw inspiration from MGF's. For example, it turns out that:

$$\mathcal{C}^+\left[\begin{smallmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{smallmatrix}\right] = -126\beta^{\rm eqv}\left[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix}\right] - 18\beta^{\rm eqv}\left[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\right] ,$$
$$\mathcal{C}^+\left[\begin{smallmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \end{smallmatrix}\right] = \frac{279}{2}\beta^{\rm eqv}\left[\begin{smallmatrix} 5 \\ 10 \end{smallmatrix}\right] + 30\beta^{\rm eqv}\left[\begin{smallmatrix} 3 & 1 \\ 6 & 4 \end{smallmatrix}\right] + \frac{15}{2}\beta^{\rm eqv}\left[\begin{smallmatrix} 4 & 0 \\ 6 & 4 \end{smallmatrix}\right] ,$$
$$2i\,\mathrm{Im}\,\mathcal{C}^+\left[\begin{smallmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{smallmatrix}\right] = 60\left(\beta^{\rm eqv}\left[\begin{smallmatrix} 0 & 3 \\ 4 & 6 \end{smallmatrix}\right] - \beta^{\rm eqv}\left[\begin{smallmatrix} 1 & 2 \\ 6 & 4 \end{smallmatrix}\right]\right) - 270\left(\beta^{\rm eqv}\left[\begin{smallmatrix} 1 & 2 \\ 4 & 6 \end{smallmatrix}\right] - \beta^{\rm eqv}\left[\begin{smallmatrix} 2 & 1 \\ 6 & 4 \end{smallmatrix}\right]\right)$$
$$+\,390\left(\beta^{\rm eqv}\left[\begin{smallmatrix} 2 & 1 \\ 4 & 6 \end{smallmatrix}\right] - \beta^{\rm eqv}\left[\begin{smallmatrix} 3 & 0 \\ 6 & 4 \end{smallmatrix}\right]\right) - 3\zeta_3\beta^{\rm eqv}\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right] ,$$

Iterated Eisenstein Integrals

- Let us briefly consider the origin of the representations of the $\mathcal{C}^+[\,\substack{\cdots\\\cdots}\,](\tau)$ in terms of $\beta_+[\,\substack{\cdots\\\cdots}\,;\tau]$ and $\beta_-[\,\substack{\cdots\\\cdots}\,;\tau]$, which we'll rewrite as $\beta^{\mathrm{eqv}}[\,\substack{\cdots\\\cdots}\,;\tau]$.
- The main idea is that repeated actions of so-called **Maass operators** $\nabla_\tau = 2i(\mathrm{Im}\,\tau)^2\partial_\tau$ **simplify** the lattice sums.

$$(\pi\nabla_\tau)^3\,\mathcal{C}^+\left[\begin{smallmatrix}2&1&1\\2&1&1\end{smallmatrix}\right] = \frac{9}{10}\left(\pi\nabla_\tau^3\right)\mathrm{E}_4 - 6(\mathrm{Im}\,\tau)^4\mathrm{G}_4\left(\pi\nabla_\tau\right)\mathrm{E}_2$$

- By plugging in the **depth-one** integral representations for G_k and E_k, and **integrating**, we obtain representations in terms of iterated integrals.
- Unfortunately, at higher depths the collections of MGF's and $\beta^{\mathrm{eqv}}[\,\substack{\cdots\\\cdots}\,;\tau]$ are **not one-to-one**. Only particular combinations of $\beta^{\mathrm{eqv}}[\,\substack{\cdots\\\cdots}\,;\tau]$ appear in MGF's, subject to **Tsunogai's derivation algebra**.
- To investigate this point further, let us switch to the **generating series point of view**.

Generating series of Modular Graph Forms

- A **genenerating series of convergent MGFs** (that do not simplify under holomorphic subgraph reduction) was defined in [Gerken, Kleinschmidt, Schlotterer, 1911.03476, 2004.05156]:

$$Y^{\tau}_{\vec{\eta}}(\sigma \mid \rho) = (\tau - \bar{\tau})^{n-1} \int \left(\prod_{j=2}^{n} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \exp\left(\sum_{1 \le i < j}^{n} s_{ij} G(z_i - z_j, \tau) \right) \times \sigma \left[\overline{\varphi^{\tau}(z_j, \eta_j, \bar{\eta}_j)} \right] \rho \left[\varphi^{\tau}(z_j, (\tau - \bar{\tau})\eta_j, \bar{\eta}_j) \right],$$

  where the n **punctures** z_j are integrated over a **torus** of modular parameter τ, and the η_j and $\bar{\eta}_j$ are **formal variables** of the generating series.
- The integrals $Y^{\tau}_{\vec{\eta}}$ are indexed by permutations $\sigma, \rho \in \mathcal{S}_{n-1}$ that act on the subscripts $2, 3, \ldots, n$ of the $\{z_j, \eta_j\}$ variables and leave z_1 inert.
- The integrand involves **doubly-periodic functions** $\varphi^{\tau}(z_j, \ldots) = \varphi^{\tau}(z_j + 1, \ldots) = \varphi^{\tau}(z_j + \tau, \ldots)$, build out of products of the **Kronecker-Eisenstein series**:

$$\Omega(z, \eta, \tau) = \exp\left(2\pi i \eta \frac{\operatorname{Im} z}{\operatorname{Im}\tau} \right) \frac{\theta'(0,\tau)\theta(z+\eta,\tau)}{\theta(z,\tau)\theta(\eta,\tau)}.$$

- The exponent (**Koba-Nielsen factor**) features the closed-string Green function $G(z, \tau)$ on the torus.

Generating series of Modular Graph Forms

- On the one hand, these integrals may be computed by performing a **Fourier transform**, which leads to sums over discrete momenta and **which yields expressions in terms of MGFs**.

- Alternatively, we note that the (KZB-type) **differential equations** are of the form:

$$2\pi i \partial_\tau Y^\tau_{\vec{\eta}}(\sigma|\rho) = \sum_{\alpha \in S_{n-1}} \left\{ -\frac{1}{(\tau-\bar{\tau})^2} R_{\vec{\eta}}(\epsilon_0)_\rho{}^\alpha + \sum_{k=4}^{\infty} (1-k)(\tau-\bar{\tau})^{k-2} \mathrm{G}_k(\tau) R_{\vec{\eta}}(\epsilon_k)_\rho{}^\alpha \right\} Y^\tau_{\vec{\eta}}(\sigma|\alpha)\,,$$

  and can be solved in terms of a **generating series**

$$Y^\tau_{\vec{\eta}} = \sum_P R_{\vec{\eta}}(\epsilon[P]) \underbrace{\left(\sum_{P=ABC} \overline{\kappa[A;\tau]} \beta_- \left[B^t;\tau\right] \beta_+[C;\tau] \right)}_{\text{(collecting holo/antiholo. contributions)}} \underbrace{\exp\left(-\frac{R_{\vec{\eta}}(\epsilon_0)}{4\pi \, \mathrm{Im}(\tau)} \right) \hat{Y}^{i\infty}_{\vec{\eta}}}_{\text{(initial value)}}$$

- The first sum is over **words** $P = \begin{smallmatrix} j_1 & \cdots & j_\ell \\ k_1 & \cdots & k_\ell \end{smallmatrix}$ of length $\ell \geq 0$ with $k_i \geq 4$ even and $0 \leq j_i \leq k_i - 2$, while the second sum is over **deconcatenations** of P.
- The term $\overline{\kappa[X;\tau]}$ is a **purely antiholomorphic** term which carries combinations of MZV's and which can be determined through **reality properties** of the MGF's.

Generating series of Modular Graph Forms

- **The coefficients $\epsilon[P]$ are defined by**:

$$\epsilon[P] = \epsilon\begin{bmatrix} j_1 & j_2 & \dots & j_\ell \\ k_1 & k_2 & \dots & k_\ell \end{bmatrix} = \left(\prod_{i=1}^{\ell} \frac{(-1)^{j_i}(k_i-1)}{(k_i-j_i-2)!}\right) \epsilon_{k_\ell}^{(k_\ell-2-j_\ell)} \cdots \epsilon_{k_2}^{(k_2-2-j_2)} \epsilon_{k_1}^{(k_1-2-j_1)},$$

where the quantities $\epsilon_k^{(j)}$ are defined using the shorthand:

$$\epsilon_k^{(j)} = \mathrm{ad}^j_{\epsilon_0}(\epsilon_k) = \underbrace{[\epsilon_0, [\dots, [\epsilon_0, \epsilon_k]] \dots]}_{j\text{-times}}$$

- The notation $R_{\vec{\eta}}(\epsilon[P])$ indicates that we are considering a particular **matrix representation** of the generators ϵ_k. The ϵ_k-derivations satisfy various relations furnished by **Tsunogai's derivation algebra**: [Tsunogai 1995, …, Pollack 2009]

$$0 = \epsilon_k^{(k-1)}, \qquad k \geq 4 \text{ even},$$

$$0 = [\epsilon_4, \epsilon_{10}] - 3[\epsilon_6, \epsilon_8],$$

$$0 = -462\big[\epsilon_4, [\epsilon_4, \epsilon_8]\big] - 1725\big[\epsilon_6, [\epsilon_6, \epsilon_4]\big] - 280[\epsilon_8, \epsilon_8^{(1)}]$$
$$+ 125[\epsilon_6, \epsilon_{10}^{(1)}] + 250[\epsilon_{10}, \epsilon_6^{(1)}] - 80[\epsilon_{12}, \epsilon_4^{(1)}] - 16[\epsilon_4, \epsilon_{12}^{(1)}]$$

Tsunogai derivation algebra

- The Tsunogai derivation algebra has the following impact on the generating series.

1. Relations like $[\epsilon_4, \epsilon_{10}] - 3\,[\epsilon_6, \epsilon_8] = 0$ **project out** cusp-form contributions to non-holomorphic modular forms in J^{eqv}, in other words there are no $\int_\tau \mathrm{d}\tau_1 \Delta_k(\tau_1)$
2. Therefore, MGF's and the $\beta^{\mathrm{eqv}}[\,\substack{\cdots\\ \cdots}\,;\tau]$ are not one-to-one. It turns out that a 'full' set of $\beta^{\mathrm{eqv}}[\,\substack{\cdots\\ \cdots}\,;\tau]$ requires (iterated) integrals of cusp forms starting from $k \geq 14$.

Generating Series of $\beta^{\rm eqv}$

- We now consider a **generating series for the $\beta^{\rm eqv}[\substack{\cdots \\ \cdots};\tau]$**:

$$J^{\rm eqv}\left(\{\epsilon_k\};\tau\right) = \sum_P \epsilon[P]\beta^{\rm eqv}[P;\tau]$$

- The central result of our paper is that:

$$J^{\rm eqv}\left(\{\epsilon_k\};\tau\right) = J_+\left(\{\epsilon_k\};\tau\right)B^{\rm sv}\left(\{\epsilon_k\};\tau\right)\phi^{\rm sv}\left(\widetilde{J_-}(\{\epsilon_k\};\tau)\right).$$

which makes explicit a construction in [Brown, 1707.01230, 1708.03354] of these integrals. The holomorphic / antiholomorphic contributions are packaged in the following way:

$$J_\pm\left(\{\epsilon_k\};\tau\right) = \sum_P \epsilon[P]\beta_\pm[P;\tau].$$

- The tilde of $\widetilde{J_-}\left(\{\epsilon_k\};\tau\right)$ instructs us to **reverse the words**:

$$\epsilon_{k_1}^{(j_1)}\ldots\epsilon_{k_\ell}^{(j_\ell)} \to \epsilon_{k_\ell}^{(j_\ell)}\ldots\epsilon_{k_1}^{(j_1)}$$

- We furthermore have $B^{\rm sv}\left(\{\epsilon_k\};\tau\right) = \sum_P \epsilon[P]b^{\rm sv}[P;\tau]$, with

$$b^{\rm sv}\left[\begin{smallmatrix}\cdots & j_i & \cdots \\ \cdots & k_i & \cdots\end{smallmatrix};\tau\right] = \sum_{p_i=0}^{k_i-2-j_i}\sum_{\ell_i=0}^{j_i+p_i}\binom{k_i-2-j_i}{p_i}\binom{j_i+p_i}{\ell_i}\frac{(-2\pi i\bar{\tau})^{\ell_i}}{(4\pi{\rm Im}(\tau))^{p_i}}c^{\rm sv}\left[\begin{smallmatrix}\cdots & j_i-\ell_i+p_i & \cdots \\ \cdots & k_i & \cdots\end{smallmatrix}\right],$$

- The new ingredient $B^{\mathrm{SV}}(\epsilon_k)$ is specified by the c^{sv} which are composed out of **single-valued MZV's**. For example:

$$c^{\mathrm{sv}}\begin{bmatrix}0&1\\4&6\end{bmatrix}=\frac{\zeta_3}{907200}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}1&0\\4&6\end{bmatrix}=-\frac{\zeta_3}{226800}\,,$$

$$c^{\mathrm{sv}}\begin{bmatrix}0&3\\4&6\end{bmatrix}=-\frac{\zeta_5}{7200}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}1&2\\4&6\end{bmatrix}=\frac{\zeta_5}{21600}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}2&1\\4&6\end{bmatrix}=-\frac{\zeta_5}{21600}\,,$$

$$c^{\mathrm{sv}}\begin{bmatrix}0&4\\4&6\end{bmatrix}=-\frac{\zeta_3^2}{315}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}1&3\\4&6\end{bmatrix}=\frac{\zeta_3^2}{1260}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}2&2\\4&6\end{bmatrix}=-\frac{\zeta_3^2}{1890}\,,$$

$$c^{\mathrm{sv}}\begin{bmatrix}1&4\\4&6\end{bmatrix}=\frac{7\zeta_7}{360}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}2&3\\4&6\end{bmatrix}=-\frac{7\zeta_7}{720}\,,\qquad c^{\mathrm{sv}}\begin{bmatrix}2&4\\4&6\end{bmatrix}=\frac{2\zeta_3\zeta_5}{15}\,.$$

$$c^{\mathrm{sv}}\begin{bmatrix}2&2&4\\4&4&6\end{bmatrix}=-\frac{1}{450}\zeta^{\mathrm{sv}}_{3,5,3}-\frac{2}{45}\zeta_3^2\zeta_5-\frac{221}{21600}\zeta_{11}\,,$$

$$c^{\mathrm{sv}}\begin{bmatrix}2&4&4\\4&6&6\end{bmatrix}=\frac{1}{3750}\zeta^{\mathrm{sv}}_{5,3,5}+\frac{2}{375}\zeta_3\zeta_5^2+\frac{1804427}{124380000}\zeta_{13}\,,$$

$$c^{\mathrm{sv}}\begin{bmatrix}2&2&6\\4&4&8\end{bmatrix}=-\frac{1}{1764}\zeta^{\mathrm{sv}}_{3,7,3}+\frac{1}{1470}\zeta^{\mathrm{sv}}_{5,3,5}-\frac{2}{63}\zeta_3^2\zeta_7-\frac{137359}{24378480}\zeta_{13}\,,$$

- **Conjecturally**:

$$c^{\mathrm{sv}}\begin{bmatrix}k_1-2&\dots&k_\ell-2\\k_1&\dots&k_\ell\end{bmatrix}=\left(\prod_{i=1}^{\ell}\frac{1}{1-k_i}\right)\mathrm{sv}\left(f_{k_1-1}\dots f_{k_\ell-1}\right)\quad\text{mod fewer } f_i$$

The change of alphabet ϕ^{sv}

- The map ϕ^{sv} applies a **change of alphabet** to the $\epsilon[P]$. For example:

$$\phi^{\mathrm{sv}}(\epsilon_2) = \epsilon_4 + \frac{\zeta_3}{252}\left(\left[\epsilon_6^{(2)}, \epsilon_4\right] - 3\left[\epsilon_6^{(1)}, \epsilon_4^{(1)}\right] + 6\left[\epsilon_6, \epsilon_4^{(2)}\right]\right) + \ldots$$

- More generally, the map ϕ^{sv} can be described through a **conjugation** with another generating series: $\phi^{\mathrm{sv}}(\epsilon_k) = \mathbb{M}^{\mathrm{SV}} \epsilon_k \left(\mathbb{M}^{\mathrm{SV}}\right)^{-1}$, which is given by:

$$\mathbb{M}^{\mathrm{SV}}(z_i) = \sum_{\ell \geq 0} \sum_{m_1, \ldots, m_\ell \in 2\mathbb{N}+1} \mathrm{sv}\left(f_{m_1} f_{m_2} \ldots f_{m_\ell}\right) z_{m_1} z_{m_2} \ldots z_{m_\ell}$$

- Here the f_i are letters in the so-called ***f*-alphabet** of (motivic) multiple zeta values, and the z_j are a new class of operators in the derivation algebra which **normalize** the set $\{\epsilon_k\}$. For example:

$$[z_3, \epsilon_4] = \frac{1}{504}\left(\left[\epsilon_6^{(2)}, \epsilon_4\right] - 3\left[\epsilon_6^{(1)}, \epsilon_4^{(1)}\right] + 6\left[\epsilon_6, \epsilon_4^{(2)}\right]\right)$$

- Putting things together, we find (in shorthand notation):

$$J^{\mathrm{eqv}} = J_+ B^{\mathrm{sv}} \mathbb{M}^{\mathrm{sv}} \widetilde{J}_- \left(\mathbb{M}^{\mathrm{sv}}\right)^{-1}$$

Iterated Integrals of Holomorphic Cusp Forms

- We may generalize the construction by relaxing the constraints from Tsunogai's derivation algebra. In this case **we also require contributions from holomorphic cusp forms $\Delta_k(\tau) = q + \mathcal{O}\left(q^2\right)$ in the modular completion**. For example: [Brown, 1407.5167, 1707.01230]

  [Dorigoni, Kleinschmidt, Schlotterer, 2109.05018]

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1 & 4\\ 6 & 8\end{smallmatrix};\tau\right] = (\beta_\pm \text{and MZVs}) + \frac{1}{52920000}\frac{\Lambda(\Delta_{12},12)}{\Lambda(\Delta_{12},10)}\left(\beta_+\left[\begin{smallmatrix}5\\ \Delta_{12}\end{smallmatrix};\tau\right] - \beta_-\left[\begin{smallmatrix}5\\ \Delta_{12}\end{smallmatrix};\tau\right]\right)$$

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2 & 3\\ 4 & 10\end{smallmatrix};\tau\right] = (\beta_\pm \text{and MZVs}) - \frac{1}{122472000}\frac{\Lambda(\Delta_{12},12)}{\Lambda(\Delta_{12},10)}\left(\beta_+\left[\begin{smallmatrix}5\\ \Delta_{12}\end{smallmatrix};\tau\right] - \beta_-\left[\begin{smallmatrix}5\\ \Delta_{12}\end{smallmatrix};\tau\right]\right)$$

- The coefficients contain ratios of (critical and non-critical) **L-values** $\frac{\Lambda(\Delta_k,\text{n.c.})}{\Lambda(\Delta_k,\text{crit.})}$.

Conclusion and outlook

- MGFs are an interesting class of non-holomorphic modular forms, which have (conjecturally s.v.) MZV's in the coefficients of their q-expansions.
- We provided the dictionary between MGF's and Brown's equivariant iterated Eisenstein integrals, and provide evidence for Brown's conjecture that equivariant iterated Eisenstein integrals contain all modular graph forms.

- Future work: explore similar generating-function approach to z-dependent elliptic MGFs / single-valued elliptic polylogarithms and their iterated-integral representation.
- Future work: explore connections to the recent one-loop KLT formula? [Stieberger, 2212.06816]