An open twistor sigma model for 4d gravity and celestial $Lw_{1+\infty}$ symmetry

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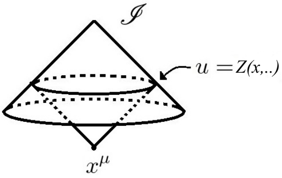
We consider 4d pure gravity in *split signature*:

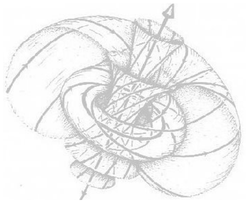
- Adapt Math.DG/0504582, Duke Math (2007), LeBrun & M. to formulate global SD gravity in split signature manifesting Strominger's celestial $Lw_{1+\infty}$ symmetry.
- Adapt Adamo, M. & Sharma 2103.16984, to construct full gravity tree-level S-matrix from open chiral sigma model built from from $\mathscr{I}$ with $Lw_{1+\infty}$ vertex operators.

Holography from null infinity, and amplitudes

- Celestial Holography seeks to find boundary theory that constructs 4d gravity from $\mathscr{I}$.
- Newman '70's: tries to rebuild space-time from 'cuts' of $\mathscr{I}$.
- Yields instead '$\mathcal{H}$-space' a complex self-dual space-time.



- Penrose: $\rightsquigarrow$ asymptotic Twistor space $P\mathscr{T} \sim \mathbb{CP}^3$, the *nonlinear graviton*.
- Embodies integrability of SD sector.
- Chiral sigma models in twistor space give full 4d gravity S-matrix expanding around SD sector; manifests $Lw_{1+\infty}$ symmetry.



Gravity amplitudes at MHV ($- - + \ldots +$ helicity)

Scatter n gravitons with momenta k_i, $i = 1, \ldots n$.

- In 2-component spinors, null momenta $k_{i\alpha\dot\alpha} = \kappa_{i\alpha}\kappa_{i\dot\alpha}$. Spinor helicity notation:

$$\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}\,,\ [1\,2] := \kappa_{1\dot\alpha}\kappa_2^{\dot\alpha}\,,\ \langle 1|2|3] = \kappa_{1\alpha}k_2^{\alpha\dot\alpha}\kappa_{3\dot\alpha}\,.$$

- Hodges 2012 MHV formula, define $n \times n$ matrix:

$$\mathbb{H}_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & i \neq j \\ -\sum_k \frac{[ik]}{\langle ik\rangle} & i = j\,. \end{cases}$$

Then: $$\mathcal{M}(1, \ldots, n) = \langle 12\rangle^6 \det{}'\mathbb{H}\, \delta^4(\textstyle\sum_i k_i)$$

- $\mathbb{H}$ is Laplace matrix for matrix-tree theorem $\rightsquigarrow_{\text{[Feng,He 2012]}}$
- Sum of trees [Bern,Dixon,Perelstein,Rosowski '98, Nguyen, Spradlin, Volovich, Wen '10]

Why??? $\mathcal{M} = \langle V_1 \ldots V_{n-2}\rangle_{\text{tree}}$ from Sigma model.

Conformal geometry in 4d split signature & self-duality

Conformal group $= SO(3,3)$ acts globally on:

- Conformally flat models: $S^2 \times S^2$ or $S^2 \times S^2/\mathbb{Z}_2$:

$$ds^2 = \Omega^2 (ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}) ,$$

  Coordinates $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $|\mathbf{x}| = |\mathbf{y}| = 1$.
- $\mathbb{Z}_2$ acts by $(\mathbf{x}, \mathbf{y}) \to (-\mathbf{x}, -\mathbf{y})$.
- For flat $\Lambda = 0 : \Omega \sim \frac{1}{x_3 - y_3}$, and

$$\mathscr{I} = \{x_3 = y_3\} = \mathbb{R} \times S^1 \times S^1 .$$

(For $\Lambda \neq 0$: $\Omega \sim 1/y_3$, and $\mathscr{I} = S^2 \times S^1$.)

Curvature: For curved (M^4, g), 2-forms split: $\Omega^2_M = \Omega^{2+} \oplus \Omega^{2-}$

$$\text{Riem} = \begin{pmatrix} \text{Weyl}^+ + S\delta & \text{Ricci}_0 \\ \text{Ricci}_0 & \text{Weyl}^- + S\delta \end{pmatrix} .$$

This talk: expand around $\text{Weyl}^- = 0 = \text{Ricci}$, so Ω^{2-} is flat.

α and β-surfaces and the Zollfrei condition

The split signature conformally flat metric

$$ds^2 = \Omega^2 (ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}) \,,$$

admits a 3-parameter family of β-planes denoted by $\mathbb{PT}_{\mathbb{R}}$:

- respectively totally null ASD S^2s given by

$$\mathbf{x} = A\mathbf{y} \,, \qquad A \in SO(3) = \mathbb{RP}^3 \,.$$

- $\mathrm{Weyl}^- = 0 \Rightarrow \beta$-planes survive as β-surfaces.
- β-surfaces are projectively flat.
- If compact, β-surfaces are necessarily S^2 or $\mathbb{RP}^2$.
- Null geodesics are projectively $\mathbb{RP}^1$s or double cover.

Following Guillemin we define:

Definition

(M^d, g) is Zollfrei if all null geodesics are embedded S^1s.

Conformally self-dual case

Theorem (LeBrun & M. [Duke Math J. 2007, math.dg/0504582.]**)**

Let $(M^4, [g])$ be Zollfrei with nontrivial SD Weyl-curvature. Then $M = S^2 \times S^2$ and there is a $1:1$-correspondence between

1. *SD conformal structures on $S^2 \times S^2$ near flat model &*
2. *Deformations $\mathbb{PT}_{\mathbb{R}}$ of standard embedding of $\mathbb{RP}^3 \subset \mathbb{CP}^3$.*

Let $U \simeq \mathbb{R}^3 \times \mathbb{RP}^3 \subset \mathbb{CP}^3$ be a neighbourhood of of $\mathbb{RP}^3$ in $\mathbb{CP}^3$,
$\rightsquigarrow \mathbb{PT}_{\mathbb{R}} := \{\beta \text{ surfaces in } M\} = \{\text{graph of map } F : \mathbb{RP}^3 \to \mathbb{R}^3\}$

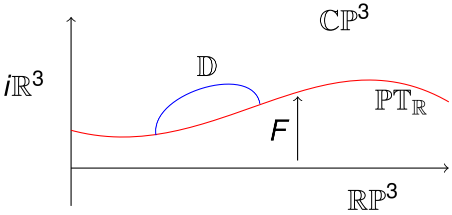


Figure: $\mathbb{D}$ = hol. disc $\subset \mathbb{CP}^3$ with $\partial\mathbb{D} \subset \mathbb{PT}_{\mathbb{R}}$.

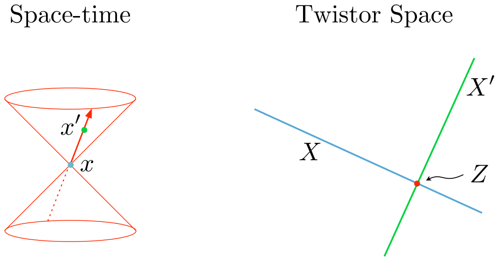
Reconstruction of M^4 from twistor space $\mathbb{PT}_\mathbb{R}$

Each $x \in M^4 \leftrightarrow$ holomorphic disc $\mathbb{D}_x \subset \mathbb{CP}^3$ with $\partial\mathbb{D}_x \subset \mathbb{PT}_\mathbb{R}$.

- $\mathbb{D}_x$ has topological degree one.
- Reconstruct M^4 from $\mathbb{PT}_\mathbb{R}$ space of all such disks:

$$M = \{\text{Moduli of degree-1 hol. disks: } \mathbb{D}_x \subset \mathbb{CP}^3, \partial\mathbb{D}_x \subset \mathbb{PT}_\mathbb{R}\}$$

- Gives compact 4d moduli space
- M admits a conformal structure for which $\partial\mathbb{D}_x \cap \partial\mathbb{D}_{x'} = Z$ means that x, x' sit on same β-plane:



Restriction to Einstein vacuum case

Adapting Penrose nonlinear graviton (1976) to split signature

Which $\mathbb{PT}_{\mathbb{R}} \subset \mathbb{CP}^3$ give SD Einstein $g \in [g]$ on $S^2 \times S^2$?

- Let $Z^A = (\lambda_\alpha, \mu^{\dot\alpha})$, $\alpha = 0, 1, \dot\alpha = \dot 0, \dot 1$ be homogenous coordinates for $\mathbb{CP}^3$.
- Introduce Poisson structure and 1-form

$$\{f, g\} := \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\beta}} = \left[\frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \mu}\right] ,$$

$$\theta := \epsilon^{\alpha\beta} \lambda_\alpha d\lambda_\beta = \langle \lambda d\lambda \rangle$$

  of rank 2 and homogeneity degree -2 and 2 respectively.

Theorem

A vacuum $g \in [g]$ exists when $\theta|_{\mathbb{PT}_{\mathbb{R}}}$ & $\{\,,\}_{\mathbb{PT}_{\mathbb{R}}}$ are real.

Poisson diffeos of plane & $Lw_{1+\infty}$

W_N = higher spin symmetries in 2d CFT [Zamolodchikov 1980s].
For $N \to \infty$ classical limit w_∞ = Poisson diffeos of plane [Hoppe].

- Plane has coords $\mu^{\dot\alpha}$, $\dot\alpha = \dot 0, \dot 1$ with Poisson bracket
$$\{f, g\} := \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\alpha}}\,, \qquad \varepsilon^{\dot\alpha\dot\beta} = \varepsilon^{[\dot\alpha\dot\beta]}, \quad \varepsilon^{\dot 0 \dot 1} = 1\,.$$
- Basis of $w_{1+\infty} \leftrightarrow$ polynomial hamiltonians
$$w^p_m = (\mu^{\dot 0})^{p-m-1}(\mu^{\dot 1})^{p+m-1}\,, \qquad |m| \leq p-1, \quad 2p-2 \in \mathbb{N}$$
- Poisson brackets $\leftrightarrow$ commutation relations of $w_{1+\infty}$:
$$\{w^p_m, w^q_n\} = (2(p-1)n - 2(q-1)m) w^{p+q-2}_{m+n}\,.$$
- Loop algebra $Lw_{1+\infty}$, loop coord $\frac{\lambda_1}{\lambda_0} = \tan\frac{\theta}{2}$, generators
$$g^p_{m,r} = w^p_m \mathrm{e}^{ir\theta}\,, \qquad r \in \mathbb{Z}.$$
- Poisson brackets now
$$\{g^p_{m,r}, g^q_{n,s}\} = (2(p-1)n - 2(q-1)m)\, g^{p+q-2}_{m+n,r+s}\,.$$

This is the structure preserving diffeomorphism group of $\mathbb{PT}_{\mathbb{R}}$.

Generating functions for Einstein embeddings

Explicitly in homogeneous coordinates:

- Let $Z^A = U^A + iV^A$, with $U^A, V^A \in \mathbb{R}^4$.
- Let $h(U)$ be an arbtrary function of homogeneity degree 2,

$$U \cdot \frac{\partial h}{\partial U} = 2h.$$

Proposition

All 'small' Einstein vacuum twistor data $\leftrightarrow$ $h(U)$ by setting

$$\mathbb{T}_{\mathbb{R}} = \left\{ V^A = \left\{ h, U^A \right\} \right\} = \left\{ v_\alpha = 0, v^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} \frac{\partial h}{\partial u^{\dot\beta}} \right\}$$

projectivising gives $\mathbb{PT}_{\mathbb{R}}$.
The corresponding self-dual $(2,2)$ vacuum metrics are Zollfrei on $S^2 \times S^2$ with null $\mathscr{I}$ modelled by $x_3 = y_3$.

The Poisson bracket underpins Strominger's $Lw_{1+\infty}$ structure, [Adamo, M., Sharma, 2110.06066.]. Here $Lw_{1+\infty}$ acts canonically on

$$\{\text{SD gravity phase space}\} = Lw^{\mathbb{C}}_{1+\infty}/Lw_{1+\infty} \ni h(U)$$

Flat holography: the split signature story from $\mathscr{I}$

A celestial torus

Now $\mathscr{I} = \mathbb{R} \times S^1 \times S^1$ with real coords $(u, \lambda, \tilde{\lambda})$, $\lambda = \lambda_1/\lambda_0$.

$$ds^2 = \frac{1}{R^2}\left(dudR - d\lambda d\tilde{\lambda} + R\sigma d\tilde{\lambda}^2 + R\tilde{\sigma} d\lambda^2 + \dots\right),$$

where $R = 1/r$, and $\mathscr{I} = \{R = 0\}$.

- The $\sigma, \tilde{\sigma}$ are now *real* asymptotic *shears* that encode respectively SD and ASD gravitational data.
- Twistors intersect $\mathscr{I}$ in null geodesic circles in $\lambda =$ const.:

$$u = Z(\lambda, \tilde{\lambda}), \qquad \frac{\partial^2 Z}{\partial \tilde{\lambda}^2} = \sigma(Z, \lambda, \tilde{\lambda}).$$

  $\rightsquigarrow$ Zollfrei projective structure on each $\lambda =$ const..
- In general $\exists$ nonlinear correspondence [Lebrun & M, JDiffGeom. '02]:

$$\{\text{Zollfrei proj. str.} \leftrightarrow \sigma\} \overset{1:1}{\longleftrightarrow} \{h(U)\}.$$

- In linear theory map is analogue of radon transform

$$\sigma(u, \tilde{\lambda}, \lambda) = \partial_u^2 \int_{-\infty}^{\infty} dt\, h(\mu^{\dot{\alpha}} + t\tilde{\lambda}^{\dot{\alpha}}, \lambda_\alpha).$$

Open chiral twistor sigma models

- Express disk as upper-half-plane $\mathbb{D} = \{\sigma \in \mathbb{C}, \mathrm{Im}\,\sigma \geq 0\}$.
- Represent $\mathbb{D} \subset \mathbb{PT}$ with $\partial\mathbb{D} \subset \mathbb{PT}_{\mathbb{R}}$ in hgs coords by

$$Z^A(\sigma) : \mathbb{D} \to \mathbb{T}\,, \qquad Z^A|_{\sigma=\bar\sigma} \in \mathbb{T}_{\mathbb{R}}\,.$$

- For k points $\sigma_i \in \mathbb{R}$, and $Z_i^A \in \mathbb{T}_{\mathbb{R}}$, $\exists!$ deg $k-1$ disk thru Z_i :

$$Z^A(\sigma) = \sum_{i=1}^k \frac{Z_i^A}{\sigma - \sigma_i} + M(\sigma)\,, \qquad M(\sigma) \text{ holomorphic on } \mathbb{D}.$$

- For $Z = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}_{\mathbb{R}}$ implies λ_α real.
- Therefore $M^A = (0, m^{\dot\alpha})$, but $m^{\dot\alpha} \neq 0$ unless $h = 0$.
- Action for holomorphy and boundary conditions:

$$S_D[Z(\sigma), Z_i] = \mathrm{Im} \int_{\mathbb{D}} [m\,\bar\partial m] d\sigma + \oint_{\partial\mathbb{D}} h(Z) d\sigma$$

using *spinor-helicity* notation $[\mu\,\nu] := \mu_{\dot\alpha}\nu^{\dot\alpha}$, $\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}$.

Gravity S-matrix on SD background via sigma model

Amplitudes are functionals $\mathcal{M}[h, \tilde{h}_i]$ of gravitational data:

- $h \in C^\infty(\mathbb{PT}_\mathbb{R}, \mathcal{O}(2))$for fully nonlinear SD part,
- $\tilde{h}_i \in C^\infty(\mathbb{PT}_\mathbb{R}, \mathcal{O}(-6))$, $i = 1, \ldots, k$, ASD perturbations.
- For eigenstates of momentum $k_{i\alpha\dot{\alpha}} = \kappa_{i\alpha}\tilde{\kappa}_{i\dot{\alpha}}$ take:

$$h_i = \int \frac{dt}{t^3}\delta^2(t\lambda_\alpha - \kappa_{i\alpha})e^{it[\mu,\tilde{\kappa}_i]}, \quad \tilde{h}_i = \int \frac{dt}{t^{-5}}\delta^2(t\lambda_\alpha - \kappa_{i\alpha})e^{it[\mu,\tilde{\kappa}_i]}$$

Proposition (Adapted from [Adamo, M. & Sharma, 2103.16984] to split signature.)

The amplitude for k ASD perturbations on SD background h is

$$\mathcal{M}(h, \tilde{h}_i) = \int_{(S^1 \times \mathbb{PT}_\mathbb{R})^k} S_D^{os}[h, Z_i, \sigma_i] \det{}' \tilde{\mathbb{H}} \prod_{i=1}^{k} \tilde{h}_i(Z_i) D^3 Z_i d\sigma_i \, .$$

Here $S_D^{os}[h, Z_i, \sigma_i]$ is the on-shell Sigma model action and

$$\tilde{\mathbb{H}}_{ij}(Z_i) = \begin{cases} \frac{\langle\lambda_i\lambda_j\rangle}{\sigma_i - \sigma_j} & i \neq j \\ -\sum_l \frac{\langle\lambda_i\lambda_l\rangle}{\sigma_i - \sigma_j}\, , & i = j\, . \end{cases}$$

Ideas in proof: the complete tree-level S-matrix

- Expand $h = h_{k+1} + \ldots + h_n$ to 1st order in momentum e-states h_i to give flat background perturbative amplitude.
- On shell action expands as tree correlator

$$S_D^{os}[h_{k+1} + \ldots + h_n, Z_i, \sigma_i] = \langle V_{h_{k+1}} \ldots V_{h_n} \rangle_{tree} + O(h_i^2)\,.$$

- Here the 'vertex operators' are $V_{h_i} = \int_{\partial D} h_i(\sigma_i) d\sigma_i$.
- Propagators for S_D give Poisson bracket $\{\, ,\}$

$$\langle h_i h_j \rangle_{tree} = \frac{[\partial_\mu h_i\, \partial_\mu h_j]}{\sigma_i - \sigma_j} = \frac{[i\, j]}{\sigma_i - \sigma_j} h_i h_j\,, \qquad i \neq j\,.$$

- Matrix-tree theorem then gives

$$\langle h_{k+1} \ldots h_n \rangle_{\text{tree}} = \det{}' \mathbb{H} \prod_{i=k+1}^{n} h_i\,, \qquad \mathbb{H}_{ij} = \frac{[ij]}{\sigma_i - \sigma_j}\,, \quad i \neq j \text{ etc.}$$

$$\rightsquigarrow \qquad \mathcal{M}(h_i, \tilde{h}_i) = \int_{(S^1)^n \times (\mathbb{RP}^3)^k} \det{}' \mathbb{H} \det{}' \tilde{\mathbb{H}} \prod_{j=k+1}^{n} h_j d\sigma_j \prod_{i=1}^{k} \tilde{h}_i(Z_i) D^3 Z_i d\sigma_i\,.$$

This is now equivalent to the Cachazo-Skinner formula.

Summary & conclusions

Geometry:

- Split signature SD vacuum metrics on $S^2 \times S^2$ with $\mathscr{I} \simeq S^1 \times S^1 \times \mathbb{R} \leftrightarrow C^\infty$ generating functions h on $\mathbb{RP}^3$, defines deformed real slice $\mathbb{PT}_\mathbb{R} \subset \mathbb{CP}^3$.
- Split signature twistors avoid 'lightray' or Čech-Dolbeault transform manifesting $Lw_{1+\infty}$ directly.
  Slogan: SD gravity phase space $= Lw^{\mathbb{C}}_{1+\infty}/Lw_{1+\infty}$
- Similar results follow for $\Lambda \neq 0$ where $h \leftrightarrow 2+1$ signature conformal structure of $\mathscr{I} = S^2 \times S^1$.

Sigma model:

- Reconstruction via open holomorphic discs leads to chiral open sigma model that computes gravity amplitudes.
- MHV formula gives theory underlying tree formalism of Bern et. al. from 1998 & equals the Einstein-Hilbert action.
- Framework gives $Lw^{\mathbb{C}}_{1+\infty}$ action on *full amplitude* via vertex operators that generate gravitons.
  (Real generators are passive with vanishing charges).

Discussion

Double copy

- Yang-Mills Parke Taylor $\leftrightarrow \det{}'\mathbb{H}\det{}'\tilde{\mathbb{H}}$ for GR.
- This comes from $Lw_{1+\infty} \otimes \widetilde{Lw_{1+\infty}}$.
- $Lw_{1+\infty}$ is sufficient at MHV, but both are needed for kinematic algebra beyond MHV but dont play so nicely!
- In ambitwistor-string, we can analyse vertex operators for both sectors V_h, $V_{\tilde{h}}$ with OPEs

$$V_h \cdot V_{h'} \sim V_{\{h,h'\}} + \dots, \qquad \text{but} \qquad V_h \cdot V_{\tilde{h}} \sim \text{ mess} \qquad (1)$$

But see Wei's talk.

1-loop all $+$

- 1-loop all + becomes bubble in background on disc.
- Remains $Lw_{1+\infty}$ invariant.

Thank you!

$Lw_{1+\infty}$ symmetries of sigma model

Recall, $Lw_{1+\infty} = \lambda$-dependent Poisson diffeos of $\mu^{\dot\alpha}$-plane.

- For sigma model action

$$S[Z\sigma)] = \mathrm{Im} \int_{\mathbb{D}} d\sigma \left[\mu\bar\partial_\sigma\mu\right] + \oint_{\partial\mathbb{D}} 2hd\sigma$$

Poisson diffeo with Hamiltonian $g(W)$ gives

$$\delta\mu^{\dot\alpha} = \{g, \mu^{\dot\alpha}\} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial g}{\partial\mu^{\dot\beta}}\,, \qquad \delta h = \{h, g\}\,.$$

- Symmetry $\Rightarrow \delta h = 0 \Rightarrow g$ holomorphic, $\bar\partial_h g = 0$.
- If g is real $\rightsquigarrow$ diffeomorphism of $\mathbb{PT}_{\mathbb{R}}$, coordinate freedom. Cf, supertranslations, BMS etc..
- If g is imaginary, defines an infinitesimal generating function $\rightsquigarrow$ perturbation i.e. graviton, perturbation of metric.

Quantization

- Einstein gravity tree $=$ tree sigma model correlator (MHV).
- Does full quantum sigma model correlator $\leftrightarrow$ gravity loops?

$$\langle 1\,2\rangle^{2n} \prod_{i=3}^{n} \frac{1}{\langle 1\,i\rangle^2 \langle 2\,i\rangle^2} \exp\left[-\frac{\mathrm{i}\,\alpha}{8\pi} \sum_{j\neq i} \frac{[i\,j]}{\langle i\,j\rangle} \frac{\langle 1\,i\rangle^2 \langle 2\,j\rangle^2}{\langle 1\,2\rangle^2} \right] .$$

- Does quantum sigma model realize $W_{1+\infty}$ or W-gravity?
- Moyal quantization of $\mu^{\dot{\alpha}}$-plane and 'palatial twistors'?

Questions:

- $N = 8$ formulation with S_n symmetry?
- Axiomatize correspondence between celestial OPES and twistor sigma model/ambitwistor string OPEs.

[Adamo, Casali, Sharma, Wei, arxiv: 2111.02279]