The ϵ-form of the 4PM Feynman integral differential equations

based on
[CD, Kälin, Liu, Porto, '21] 2112.11296
[CD, Kälin, Liu, Neef, Porto, '22] 2210.05541 + to appear
[CD, Henn, Wagner, '22] 2211.16357

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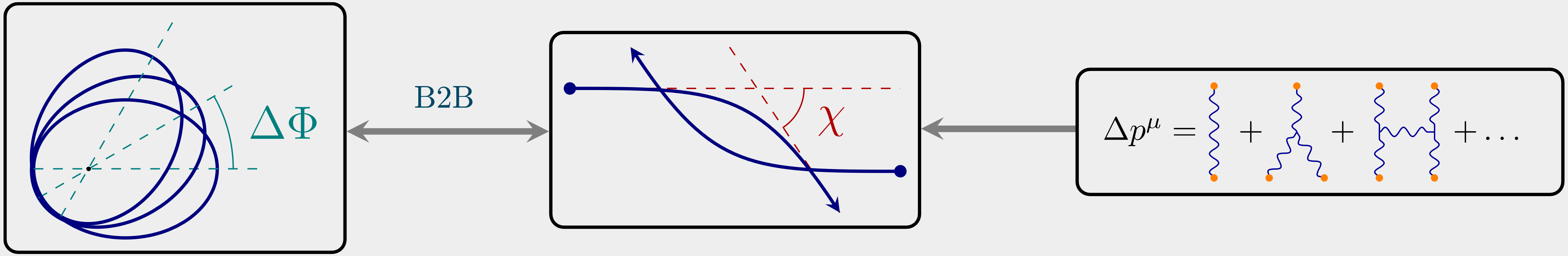
Christoph Dlapa

Gravitational two-body problem

[Kälin, Porto, 19' /
Kälin, Porto, 20' /
Kälin, Neef, Porto, 21' /
Cho, Kälin, Porto, '21]



- EFT approach: $e^{iS_{\text{eff}}[x_a]} = \int \mathcal{D}h_{\mu\nu} e^{iS_{\text{EH}}[h]+iS_{\text{GF}}[h]+iS_{\text{pp}}[x_a,h]}$
- Post-Minkowskian expansion: $S_{\text{eff}} = \sum_{n=0}^{\infty} \int \mathrm{d}\tau_1 G^n \mathcal{L}_n[x_1(\tau_1), x_2(\tau_2)]$

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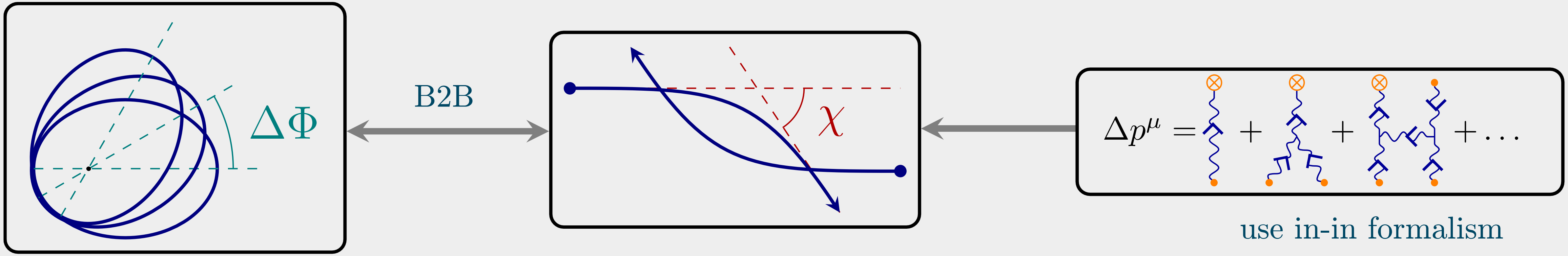

use in-in formalism

$S[h_1, h_2] = S[h_1] - S[h_2]$

$\Rightarrow$ ret/adv propagators

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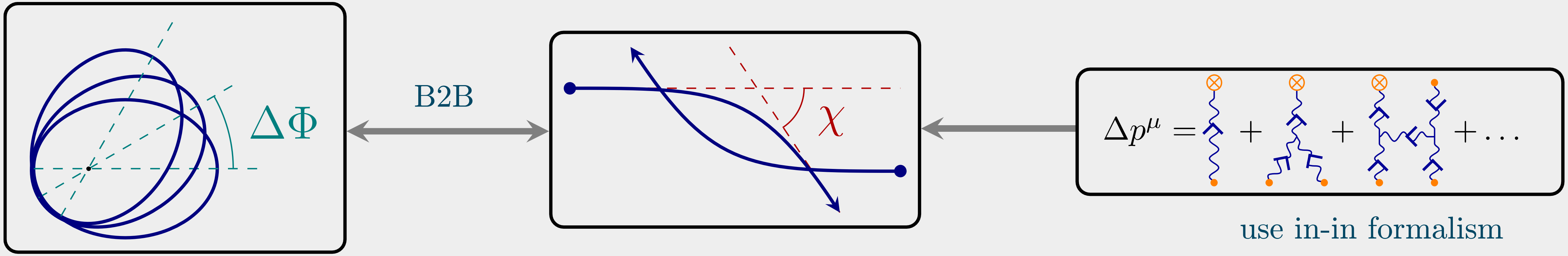

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- Total momentum change: $\Delta p_1^\mu = m_1 \Delta v_1^\mu = -\eta^{\mu\nu} \sum_n \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \frac{\partial \mathcal{L}_n}{\partial x_1^\nu}$

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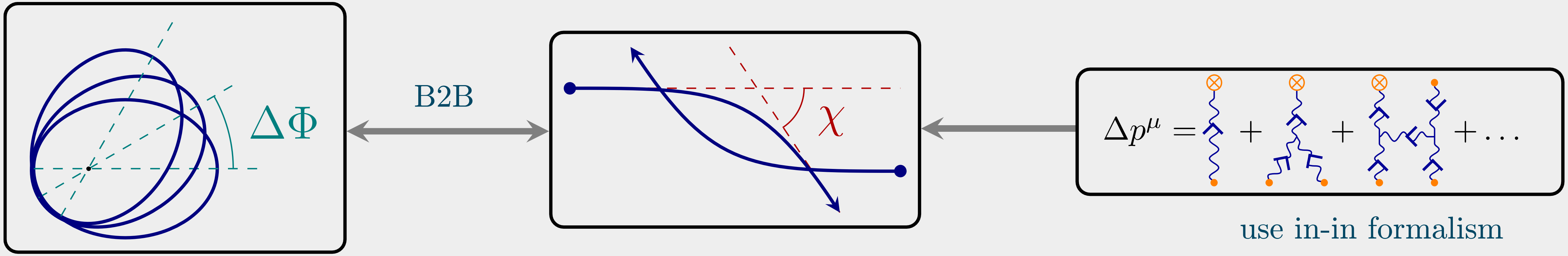

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- Corrections to trajectories: $x_a^\mu = b_a^\mu + u_a^\mu \tau_a + \sum_n G^n \delta^{(n)} x_a^\mu(\tau_a)$

solve Euler-Lagrange equations iteratively

Loop integrals at 4PM

- Example integral family:

cut linear propagators

$$D = 4 - 2\epsilon$$

$$I_{a_1 \dots a_{15}}(x) = \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3 \frac{\delta^{(a_1)}(k_1 \cdot u_1)\delta^{(a_2)}(k_2 \cdot u_1)\delta^{(a_3)}(k_3 \cdot u_2)}{(k_1 \cdot u_2 - i0)^{a_4}(k_2 \cdot u_2 - i0)^{a_5}(k_3 \cdot u_1 - i0)^{a_6}} \frac{1}{\prod_{i=7}^{15} D_i^{a_i}}$$

$$\begin{aligned} \{D_1, \dots, D_{15}\} = \{ & -k_1^2, & & -k_2^2, & & -k_3^2, \\ & -(k_1 - q)^2, & & -(k_2 - q)^2, & & -(k_3 - q)^2, \\ & -(k_1 - k_2)^2, & & -(k_2 - k_3)^2, & & -(k_1 - k_3)^2\} \end{aligned}$$

$$u_1^2 = u_2^2 = 1,$$
$$u_1 \cdot q = u_2 \cdot q = 0$$
$$u_1 \cdot u_2 = \gamma$$

  - Treat cut propagators just as linear propagators

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$$u_1^2 = u_2^2 = 1,$$

$$u_1 \cdot q = u_2 \cdot q = 0$$

$$u_1 \cdot u_2 = \gamma$$

  - Treat cut propagators just as linear propagators
  - Rationalize square-root:

only one variable

$$\gamma = \frac{1}{2}(1/x + x), \qquad v \equiv \sqrt{\gamma^2 - 1} = \frac{1}{2}(1/x - x)$$

Differential equations method

[Kotikov, '91 / Remiddi, '97 /
Gehrmann, Remiddi, '00 /
Argeri, Mastrolia, '07]

- Integration-by-parts reduction to basis

[FIRE6, Smirnov, Chukharev, '19 /
LiteRed, Lee, '13]

$$\vec{f} = \begin{pmatrix} I_{111000010101110} \\ \vdots \\ I_{111000111111111} \end{pmatrix}, \qquad I_{a_1 \ldots a_{15}} = \sum_i c_i(x,\epsilon) f_i$$

ret/adv ⇒ turn off symmetry detection

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ret/adv ⇒ turn off symmetry detection

- Solve by transforming to canonical form:

[Henn, '13]

$$\partial_x \vec{f} = A(x,\epsilon)\vec{f} \qquad \xrightarrow{\vec{g} = T\vec{f}} \qquad \partial_x \vec{g} = \epsilon\, \tilde{A}(x)\vec{g}$$

rational

[Lee, '15]

[Libra, epsilon, Fuchsia, *CANONICA*]

Dyson series

$$\vec{g} = \mathrm{P}\, e^{\epsilon \int_{x_0}^{x} \tilde{A}(\tilde{x})\, \mathrm{d}\tilde{x}}\, \vec{g}_0(\epsilon) = \left[1 + \epsilon \int_{x_0}^{x} \mathrm{d}x_1 \tilde{A}(x_1) + \epsilon^2 \int_{x_0}^{x} \mathrm{d}x_1 \tilde{A}(x_1) \int_{x_0}^{x_1} \mathrm{d}x_2 \tilde{A}(x_2) + \mathcal{O}(\epsilon^3)\right] \vec{g}_0(\epsilon)$$

Elliptic integrals

- Polylogarithms for most sectors

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x) \vec{g}, \qquad \tilde{A}(x) = \sum_i \underset{\uparrow}{M_i} a_i(x)$$

constant matrices

$$a_i(x) \in \left\{ \frac{1}{x}, \frac{1}{x-1}, \frac{1}{x+1}, \frac{x}{1+x^2} \right\}$$

new at 4PM

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new at 4PM

- Algorithms do not work for one sector

$$\partial_x \vec{f} = A(x,\epsilon) \vec{f}$$

  - analyze ϵ^0-part:

$$A(x,\epsilon) = A^{(0)}(x) + \epsilon A^{(1)}(x) + \mathcal{O}(\epsilon^2), \qquad \partial_x T^{(0)}(x) = A^{(0)}(x) T^{(0)}(x)$$

remove with transformation

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new at 4PM

- Algorithms do not work for one sector
  - analyze ϵ^0-part:

$$\partial_x \vec{f} = A(x,\epsilon)\vec{f}$$

$$\partial_\gamma \vec{f} = A(\gamma,\epsilon)\vec{f} \longrightarrow \sqrt{\gamma^2-1}$$

$$T^{(0)}(\gamma)$$

$$A(x,\epsilon) = A^{(0)}(x) + \epsilon A^{(1)}(x) + \mathcal{O}(\epsilon^2), \qquad \partial_x T^{(0)}(x) = A^{(0)}(x) T^{(0)}(x)$$

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- Algorithms do not work for one sector
  - Third-order DE: $$\left[\partial^3 - \frac{6x}{1-x^2}\partial_x^2 - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\partial_x - \frac{1+x^2}{x^3(1-x^2)}\right]\Psi_{1,2,3} = 0$$
  - Solve with Mathematica: $\Psi_1 = x\mathrm{K}^2(1-x^2), \quad \Psi_2 = x\mathrm{K}(1-x^2)\mathrm{K}(x^2), \quad \Psi_3 = x\mathrm{K}^2(x^2)$

$$\mathrm{K}(z) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}},$$

$$\mathrm{E}(z) = \int_0^1 \frac{1-zt^2}{\sqrt{1-t^2}}\mathrm{d}t$$

Elliptic integrals

- Polylogarithms for most sectors $\partial_x \vec{g} = \epsilon \tilde{A}(x)\vec{g}, \quad \tilde{A}(x) = \sum_i \underset{\substack{\uparrow \\ \text{constant matrices}}}{M_i} a_i(x)$
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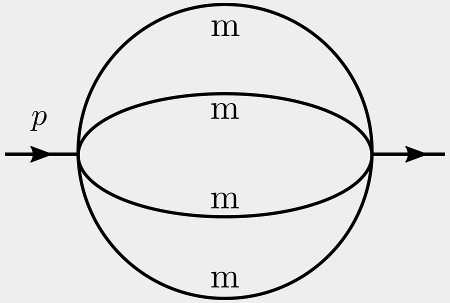

[Broedel, Duhr, Dulat, Marzucca, Penante, Tancredi, '19 /
Broedel, Duhr, Matthes, '21 /
Pögel, Wang, Weinzierl, '22]

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ϵ-form using INITIAL

[CD, Henn, Yan, '20 /
CD, Henn, Wagner, '22]

- Elliptic differential equations

68 master integrals:
$\Rightarrow$ constant 68 × 68 matrices

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x) \vec{g}, \qquad \tilde{A}(x) = \sum_i M_i a_i(x)$$

$$\begin{aligned} a_i(x) \in \Big\{ & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)}, \quad \frac{1}{1-x}, \quad \frac{1}{x}, \quad \frac{1}{1+x}, \quad \frac{x}{1+x^2}, \\ & \frac{\mathrm{K}^2(1-x^2)}{\pi^2 x(1-x^2)}, \quad \frac{\mathrm{K}^2(1-x^2)}{\pi^2(1-x^2)}, \quad \frac{\mathrm{K}^2(1-x^2)}{\pi^2 x}, \quad \frac{\mathrm{K}^2(1-x^2)}{\pi^2}, \quad \frac{(1-x^2)\mathrm{K}^2(1-x^2)}{\pi^2 x}, \\ & \frac{\mathrm{K}^4(1-x^2)}{\pi^4 x(1-x^2)}, \quad \frac{\mathrm{K}^4(1-x^2)}{\pi^4 x}, \quad \frac{(1-x^2)\mathrm{K}^4(1-x^2)}{\pi^4 x}, \quad \frac{(1-x^2)^2\mathrm{K}^4(1-x^2)}{\pi^4 x} \quad \Big\} \end{aligned}$$

  - Differential equations not problematic for us, despite elliptics
  - Canonical form just as in polylogarithmic 3PM case

ϵ-form using INITIAL

[CD, Henn, Yan, '20 /
CD, Henn, Wagner, '22]

constant matrices
68×68

- Elliptic differential equations

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x)\vec{g}, \quad \tilde{A}(x) = \sum_i M_i a_i(x)$$

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  - Simple integration kernels $\Rightarrow$ easy to handle

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  - Simple integration kernels $\Rightarrow$ easy to handle
  - Eisenstein kernels: $E_{2,8,1,1,2}, E_{2,8,1,1,4}, E_{2,8,1,1,8} \longrightarrow$ efficient numerical evaluation in GiNaC

[Walden, Weinzierl, '20]

ϵ-form using INITIAL

[CD, Henn, Yan, '20 /
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[Walden, Weinzierl, '20]

efficient numerical
evaluation in GiNaC

- Observables: No iterated integrals of elliptic kernels
  - in ϵ-form, one can simply remove the kernels
  - otherwise $\int \mathrm{d}x \frac{2\mathrm{K}(1-x^2)\mathrm{E}(1-x^2)}{\pi^2 x} = \frac{(1-x^2)\mathrm{K}^2(1-x^2)}{\pi^2}$

$$\vec{g} = T\vec{f}$$

elliptics in the result come
from transformation only

[CD, Kälin, Liu, Porto, '21]

Boundary conditions

$$\vec{f} = T^{-1}\mathrm{P}\, e^{\epsilon \int \tilde{A}(x)\,\mathrm{d}x}\, \vec{g}_0(\epsilon)$$

- Compare series expansions around singular point
  - Small velocity expansion: $v \equiv \sqrt{\gamma^2 - 1}$ $\quad v \to 0 \Longleftrightarrow \gamma \to 1 \Longleftrightarrow x \to 1$

1) Solution of differential equations:

[Lee, Smirnov, Smirnov, Steinhauser, '19]

[Libra: Lee, '20]

2) Explicit integral expansions:

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Frobenius/Wasow:

$$\partial_x \vec{f} = A(x,\epsilon)\vec{f}, \qquad \vec{f}(v,\epsilon) \simeq \sum_{n_1,n_2,k} v^{n_1+n_2\epsilon} \log^k(v) H_{n_1,n_2,k}(\epsilon)\vec{g}_0(\epsilon), \qquad n_1, n_2, k \in \mathbb{Z}$$

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2) Explicit integral expansions:

method of regions:

$$I = \int \mathrm{d}k \frac{\delta(k\cdot u_1)\ldots}{k^2(k\cdot u_2)\ldots}, \qquad \vec{f}(v,\epsilon) \simeq \sum_{n_1,n_2,k} v^{n_1+n_2\epsilon} \log^k(v) \vec{h}_{n_1,n_2,k}(\epsilon) \longleftarrow$$

PN-like integrals

Boundary conditions

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2) Explicit integral expansions:

$$\vec{h}_{n_1,n_2,k}(\epsilon) = H_{n_1,n_2,k}(\epsilon)\vec{g}_0(\epsilon)$$

relations between infinite set of PN-like integrals

method of regions:

$$I = \int \mathrm{d}k \frac{\delta(k\cdot u_1)\dots}{k^2(k\cdot u_2)\dots}, \qquad \vec{f}(v,\epsilon) \simeq \sum_{n_1,n_2,k} v^{n_1+n_2\epsilon} \log^k(v) \vec{h}_{n_1,n_2,k}(\epsilon)$$

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relations between infinite set of PN-like integrals

- Identify an independent set: $\qquad \vec{g}_0(\epsilon) = H_{\text{indep}}(\epsilon)^{-1}\vec{h}_{\text{indep}}(\epsilon)$

Static boundary integrals

$$\vec{f}(v,\epsilon) \simeq \sum_{n_1,n_2,k} v^{n_1+n_2\epsilon} \log^k(v) \vec{h}_{n_1,n_2,k}(\epsilon), \qquad \vec{f} = \begin{pmatrix} I_{111000010101110} \\ \vdots \\ I_{111000111111111} \end{pmatrix}$$

- 3PM example:

$$v \to 0$$

$$I_N(x) = \int \mathrm{d}^D\ell \mathrm{d}^D k \frac{\delta((k-\ell)\cdot u_1)\delta(\ell \cdot u_2)}{(k-\ell+q)^2\ell^2 k^2}$$

[Beneke, Smirnov, '97]

[asy2.m: Pak, Smirnov, '10 /
Jantzen, Smirnov, Smirnov, '12]

[FIESTA3: Smirnov, '13]

- For Feynman: possible in parametric space

$$\alpha_i \to v^{2c_i}\alpha_i, \quad i = 1, \ldots, 3$$

$$\mathbf{c}_{\text{pot}} = (0,0,0), \quad \mathbf{c}_{\text{rad}} = (0,0,-1)$$

$$I_N(x) \simeq v^0 h_{0,0,0} + v^{1-2\epsilon} h_{1,-2,0} + \ldots$$

Static boundary integrals

$$\vec{f}(v,\epsilon) \simeq \sum_{n_1,n_2,k} v^{n_1+n_2\epsilon} \log^k(v) \vec{h}_{n_1,n_2,k}(\epsilon), \qquad \vec{f} = \begin{pmatrix} I_{111000010101110} \\ \vdots \\ I_{111000111111111} \end{pmatrix}$$

- 3PM example: $v \to 0$

[Beneke, Smirnov, '97]

[asy2.m: Pak, Smirnov, '10 /
Jantzen, Smirnov, Smirnov, '12]

[FIESTA3: Smirnov, '13]

$$I_N(x) = \int \mathrm{d}^D\ell \mathrm{d}^D k \frac{\delta((k-\ell)\cdot u_1)\delta(\ell \cdot u_2)}{(k-\ell+q)^2\ell^2 k^2}$$

- For Feynman: possible in parametric space

$$\alpha_i \to v^{2c_i}\alpha_i, \quad i = 1, \dots, 3 \qquad \mathbf{c}_{\text{pot}} = (0,0,0), \quad \mathbf{c}_{\text{rad}} = (0,0,-1)$$

$$I_N(x) \simeq v^0 h_{0,0,0} + v^{1-2\epsilon} h_{1,-2,0} + \dots$$

- Momentum space: radiation: $(\ell^0, \boldsymbol{\ell}) \sim (v, 1), \quad (k^0, \boldsymbol{k}) \sim (v, v),$

$$h_{1,-2,0}(\epsilon) = -\int \mathrm{d}^d\boldsymbol{\ell} \frac{1}{(\boldsymbol{\ell}-\boldsymbol{q})^2\boldsymbol{\ell}^2} \int \mathrm{d}^d\boldsymbol{k} \frac{1}{\boldsymbol{k}^2 - (\boldsymbol{\ell}\cdot\boldsymbol{n})^2}$$

$$d = 3 - 2\epsilon$$
$$\boldsymbol{n}^2 = 1, \quad \boldsymbol{n}\cdot\boldsymbol{q} = 0$$

Static boundary integrals

$I_N(x) \simeq v^0 h_{0,0,0} + v^{1-2\epsilon} h_{1,-2,0} + \dots$ $\qquad d = 3 - 2\epsilon, \quad \boldsymbol{n}^2 = 1, \quad \boldsymbol{n} \cdot \boldsymbol{q} = 0$

- Momentum space: radiation: $(\ell^0, \boldsymbol{\ell}) \sim (v, 1), \quad (k^0, \boldsymbol{k}) \sim (v, v),$

$$h_{1,-2,0}(\epsilon) = -\int \mathrm{d}^d \boldsymbol{\ell} \frac{1}{(\boldsymbol{\ell} - \boldsymbol{q})^2 \boldsymbol{\ell}^2} \underbrace{\int \mathrm{d}^d \boldsymbol{k} \frac{1}{\boldsymbol{k}^2 - (\boldsymbol{\ell} \cdot \boldsymbol{n})^2}}$$

[Galley, Leibovich, Porto, Ross, ‘16]

$$\int \mathrm{d}^d \boldsymbol{k} \frac{1}{[\boldsymbol{k}^2 - (\boldsymbol{\ell} \cdot \boldsymbol{n})^2 \pm i0]} = \frac{\Gamma(1 - d/2)}{[-(\boldsymbol{n} \cdot \boldsymbol{\ell})^2 \pm i0]^{1-d/2}} \longleftarrow \text{Feynman} \Rightarrow \text{conservative}$$

$$\int \mathrm{d}^d \boldsymbol{k} \frac{1}{[\boldsymbol{k}^2 - (\boldsymbol{\ell} \cdot \boldsymbol{n} \pm i0)^2]} = \frac{\Gamma(1 - d/2)}{[-(\boldsymbol{n} \cdot \boldsymbol{\ell} \pm i0)^2]^{1-d/2}} \longleftarrow \text{ret/adv} \quad \Rightarrow \text{dissipative}$$

Static boundary integrals

$$I_N(x) \simeq v^0 h_{0,0,0} + v^{1-2\epsilon} h_{1,-2,0} + \dots \qquad d = 3 - 2\epsilon, \quad \boldsymbol{n}^2 = 1, \quad \boldsymbol{n}\cdot\boldsymbol{q} = 0$$

- Momentum space: radiation: $(\ell^0, \boldsymbol{\ell}) \sim (v, 1), \quad (k^0, \boldsymbol{k}) \sim (v, v),$

$$h_{1,-2,0}(\epsilon) = -\int \mathrm{d}^d\boldsymbol{\ell} \frac{1}{(\boldsymbol{\ell}-\boldsymbol{q})^2\boldsymbol{\ell}^2} \underbrace{\int \mathrm{d}^d\boldsymbol{k} \frac{1}{\boldsymbol{k}^2 - (\boldsymbol{\ell}\cdot\boldsymbol{n})^2}}_{}$$

[Galley, Leibovich, Porto, Ross, '16]

$$\int \mathrm{d}^d\boldsymbol{k} \frac{1}{[\boldsymbol{k}^2 - (\boldsymbol{\ell}\cdot\boldsymbol{n})^2 \pm i0]} = \frac{\Gamma(1-d/2)}{[-(\boldsymbol{n}\cdot\boldsymbol{\ell})^2 \pm i0]^{1-d/2}}$$

Feynman ⇒ conservative

$$\int \mathrm{d}^d\boldsymbol{k} \frac{1}{[\boldsymbol{k}^2 - (\boldsymbol{\ell}\cdot\boldsymbol{n} \pm i0)^2]} = \frac{\Gamma(1-d/2)}{[-(\boldsymbol{n}\cdot\boldsymbol{\ell} \pm i0)^2]^{1-d/2}}$$

ret/adv ⇒ dissipative

$$h^{\mathrm{F}}_{1,-2,0}(\epsilon) = \frac{i^{-d}2^{5-2d}\Gamma(3-d)\Gamma(1-\frac{d}{2})\Gamma(d-2)\Gamma(\frac{d-1}{2})}{\Gamma(d-\frac{3}{2})}$$

$$h^{\mathrm{ret}}_{1,-2,0}(\epsilon) = \frac{i^{-2d}8^{2-d}\sqrt{\pi}\Gamma^2(1-\frac{d}{2})\Gamma(d-1)}{\Gamma(d-\frac{3}{2})}$$

compute boundary constants:

$$\vec{g}_0(\epsilon) = H_{\mathrm{indep}}(\epsilon)^{-1}\vec{h}_{\mathrm{indep}}(\epsilon)$$

Static boundary integrals

- Regions at 4PM

$$\alpha_i \to v^{2c_i}\alpha_i, \quad i = 1, \ldots, 6$$

$$\begin{array}{cccccl}
(\ell^0, \boldsymbol{\ell}) & (k_1^0, \boldsymbol{k}_1) & (k_2^0, \boldsymbol{k}_2) & & & \\
(v,1) & (v,1) & (v,1) & \longleftrightarrow & \mathbf{c}_{\text{pot}} = (0,0,0,0,0,0), & \nearrow v^{n_1} \\
(v,1) & (v,1) & (v,v) & \longleftrightarrow & \mathbf{c}_{1\text{rad}} = (0,0,0,0,0,-1), & \nearrow v^{n_1-2\epsilon} \\
(v,1) & (v,v) & (v,v) & \longleftrightarrow & \mathbf{c}_{2\text{rad}} \in \{(0,0,0,0,-1,-1), (0,0,0,-1,0,-1), & \nearrow v^{n_1-4\epsilon} \\
 & & & & \quad (0,0,0,-1,-1,-1)\} &
\end{array}$$

- Momentum space:

$$I(x) = \int \mathrm{d}^D\ell \mathrm{d}^D k_1 \mathrm{d}^D k_2 \frac{\cdots \quad \overbrace{\delta(k_1^0 - \ell^0)\delta(k_2^0 + \ell^0)}}{\cdots (\ell - q)^2 (k_1 + k_2)^2 k_1^2 k_2^2}$$

Summary

- Gravitational two-body problem
- 4PM loop integrals
- Differential equations

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x) \vec{g}$$

  - Canonical form
  - Elliptic integrals
  - Function space

$$a_i(x) \in \left\{ \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)}, \frac{1}{1-x}, \frac{1}{x}, \frac{1}{1+x}, \frac{x}{1+x^2}, \frac{\mathrm{K}^2(1-x^2)}{\pi^2 x(1-x^2)}, \frac{\mathrm{K}^4(1-x^2)}{\pi^4 x(1-x^2)}, \dots \right\}$$

- Boundary constants

$$\vec{h}_{n_1,n_2,k}(\epsilon) = H_{n_1,n_2,k}(\epsilon)\vec{g}_0(\epsilon)$$

  - Relations
  - Static integrals

$$\ell \sim (v,1), \quad k_1 \sim (v,v), \quad k_2 \sim (v,v)$$

$$\int \mathrm{d}^d\boldsymbol{k} \frac{1}{[\boldsymbol{k}^2 - (\boldsymbol{\ell}\cdot\boldsymbol{n} \pm i0)^2]} = \frac{\Gamma(1-d/2)}{[-(\boldsymbol{n}\cdot\boldsymbol{\ell} \pm i0)^2]^{1-d/2}}$$

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Outlook

- Spin
- 5PM
    - Integrand
    - IBPs (Master integrals)
    - DEs
    - Boundary conditions