Classical scattering from coherent-spin amplitudes

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QCD meets Gravity - 2022
Zürich

Based on

Mainly on…

- **Classical observables from coherent-spin amplitudes**

  RA and Alexander Ochirov [hep-th/2108.01649]

But also…

- **Searching for Kerr at 2PM**

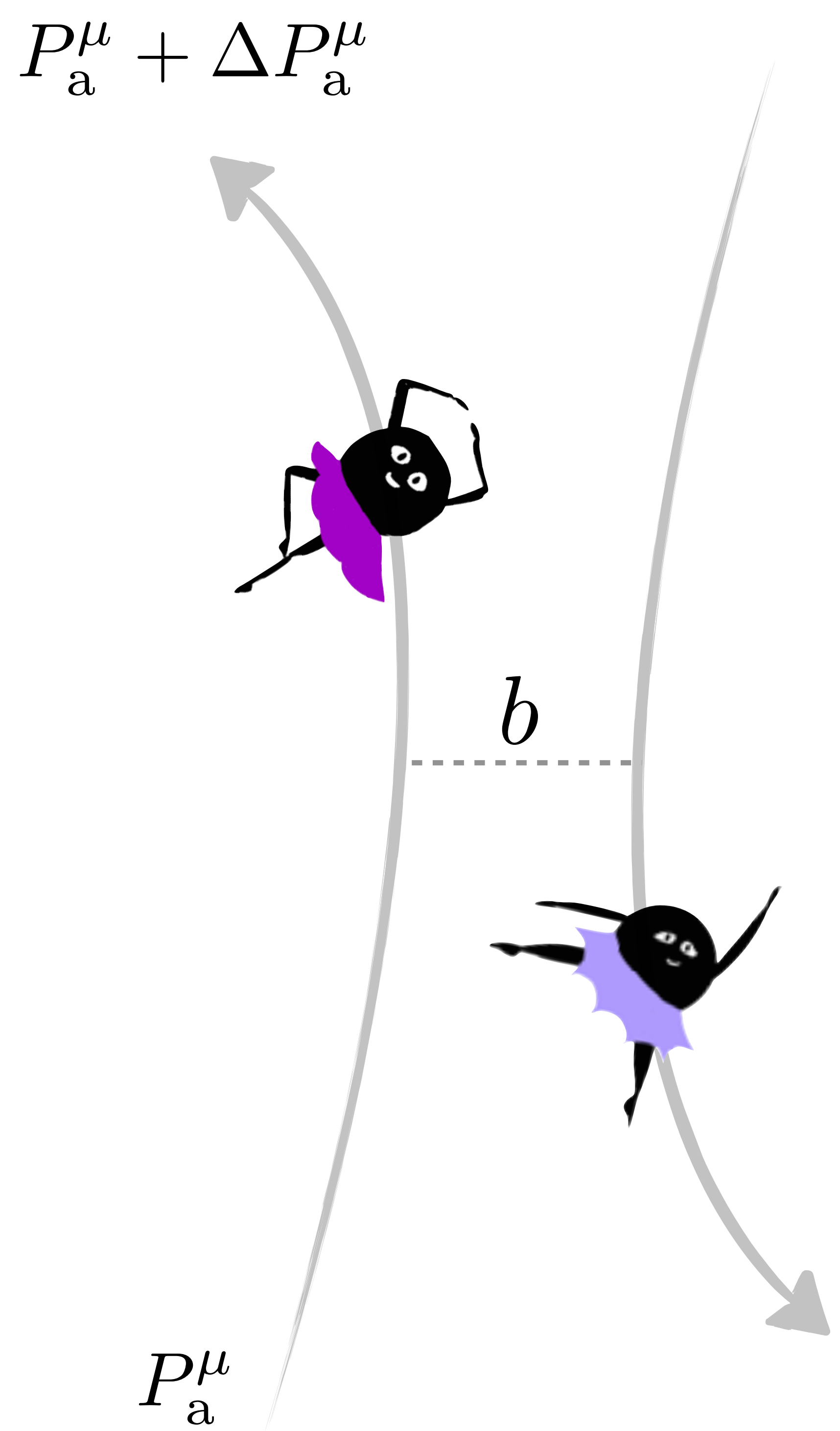
  RA, Kays Haddad and Andreas Helset [hep-th/2203.06197]

- **Classical gravitational spinning-spinless scattering at $\mathcal{O}(G^2 S^\infty)$**

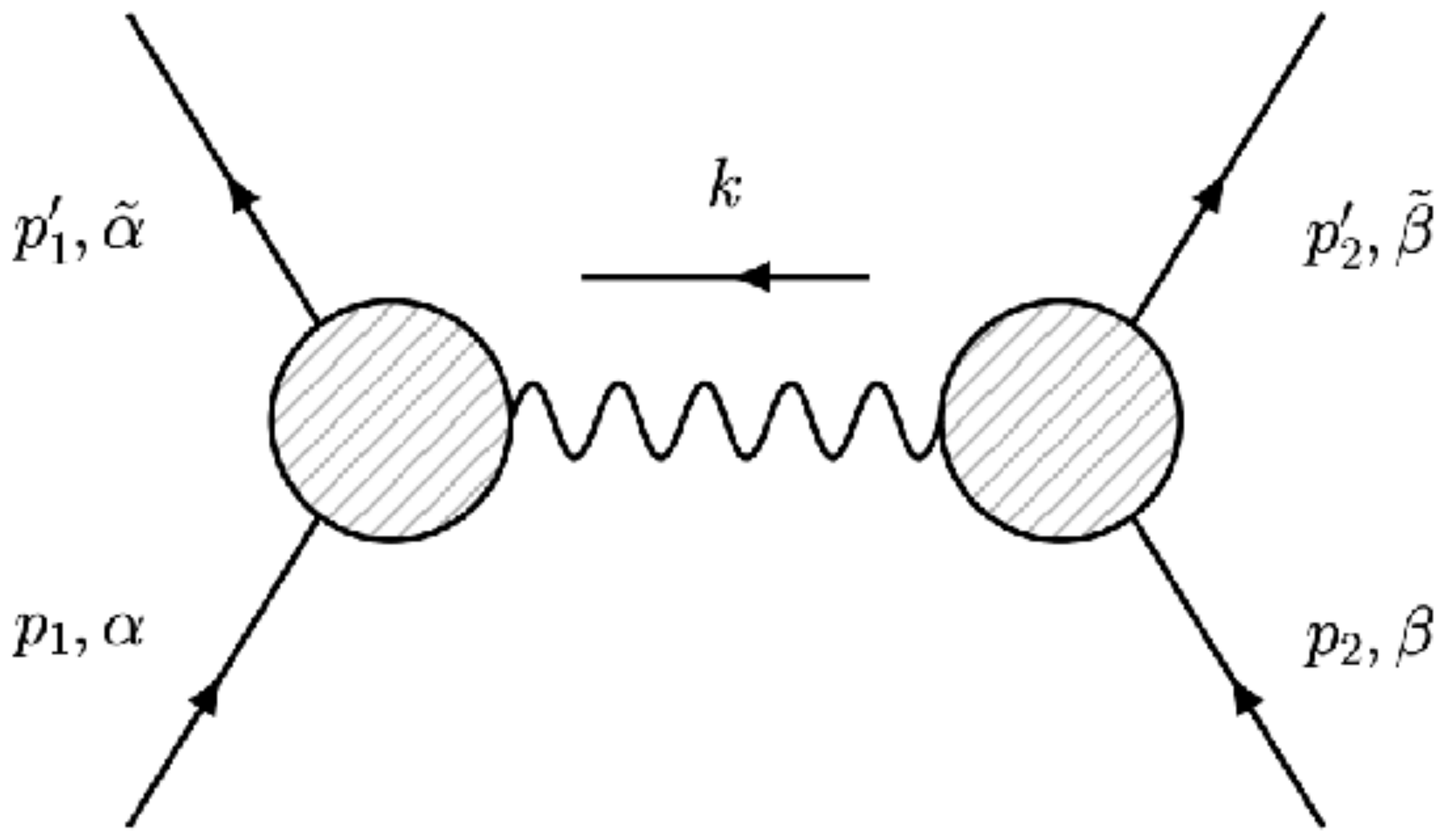
  RA, Kays Haddad and Andreas Helset [hep-th/2205.02809]

Pictorically…

Classical limit: $\hbar \to 0$



$$\Delta P^\mu_{\mathrm{a}} = -\hbar \frac{\partial}{\partial b_\mu} \int_{p_{\mathrm{a}}, p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \int_k e^{-i\bar{k}\cdot b}$$



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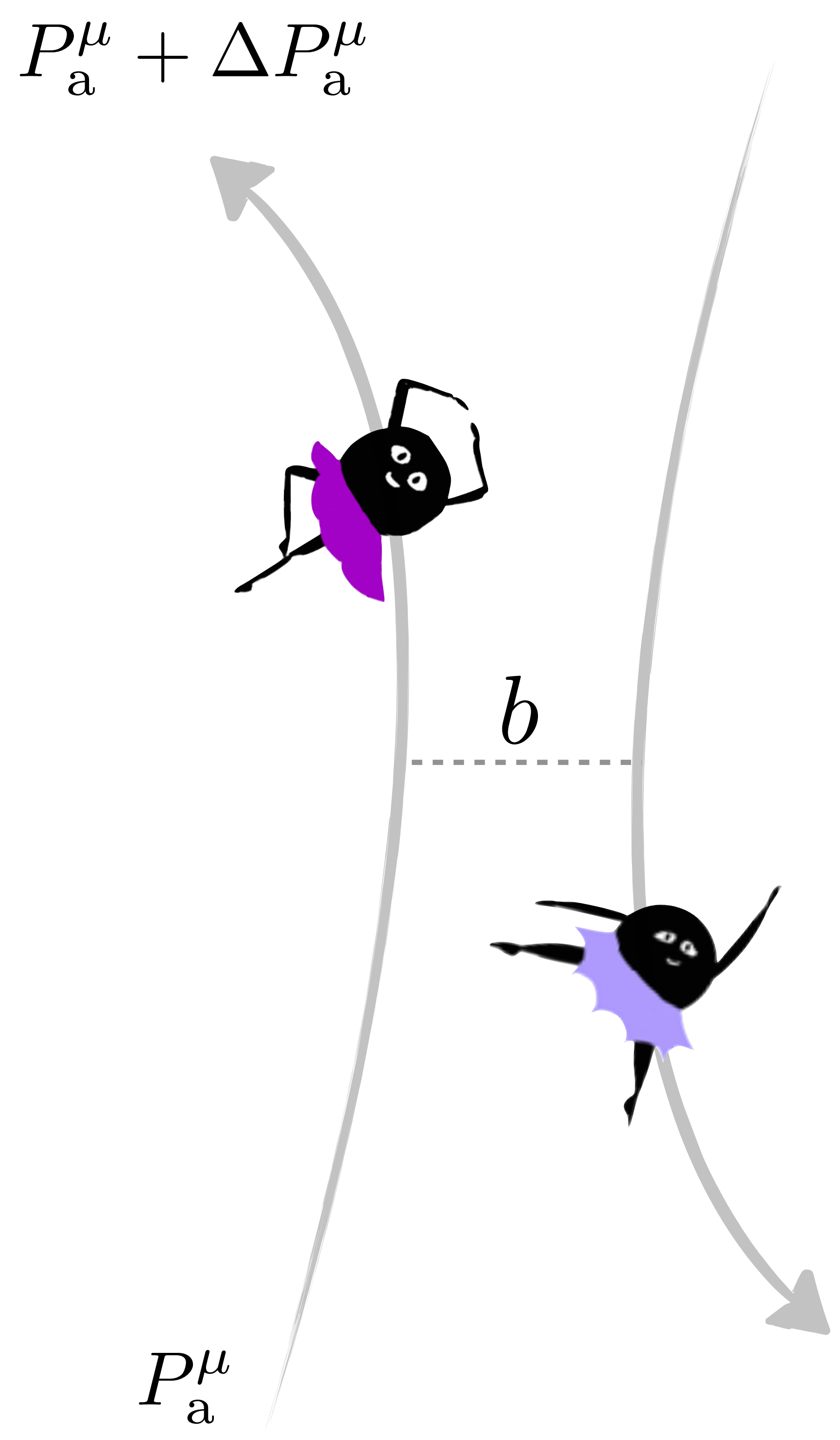


Classical limit: $\hbar \to 0$

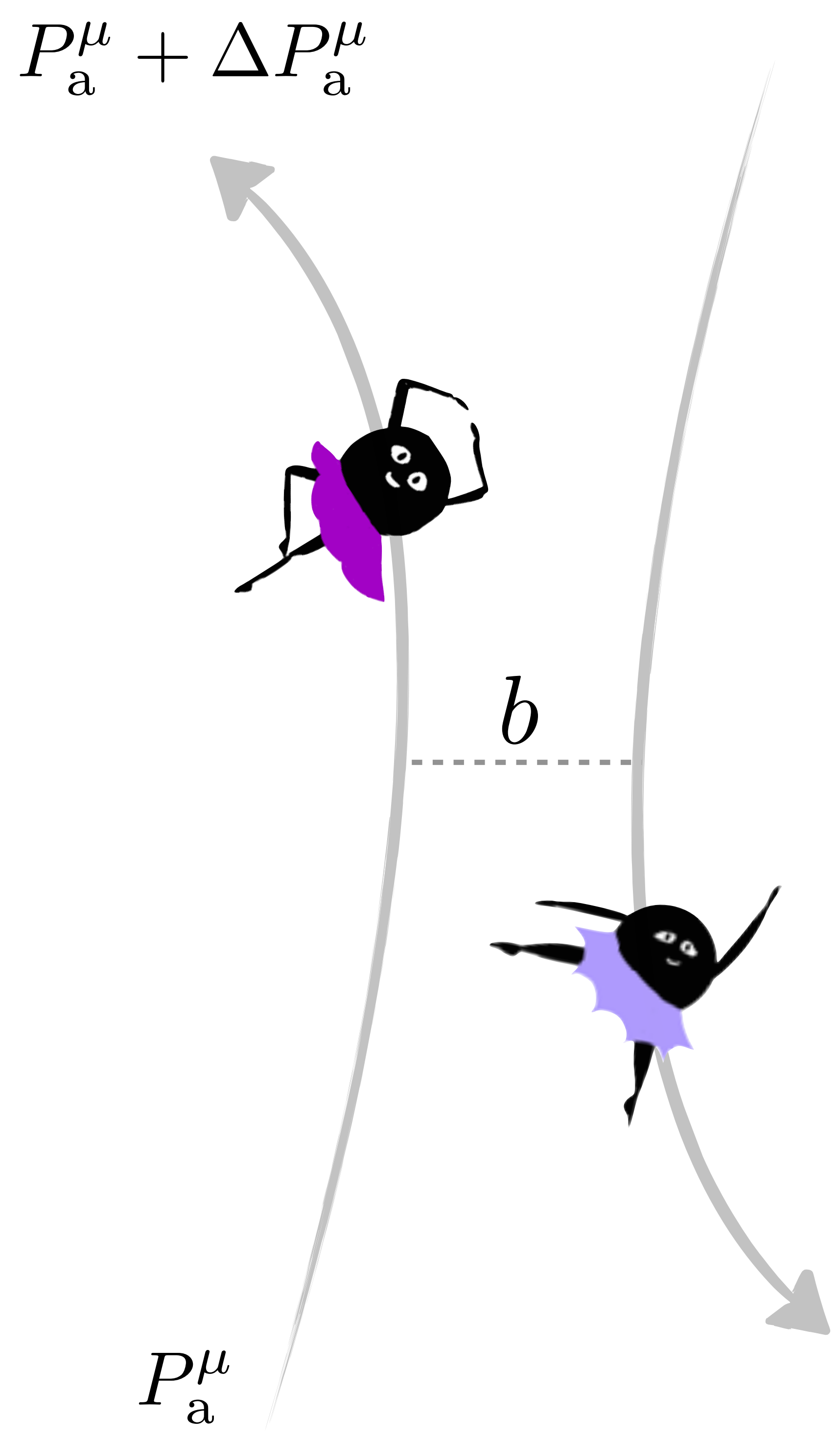
$$\Delta P^\mu_{\rm a} = -\hbar \frac{\partial}{\partial b_\mu} \int_{p_{\rm a}, p_{\rm b}} |\psi_{\rm a}(p_{\rm a})|^2 |\psi_{\rm b}(p_{\rm b})|^2 \int_{k} e^{-i\bar{k}\cdot b}$$

The KMOC formalism:

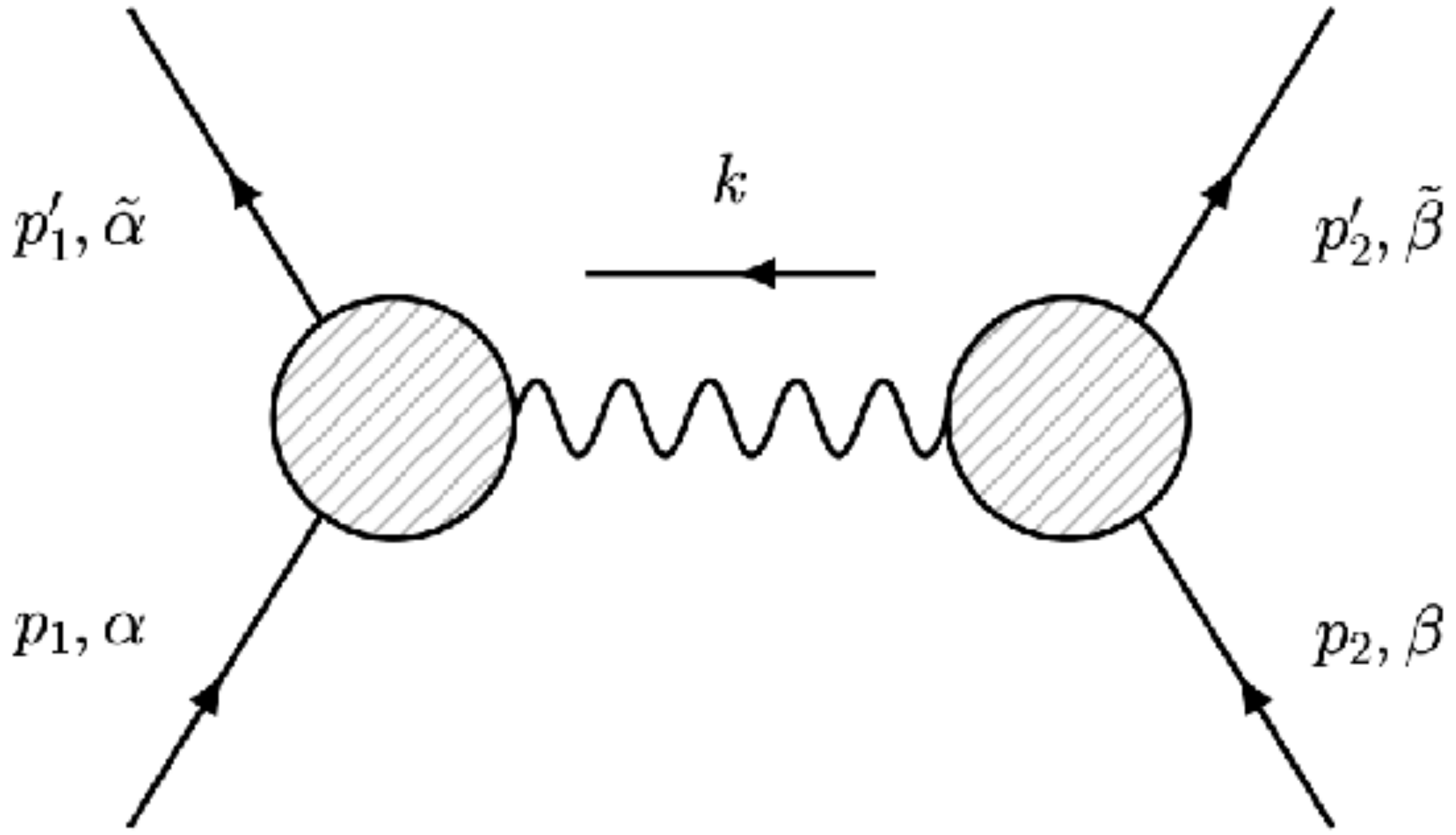
- quantum expectation values
- chosen initial quantum states
- classical observables when $\hbar \to 0$

Pictorically…

Classical limit: $\hbar \to 0$



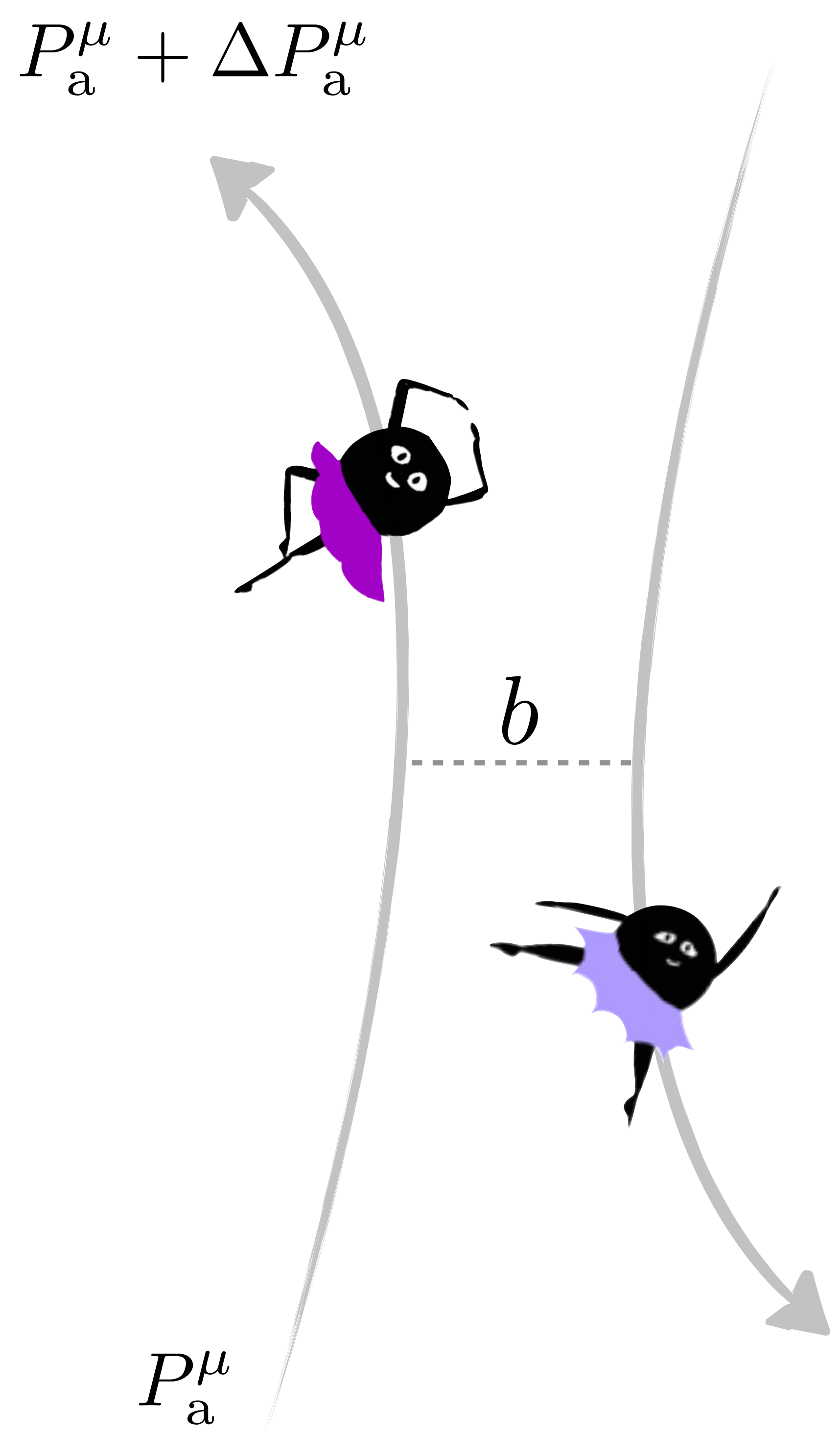
$$\Delta P_{\mathrm{a}}^{\mu} = -\hbar \frac{\partial}{\partial b_{\mu}} \int_{p_{\mathrm{a}}, p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \int_k e^{-i\bar{k}\cdot b}$$



Scattering of two coherent-spin states mediated by a graviton

Factorizes into two 3pts.

Pictorically…



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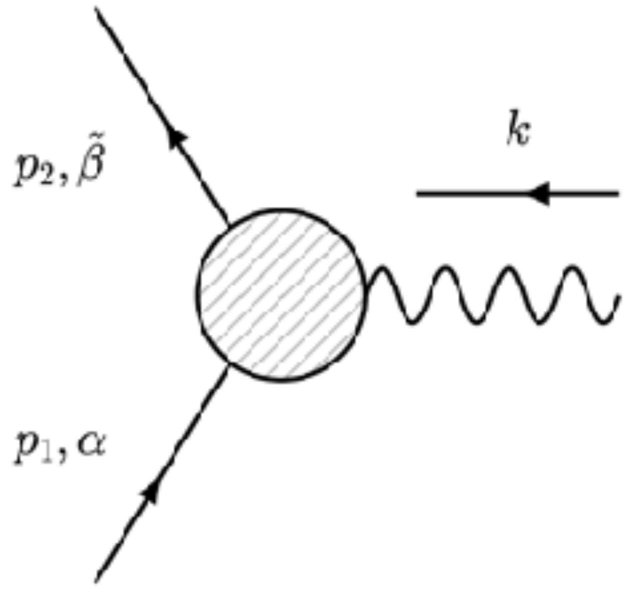
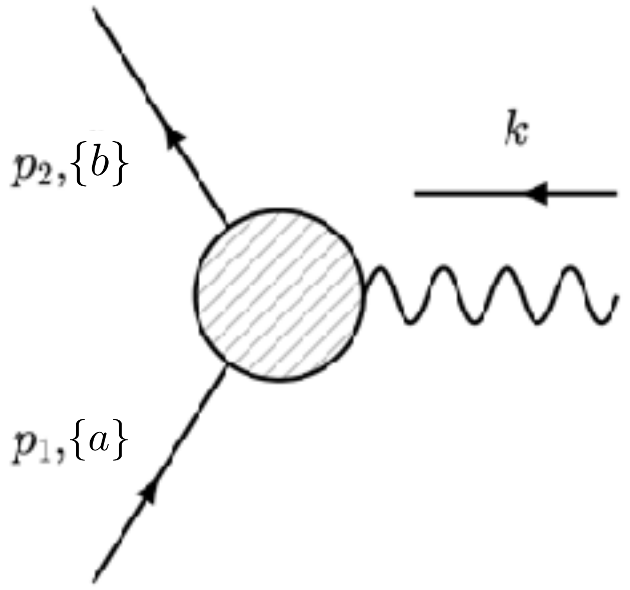
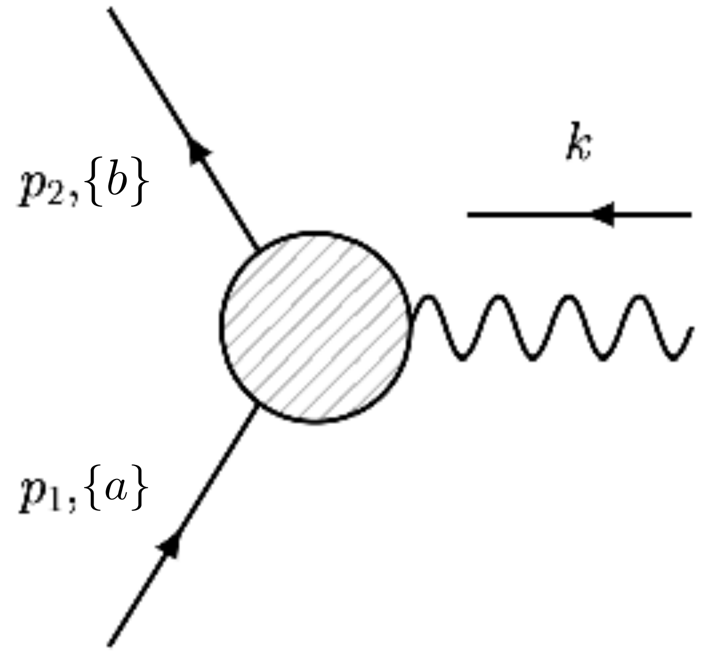
Coherent-spin amplitudes as a linear combination of definite-spin amplitudes



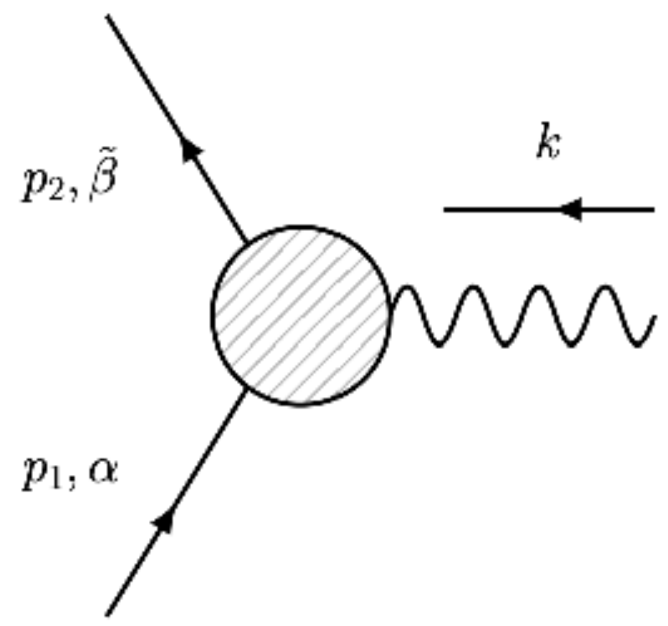
$$= e^{-(\|\alpha\|^2 + \|\beta\|^2)/2} \sum_{s_1, s_2} \frac{(\tilde{\beta}_b)^{\odot 2s_2} (\alpha^a)^{\odot 2s_1}}{\sqrt{(2s_1)!(2s_2)!}} \cdot$$



Pipeline to Classical Observables with Spin



$\Rightarrow$



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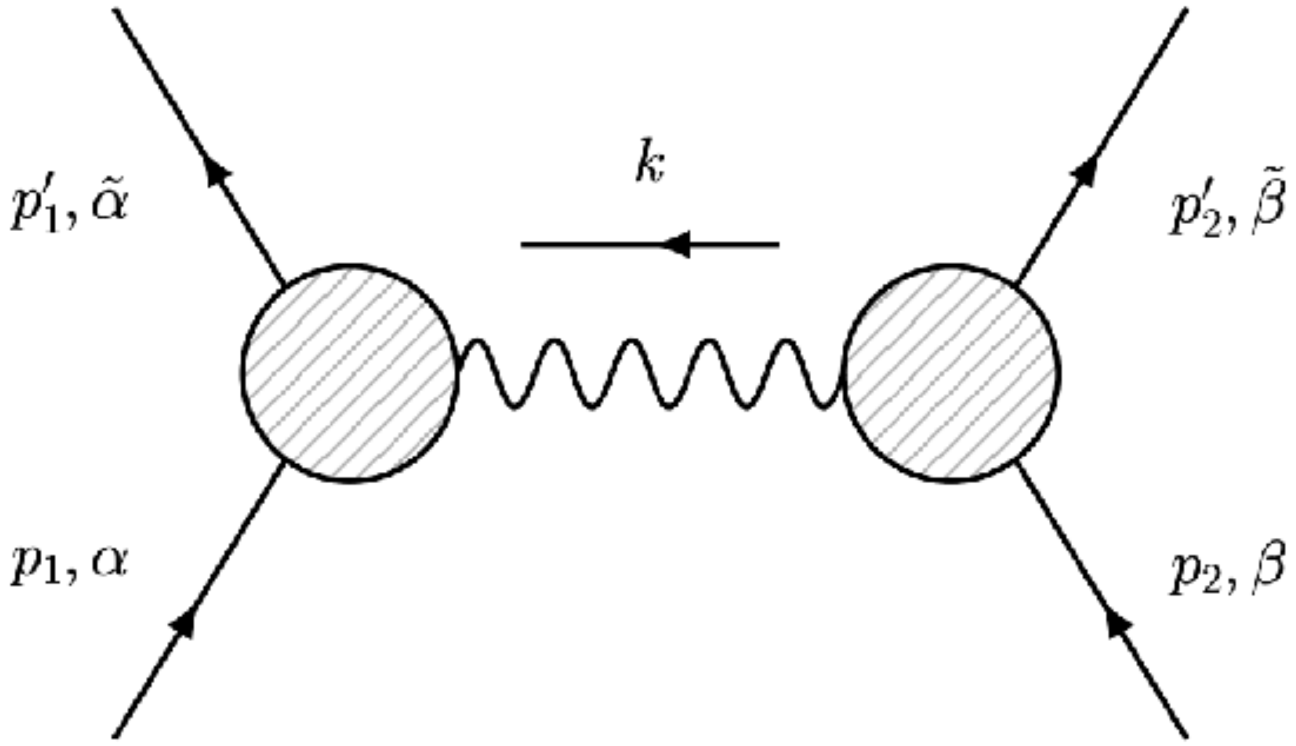


$\Rightarrow$

$$\Delta O$$
$$V$$

Definite-spin
Three-points

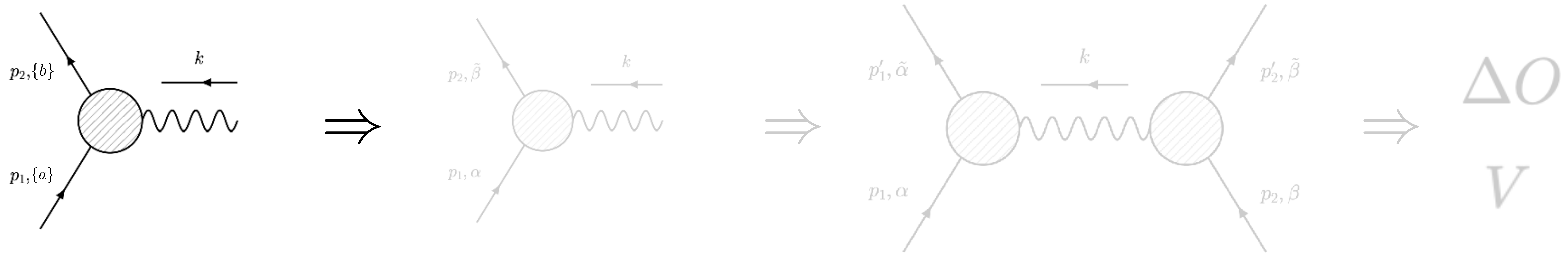
Spin-coherent
Three-points

Spin-coherent
Four-points

Observables

+ higher PM orders

Pipeline to Classical Observables with Spin



Definite-spin
Three-points

On-shell amplitudes

Definite-spin scattering amplitudes

[Arkani-Hamed, Huang, Huang '17]

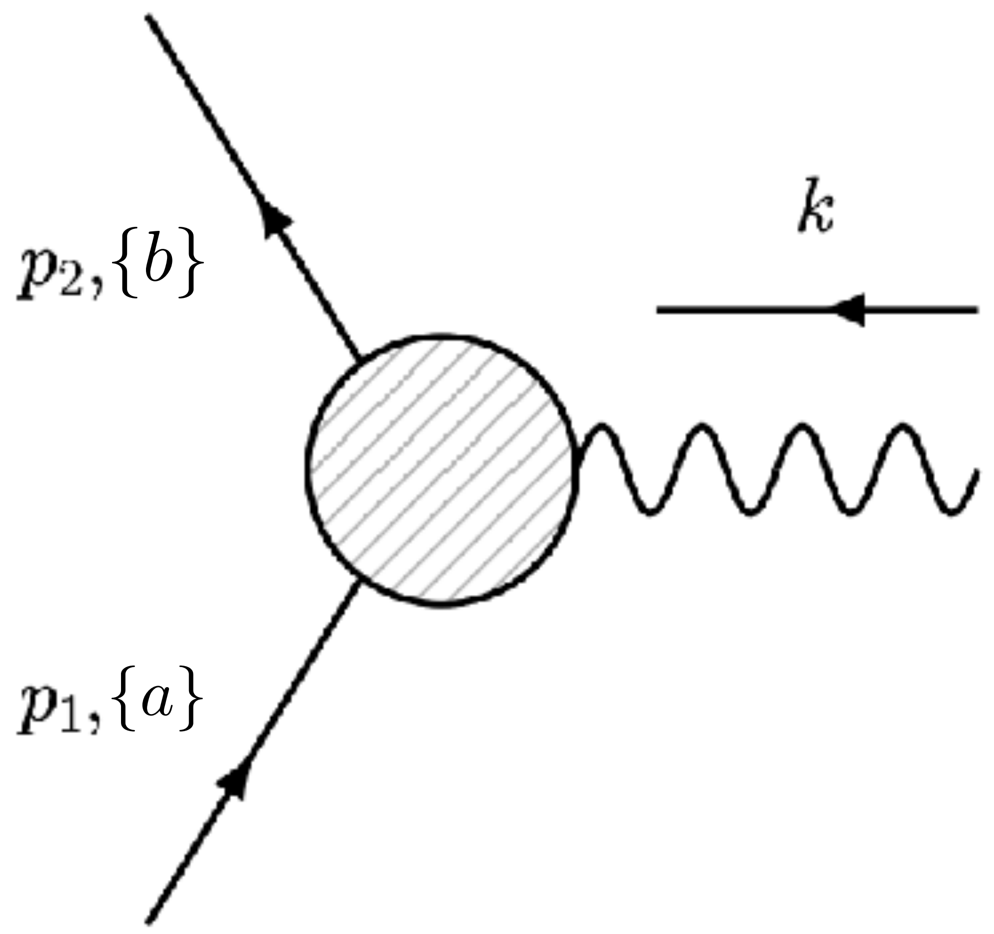
Spin-s: 2s indices

$\{a\} = \{a_1, a_2, ..., a_{2s}\}$

$\odot 2s$ symmetrization

Using the particles' little group: minimal coupling with a graviton

Best behavior in the high-energy limit



$$\mathcal{A}_{\text{min}}^{(0)\{b\}}{}_{\{a\}}(p_2, s|p_1, s; k, +) = -\frac{\kappa}{2}\frac{\langle 2^b 1_a\rangle^{\odot 2s}}{m^{2s-2}}x^2,$$

$$x = \frac{[k|p_1|r\rangle}{m\langle kr\rangle} = -\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^+)$$

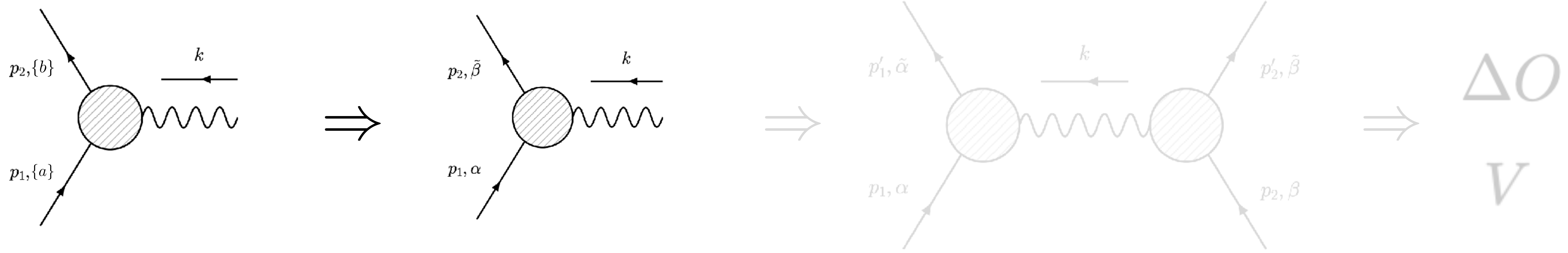
Contract all definite-spin LG indices $\longrightarrow$ Coherent-spin amps.

Non-minimal later!

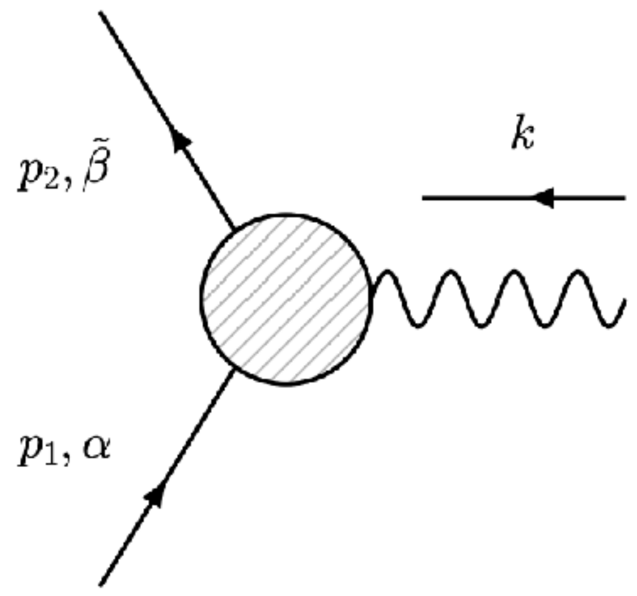
See also [Chiodaroli, Johansson, Pichini '21] for aux. vars.

*Similar for negative helicities

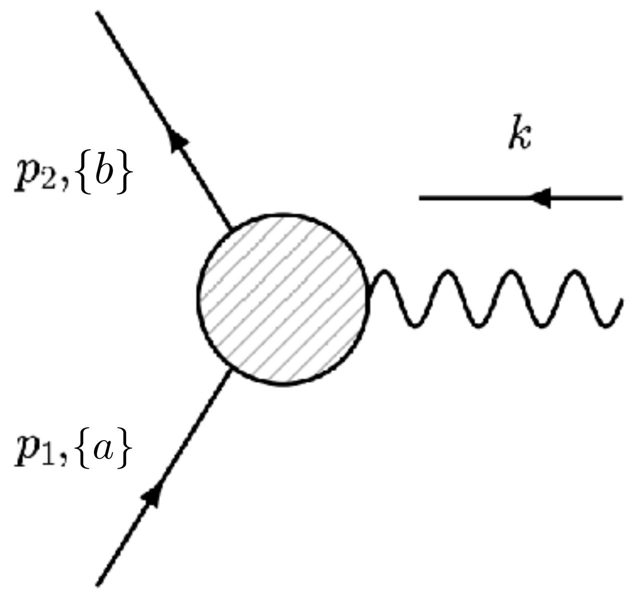
Pipeline to Classical Observables with Spin



Coherent-spin states and amplitudes



$$= e^{-(\|\alpha\|^2+\|\beta\|^2)/2} \sum_{s_1,s_2} \frac{(\tilde{\beta}_b)^{\odot 2s_2}(\alpha^a)^{\odot 2s_1}}{\sqrt{(2s_1)!(2s_2)!}} \cdot$$



Why coherent-states?

Rigorous framework for quantum-classical transitions

Schwinger's construction for spin-coherent states [Schwinger '52]

$\xleftrightarrow{SU(2)}$

Massive little-group of definite-momenta amplitudes [Arkani-Hamed, Huang, Huang '17]

Using coherent-spin states:

- contract with the little-group indices
- Identify the classical spin
- KMOC formalism with the aid of coherent states

See also [Bern, Luna, Roiban, Shen, Zeng '20] for spin-coherent states

See also [Cristofoli, Gonzo, Kosower, O'Connell '21] for massless coherent states

Textbook coherent states

Quantum Harmonic Oscillator

$$H = \hbar\omega(a^\dagger a + 1/2) \longrightarrow E_n = \hbar\omega(n + 1/2)$$

$\hbar \to 0$ $\quad n \to \infty$ $\quad$ classical limit

Uncertainties: $\Delta_n x = \sqrt{\frac{\hbar}{m\omega}(n + 1/2)}$ $\qquad \Delta_n p = \sqrt{m\omega\hbar(n + 1/2)}$

Finite errors in the classical limit :(

Textbook coherent states

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Finite errors in the classical limit :(

Coherent states: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \longrightarrow |\alpha\rangle = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}|0\rangle$

Uncertainties: $\Delta_\alpha x = \sqrt{\frac{\hbar}{2m\omega}}$ $\quad \Delta_\alpha p = \sqrt{\frac{m\omega\hbar}{2}}$

Vanishing errors in the classical limit :)

Textbook coherent states

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Uncertainties: $\Delta_\alpha x = \sqrt{\frac{\hbar}{2m\omega}}$ $\quad \Delta_\alpha p = \sqrt{\frac{m\omega\hbar}{2}}$

Vanishing errors in the classical limit :)

$$E_\alpha = \hbar\omega(\,||\alpha||^2 + 1/2)$$

For the energy to be finite $\quad ||\alpha||^2 \to \infty$

in the classical limit

Saturates the uncertainty principle

Expectation values evolve classically

[Schwinger, 1952]

Spin-states

Schwinger's construction: general spin from zero-spin with 2 creation ops.

along the z-axis

$$|s, s_z\rangle = \frac{\left(a_1^\dagger\right)^{s+s_z}\left(a_2^\dagger\right)^{s-s_z}}{\sqrt{(s+s_z)!(s-s_z)!}}|0\rangle, \qquad s_z = -s, -s+1, \ldots, s-1, s.$$

Covariantize it:

$$[a^a, a_b^\dagger] = \delta_b^a, \qquad \boldsymbol{S} = \frac{\hbar}{2} a_a^\dagger \boldsymbol{\sigma}^a{}_b a^b \qquad \Rightarrow \qquad [S^i, S^j] = i\hbar\epsilon^{ijk}S^k.$$

SU(2)-covariant s-spin states

$$|s, \{a\}\rangle \equiv |s, \{a_1 \ldots a_{2s}\}\rangle = \frac{1}{\sqrt{(2s)!}} a_{a_1}^\dagger a_{a_2}^\dagger \ldots a_{a_{2s}}^\dagger |0\rangle \equiv \frac{\left(a_a^\dagger\right)^{\odot 2s}}{\sqrt{(2s)!}}|0\rangle.$$

Coherent Spin-states

Coherent-spin states defined as

$$|\alpha\rangle = e^{-\tilde{\alpha}_a \alpha^a/2} e^{\alpha^a a_a^\dagger}|0\rangle \qquad \Rightarrow \qquad a^a|\alpha\rangle = \alpha^a|\alpha\rangle,$$

In terms of definite spin:

$$|\alpha\rangle = e^{-(\tilde{\alpha}\alpha)/2} \sum_{2s=0}^{\infty} \frac{(\alpha^a)^{\odot 2s}}{\sqrt{(2s)!}} \cdot |s, \{a\}\rangle,$$

We want the coherent state in terms of definite spin…

because we know the general definite-spin amplitudes

See also [Chiodaroli, Johansson, Pichini '21] for aux. variables.

See Paolo's Talk!

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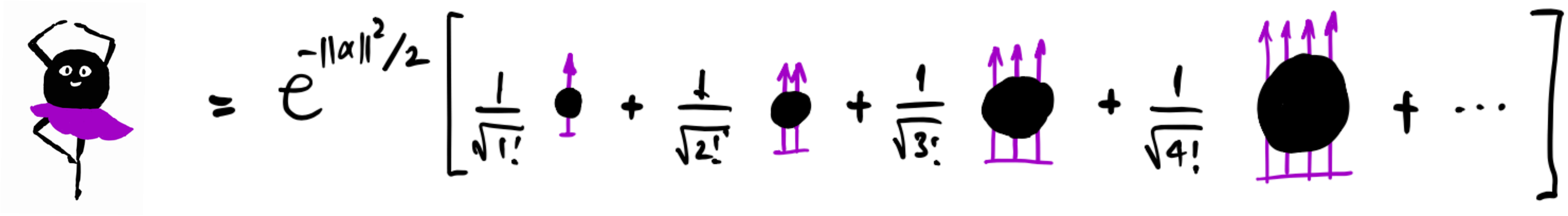

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Lorentz-covariant SU(2) spin operator

$$\sigma_{p\mu,a}{}^{b} = -\frac{1}{2m}\Big(\langle p_a|\sigma_\mu|p^b] + [p_a|\bar{\sigma}_\mu|p^b\rangle]\Big),$$

Taking the classical limit (KMOC + coherent)

See also [Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White '21]

$$\langle S_p^\mu\rangle_\alpha = \frac{\hbar}{2}(\tilde{\alpha}\sigma_p^\mu \alpha) \xrightarrow[\hbar\to 0]{} s_{\mathrm{cl}}^\mu,$$

$$\langle S_p^\mu S_p^\nu\rangle_\alpha = \langle S_p^\mu\rangle_\alpha \langle S_p^\nu\rangle_\alpha + \mathcal{O}(\hbar) \xrightarrow[\hbar\to 0]{} s_{\mathrm{cl}}^\mu s_{\mathrm{cl}}^\nu$$

Dressing the Minimal coupling

Minimal 3-point $\mathcal{A}_{\text{min}}^{(0)\{b\}}{}_{\{a\}} = -\frac{\kappa}{2}\frac{\langle 2^b 1_a\rangle^{\odot 2s}}{m^{2s-2}}x^2,$

$$\mathcal{A}_3^h \equiv \mathcal{A}^{(0)}(p_2, \beta | p_1, \alpha; k, h) = e^{-(\|\alpha\|^2+\|\beta\|^2)/2}\sum_{s_1;s_2}\frac{(\tilde{\beta}_b)^{\odot 2s_2}(\alpha^a)^{\odot 2s_1}}{\sqrt{(2s_1)!(2s_2)!}}\cdot \mathcal{A}^{(0)\{b\}}{}_{\{a\}}$$

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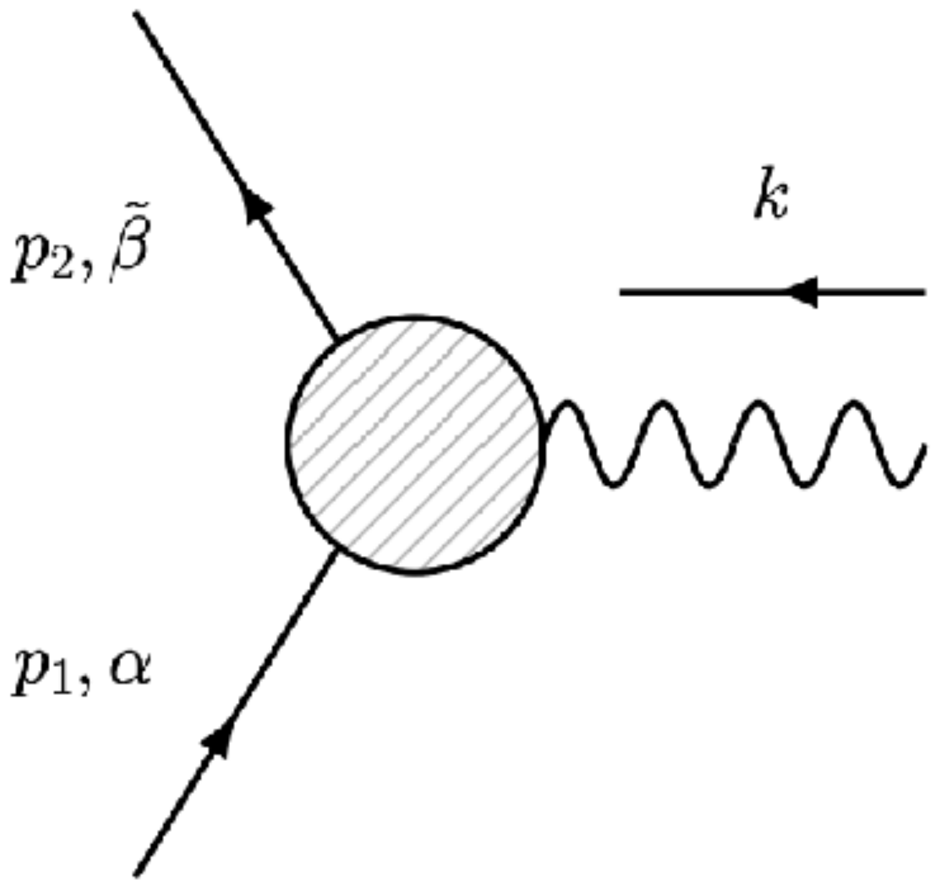


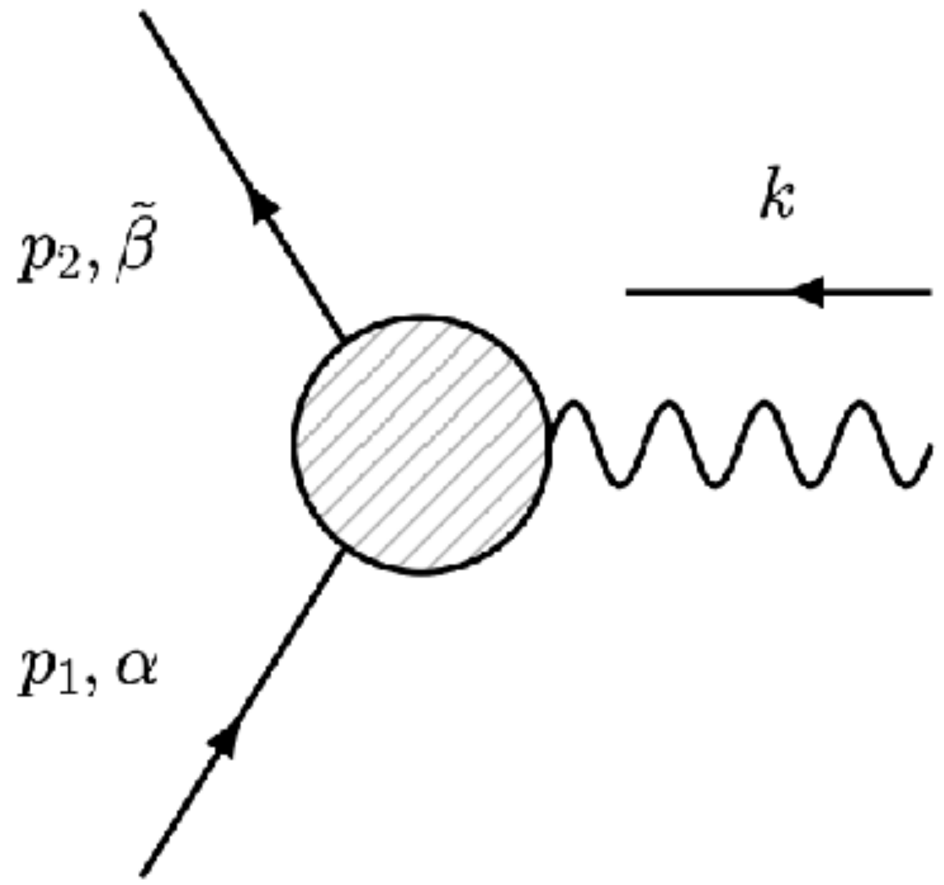
$$\begin{aligned}\mathcal{A}_{3,\min}^{+} &= -\frac{\kappa}{2}x^2 e^{-(\|\alpha\|^2+\|\beta\|^2)/2}\sum_{2s=0}^{\infty}\frac{1}{(2s)!}(\tilde{\beta}_b)^{\odot 2s}\cdot\frac{\langle 2^b 1_a\rangle^{\odot 2s}}{m^{2s-2}}\cdot(\alpha^a)^{\odot 2s}\\ &= -\frac{\kappa}{2}m^2x^2e^{-(\|\alpha\|^2+\|\beta\|^2)/2}\exp\left\{\frac{1}{m}\tilde{\beta}_b\langle 2^b 1_a\rangle\alpha^a\right\}.\end{aligned}$$

It exponentiates!

Boost to the same momenta

On-shell kinematics

$$p_{\rm a} = (p_1 + p_2)/2 = p_1 + k/2 = p_2 - k/2$$



Boost for spinors

$$|1_a\rangle = U_a{}^b(p_1, p_{\rm a})\left(|{\rm a}_b\rangle - \frac{1}{4m}|k|{\rm a}_b]\right),$$

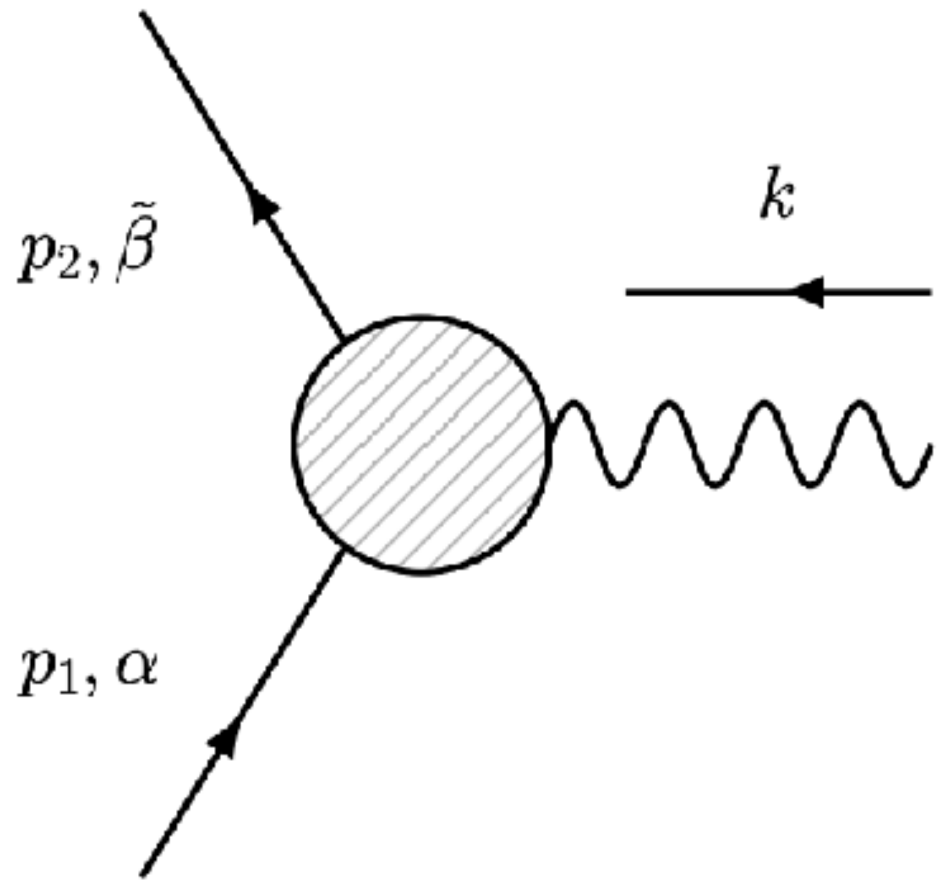
(Similar for 2)

The exponent $$\tilde{\beta}_b(p_2)\langle 2^b 1_a\rangle \alpha^a(p_1) = \tilde{\beta}_b(p_{\rm a})\left(\underbrace{\langle {\rm a}^b {\rm a}_a\rangle}_{\text{spinless term}} - \frac{1}{4m}\left(\underbrace{[{\rm a}^b|k|{\rm a}_a\rangle + \langle {\rm a}^b|k|{\rm a}_a]}_{\text{spin generator}}\right)\right)\alpha^a(p_{\rm a}).$$

This can be greatly simplified using heavy-variables (HPET)

[RA, Haddad, Helset, '20]

Boost to the same momenta



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$$\boxed{\mathcal{A}^{\pm}_{3,\rm min} = -\frac{\kappa}{2} m^2 x^{\pm 2} \underbrace{e^{-(\|\alpha\|^2 + \|\beta\|^2)/2 + \tilde{\beta}\alpha}}_{\text{overlap between coherent states}} \exp\left\{\mp \frac{\hbar}{2m}\bar{k}_\mu (\tilde{\beta}\sigma^\mu_{p_{\rm a}}\alpha)\right\}}$$

Classical limit and classical three-points

Factored out the standard coherent-state overlap: $\langle\beta|\alpha\rangle = e^{-(\|\alpha\|^2+\|\beta\|^2)/2+\tilde{\beta}\alpha}$

In the classical limit, we take: $\tilde{\beta}_a = (\alpha^a)^*$ $\longrightarrow$ Exact cancellation between the spinless term and the normalization

and we can identify the spin expectation value

$$\boxed{\mathcal{A}^{\pm}_{3,\min}\Big|_{\beta=\alpha} = -\frac{\kappa}{2}m^2 x^{\pm 2}\exp\left\{\mp\frac{1}{m}\bar{k}_\mu\langle S^\mu_{p_{\mathrm{a}}}\rangle_\alpha\right\} = -\frac{\kappa}{2}m^2 x^{\pm 2}e^{\mp\bar{k}\cdot a_{\mathrm{a}}}.}$$

Matches the Kerr BH ‘amplitude’

Can use directly to built four-points.

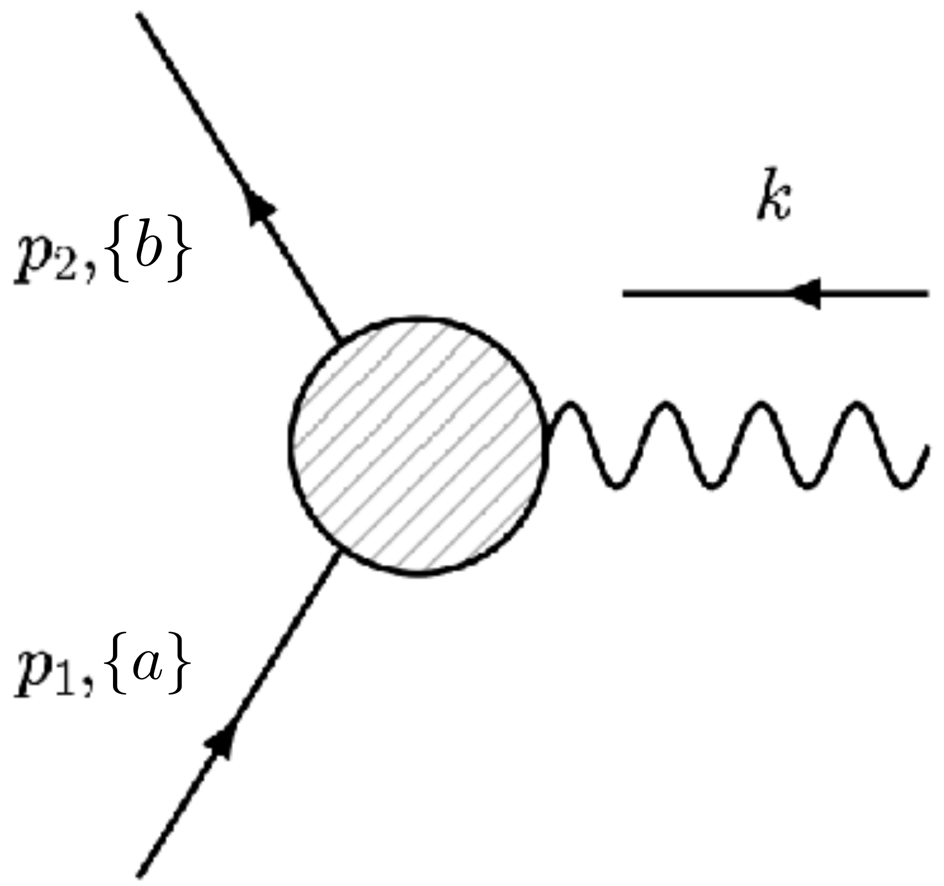
Matches 1PM results

notation: $k^\mu = \hbar\bar{k}^\mu$

$$a^\mu_{\mathrm{a}} \equiv \frac{1}{m_{\mathrm{a}}}\langle S^\mu_{p_{\mathrm{a}}}\rangle_\alpha$$

See also [Guevara, Ochirov, Vines, '19]

General (Definite-spin) three-point amplitude and Kerr BHs



Matching to the wordline

[Porto, Rothstein, '06]
[Porto, Rothstein, '08]
[Levi, Steinhoff, '15]

From the worldline, we can obtain the following classical amplitude

$$\mathcal{A}^{\pm}_{\text{gen}}(p,k) = -\kappa(p\cdot\varepsilon^{\pm}_{k})^2\left[\sum_{n=0}^{\infty}\frac{C_{\text{ES}^{2n}}}{(2n)!}(\bar{k}\cdot a)^{2n} \pm \sum_{n=0}^{\infty}\frac{C_{\text{BS}^{2n+1}}}{(2n+1)!}(\bar{k}\cdot a)^{2n+1}\right],$$

Matching with the general amplitude

$$\mathcal{A}^{(0)\{b\}}_{\text{gen}}{}_{\{a\}}(p_2,s|p_1,s;k,+) = -\frac{\kappa}{2}\sum_{n=0}^{2s} g_n^{+}\frac{x^{n+2}\langle 2^b 1_a\rangle^{\odot(2s-n)}}{m^{2s+n-2}} \odot \left(\langle 2^b k\rangle\langle k 1_a\rangle\right)^{\odot n},$$

Matching to the wordline

The Wilson coefficients

$$C_{\mathrm{ES}^{2n}} = \sum_{r=0}^{2n} \frac{(2n)!(-2)^r g_r^{\pm}}{(2n-r)!\|\alpha\|^{2r}}$$

(Same for the magnetic)

“Superclassical" rescaling:

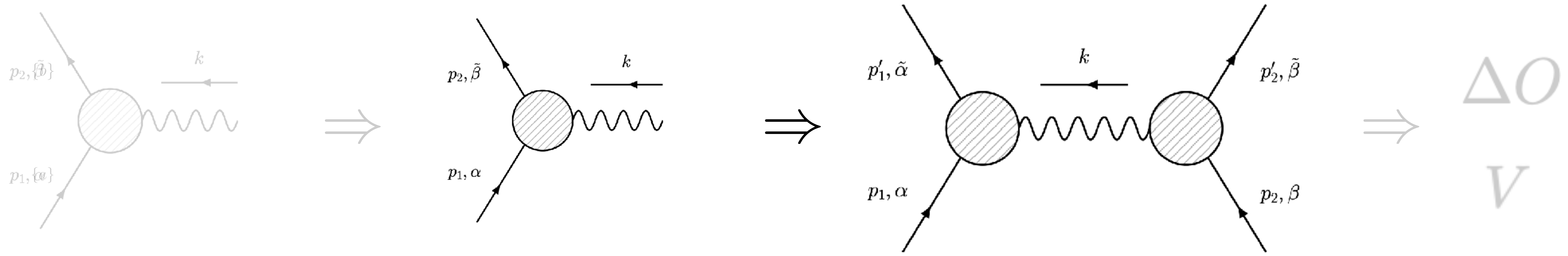
Classically suppressed unless $g_{n>0}^{\pm}$ scales with $\mathcal{O}(\hbar^{-n})$

In order to model general spinning body, non-minimal couplings depends on the spin via $\|\alpha\|^2 = \frac{2m}{\hbar}\sqrt{-a^2}.$

(Expect for a Kerr BH) $g_0^{\pm} = 1 \quad g_{n>0}^{\pm} = 0$

Unequal spin amplitudes are also suppressed by the normalization.

Pipeline to Classical Observables with Spin



Four-point coherent amplitudes (general case)

notation

$$a_{\mathrm{a}}^{\mu} \equiv \frac{1}{m_{\mathrm{a}}}\langle S_{p_{\mathrm{a}}}^{\mu}\rangle_{\alpha},$$

$$C_{2n} \equiv C_{\mathrm{ES}^{2n}}$$

$$C_{2n+1} \equiv C_{\mathrm{BS}^{2n+1}}$$

Using the three-points after matching

$$\mathcal{A}^{(0)}(p_1', \alpha | p_1, \alpha; k, \pm) = -\frac{\kappa}{2} m_{\mathrm{a}}^2 x_{\mathrm{a}}^{\pm 2} \sum_{n=0}^{\infty} \frac{C_{\mathrm{a}n}}{n!} \left(\pm \bar{k} \cdot a_{\mathrm{a}}\right)^n + \mathcal{O}(\hbar^0),$$

[Cachazo, Guevara, ’17]

[Guevara, '17]

[Guevara, Ochirov, Vines '18]

[Guevara, Ochirov, Vines '19]

and the Holomorphic Classical Limit

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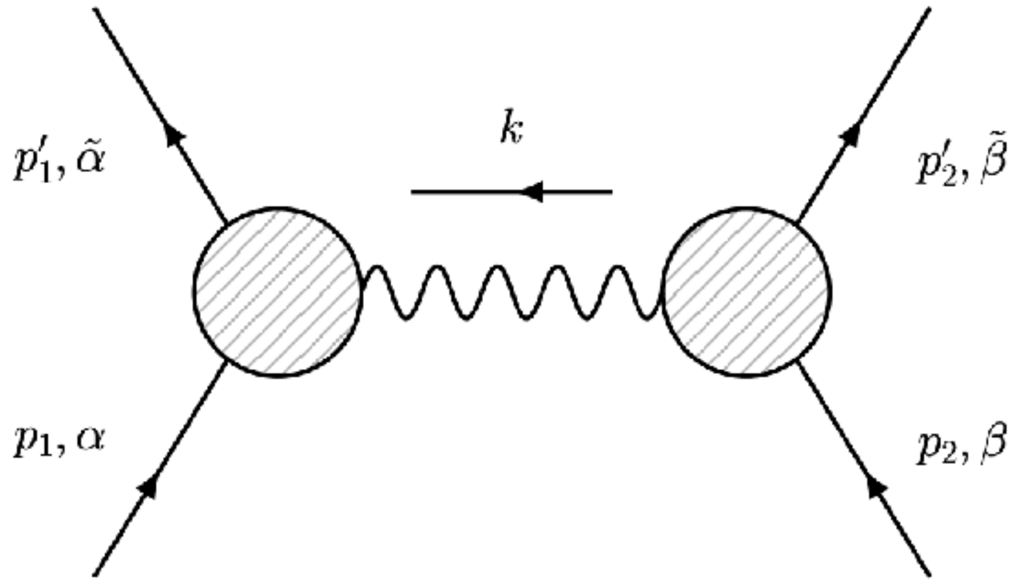


$$= -\frac{8\pi G m_{\rm a}^2 m_{\rm b}^2 \gamma^2}{\hbar^3 \bar{k}^2} \times \sum_{\pm} (1 \mp v)^2 \sum_{n_1, n_2 = 0}^{\infty} \frac{C_{{\rm a}n_1} C_{{\rm b}n_2}}{n_1! n_2!} \left(\pm i \bar{k} \cdot [w * a_{\rm a}]\right)^{n_1} \left(\pm i \bar{k} \cdot [w * a_{\rm b}]\right)^{n_2} + \mathcal{O}(\hbar^{-5/2}).$$

Multipole expansion of particle 1 and particle 2

modeled by Wilson coefficients!

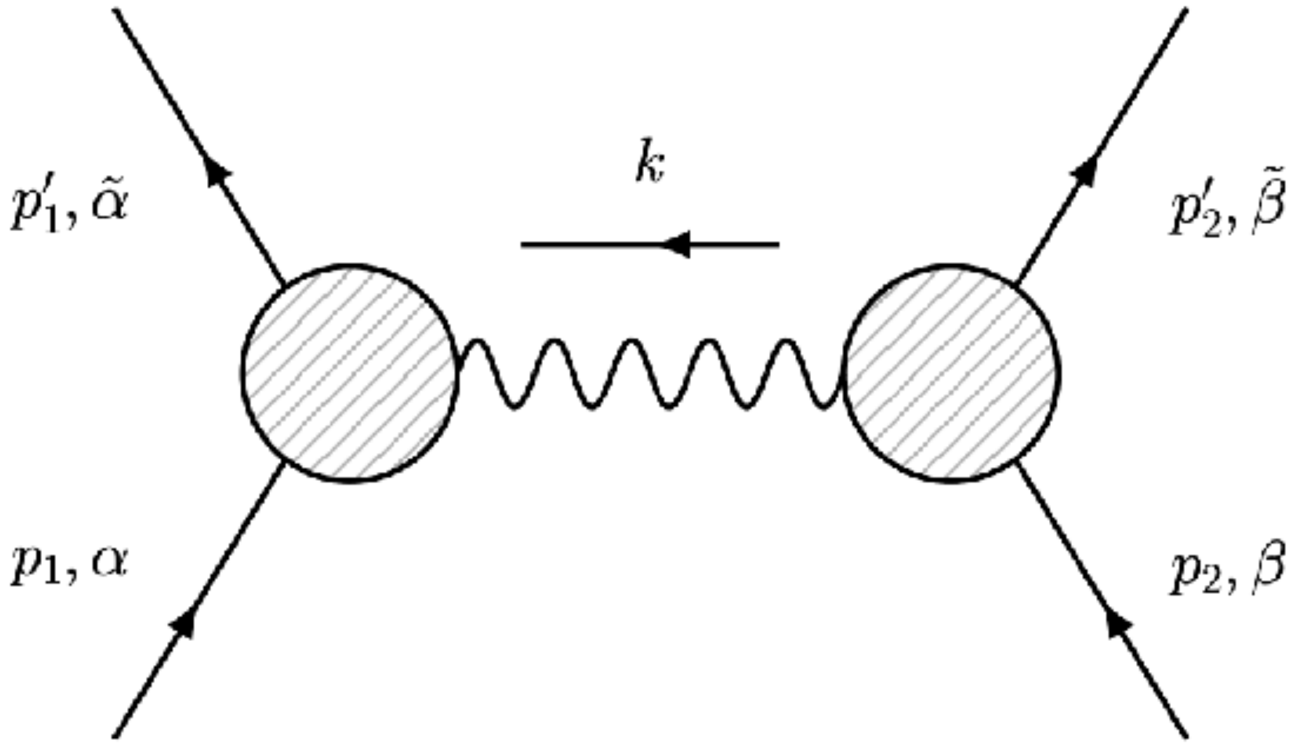
$$C_{{\rm a}n_1} C_{{\rm b}n_2}$$

$$w^{\mu\nu} = \frac{2 p_{\rm a}^{[\mu} p_{\rm b}^{\nu]}}{m_{\rm a} m_{\rm b} \gamma v},$$
$$[w * a]_{\lambda} = (*w)_{\lambda\mu} a^{\mu} = \frac{\epsilon_{\lambda\mu\nu\rho} p_{\rm a}^{\mu} p_{\rm b}^{\nu} a^{\rho}}{m_{\rm a} m_{\rm b} \gamma v},$$

Pipeline to Classical Observables with Spin

$\Rightarrow$

$\Rightarrow$



$\Rightarrow$

$$\Delta O$$

$$V$$

Kosower Maybee O'Connell (KMOC) formalism

[Kosower, Maybee, O'Connell '18]

[Maybee, O'Connell, Vines '19]

[de la Cruz, Maybee, O'Connell '20]

Changing in an operator due to scattering

$$\Delta O = \langle \mathrm{out}|O|\mathrm{out}\rangle - \langle \mathrm{in}|O|\mathrm{in}\rangle = \langle \mathrm{in}|S^\dagger O S|\mathrm{in}\rangle - \langle \mathrm{in}|O|\mathrm{in}\rangle$$

Using $S = 1 + i\,T$ and optical theorem

$$\Delta O = \langle \mathrm{in}|S^\dagger O S|\mathrm{in}\rangle - \langle \mathrm{in}|O|\mathrm{in}\rangle = \underbrace{i\langle \mathrm{in}|\,[OT - T^\dagger O]\,|\mathrm{in}\rangle}_{\Delta_1 O} + \underbrace{\langle \mathrm{in}|T^\dagger O T|\mathrm{in}\rangle}_{\Delta_2 O},$$

leading order next-to-leading order

We need to prepare well-defined the initial state states.

Kosower Maybee O'Connell (KMOC) formalism

Incoming (spinless) state: $$|\text{in}\rangle = \int_{p_1}\int_{p_2}\psi_\text{a}(p_1)\psi_\text{b}(p_2)e^{ib\cdot p_1/\hbar}\underline{|p_1;p_2\rangle}$$

[Kosower, Maybee, O'Connell '18]

definite momenta state

Incoming (spinning) state: $$|\text{in}\rangle = \sum_{a_1,a_2}\int_{p_1}\int_{p_2}\psi_\text{a}(p_1)\psi_\text{b}(p_2)\xi_{a_1}\xi_{a_2}e^{ib\cdot p_1/\hbar}|p_1,p_2;\underline{a_1,a_2}\rangle$$

[Maybee,O'Connell, Vines '19]

Quantum
spin-indices

Kosower Maybee O'Connell (KMOC) formalism

Incoming (spinless) state:

[Kosower, Maybee, O'Connell '18]

$$|\mathrm{in}\rangle = \int_{p_1}\int_{p_2} \psi_{\mathrm{a}}(p_1)\psi_{\mathrm{b}}(p_2)e^{ib\cdot p_1/\hbar}\underbrace{|p_1;p_2\rangle}_{\text{definite momenta state}}$$

Incoming (spinning) state:

[Maybee,O'Connell, Vines '19]

$$|\mathrm{in}\rangle = \sum_{a_1,a_2}\int_{p_1}\int_{p_2} \psi_{\mathrm{a}}(p_1)\psi_{\mathrm{b}}(p_2)\xi_{a_1}\xi_{a_2}e^{ib\cdot p_1/\hbar}|p_1,p_2;\underbrace{a_1,a_2\rangle}_{\text{Quantum spin-indices}}$$

Incoming (coherent) state:

[RA, Ochirov '21]

$$|\mathrm{in}\rangle = \int_{p_1}\int_{p_2} \psi_{\mathrm{a}}(p_1)\psi_{\mathrm{b}}(p_2)e^{ib\cdot p_1/\hbar}|p_1,\alpha;p_2,\beta\rangle$$

Kosower Maybee O'Connell (KMOC) formalism

Incoming (spinless) state:

[Kosower, Maybee, O'Connell '18]

$$|\mathrm{in}\rangle = \int_{p_1}\int_{p_2} \psi_\mathrm{a}(p_1)\psi_\mathrm{b}(p_2)e^{ib\cdot p_1/\hbar}\underbrace{|p_1;p_2\rangle}_{\text{definite momenta state}}$$

Incoming (spinning) state:

[Maybee,O'Connell, Vines '19]

$$|\mathrm{in}\rangle = \sum_{a_1,a_2}\int_{p_1}\int_{p_2} \psi_\mathrm{a}(p_1)\psi_\mathrm{b}(p_2)\xi_{a_1}\xi_{a_2}e^{ib\cdot p_1/\hbar}|p_1,p_2;\underbrace{a_1,a_2}_{\text{Quantum spin-indices}}\rangle$$

Incoming (coherent) state:

[RA, Ochirov '21]

$$|\mathrm{in}\rangle = \int_{p_1}\int_{p_2} \psi_\mathrm{a}(p_1)\psi_\mathrm{b}(p_2)e^{ib\cdot p_1/\hbar}|p_1,\alpha;p_2,\beta\rangle$$

$$= \underbrace{e^{-(\|\alpha\|^2+\|\beta\|^2)/2}}_{\text{normalization}}\sum_{s_1,s_2}\int_{p_1,p_2} e^{ib\cdot p_1/\hbar}\psi_\mathrm{a}(p_1)\psi_\mathrm{b}(p_2)\frac{(\alpha^a)^{\odot 2s_1}(\beta^b)^{\odot 2s_2}}{\sqrt{(2s_1)!(2s_2)!}}\cdot \underbrace{|p_1,s_1,\{a\};p_2,s_2,\{b\}\rangle}_{\text{definite spin-state}}.$$

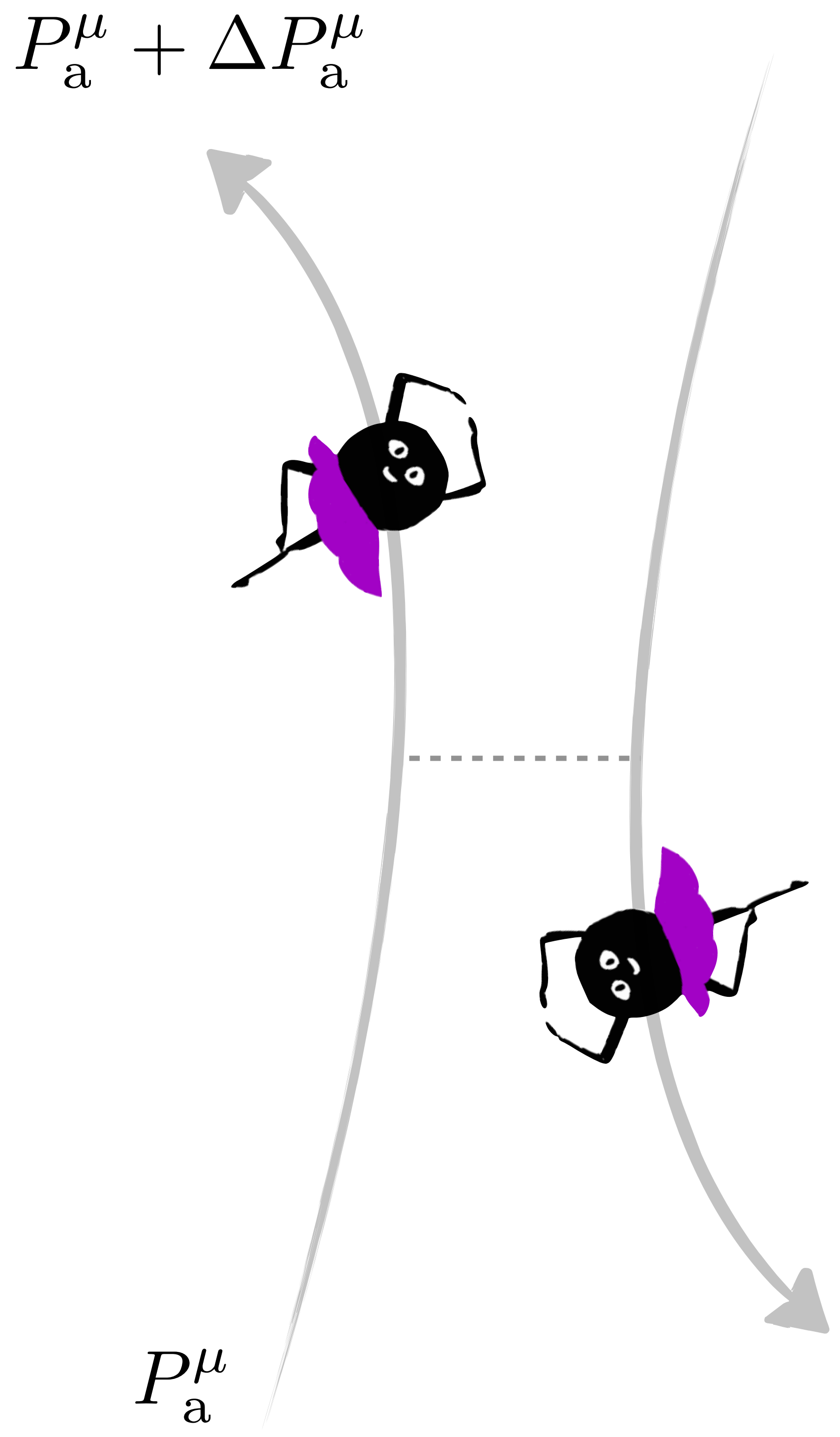
KMOC using coherent states

After the Fourier transform to the impact parameter space…

$$\Delta P_{\mathrm{a}}^{\mu} = -\hbar \frac{\partial}{\partial b_{\mu}} \int_{p_{\mathrm{a}},p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \mathcal{A}_4^{(0)}(b)$$

$$\boxed{\Delta P_{\mathrm{a}}^{\mu} = G m_{\mathrm{a}} m_{\mathrm{b}} \frac{\gamma}{v} \sum_{\pm} (1 \mp v)^2 \frac{[b \pm w * (a_{\mathrm{a}} + a_{\mathrm{b}})]^{\mu}}{[b \pm w * (a_{\mathrm{a}} + a_{\mathrm{b}})]^2} \bigg|_{\mathrm{cl}},}$$

Matches Vines 17'



$^*\mathrm{cl.}$ means initial momenta $p_{\mathrm{a,b}}^{\mu}$ localized on their classical values $m_{\mathrm{a,b}} u_{\mathrm{a,b}}^{\mu}$

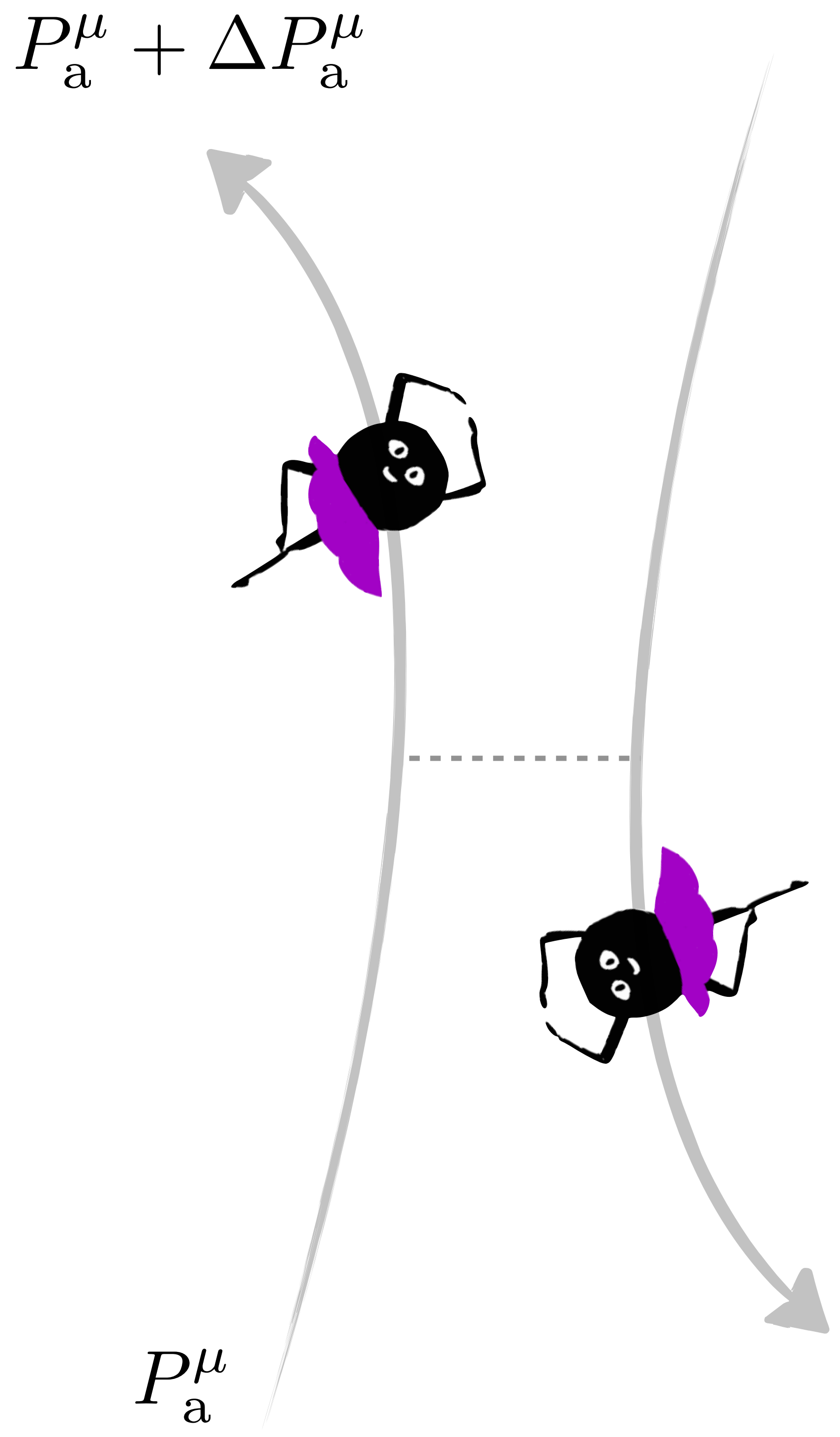
KMOC using coherent states

After the Fourier transform to the impact parameter space…

$$\Delta P_{\rm a}^{\mu} = -\hbar \frac{\partial}{\partial b_{\mu}} \int_{p_{\rm a},p_{\rm b}} |\psi_{\rm a}(p_{\rm a})|^2 |\psi_{\rm b}(p_{\rm b})|^2 \mathcal{A}_4^{(0)}(b)$$

$$\boxed{\Delta P_{\rm a}^{\mu} = G m_{\rm a} m_{\rm b} \frac{\gamma}{v} \sum_{\pm} (1 \mp v)^2 \frac{[b \pm w * (a_{\rm a} + a_{\rm b})]^{\mu}}{[b \pm w * (a_{\rm a} + a_{\rm b})]^2} \bigg|_{\rm cl},}$$

Matches Vines 17'



From the eikonal amplitude $\mathcal{A}_4^{(0)}(b)$ one can also obtain (for general bodies)

Angular impulse $$\Delta S_{\rm a}^{\mu} = \frac{\hbar}{m_{\rm a}} \int_{p_{\rm a},p_{\rm b}} |\psi_{\rm a}(p_{\rm a})|^2 |\psi_{\rm b}(p_{\rm b})|^2 \left[p_{\rm a}^{\mu} a_{\rm a}^{\nu} \frac{\partial}{\partial b^{\nu}} - \epsilon^{\mu\nu\rho\sigma} p_{{\rm a}\nu} a_{{\rm a}\rho} \frac{\partial}{\partial a_{\rm a}^{\sigma}} \right] \mathcal{A}_4^{(0)}(b)$$

*cl. means initial momenta $p_{\rm a,b}^{\mu}$ localized on their classical values $m_{\rm a,b} u_{\rm a,b}^{\mu}$

Hamiltonian at 1PM

Potential $$V^{(1)}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\mathrm{a}},\boldsymbol{S}_{\mathrm{b}}) = -\frac{\hbar^3}{4E_{\mathrm{a}}E_{\mathrm{b}}}\int\frac{d^3\bar{\boldsymbol{k}}}{(2\pi)^3}e^{i\bar{\boldsymbol{k}}\cdot\boldsymbol{r}}\mathcal{A}^{(0)}(\bar{\boldsymbol{k}},\boldsymbol{p},\boldsymbol{S}_{\mathrm{a}},\boldsymbol{S}_{\mathrm{b}}),$$

For generic objects:

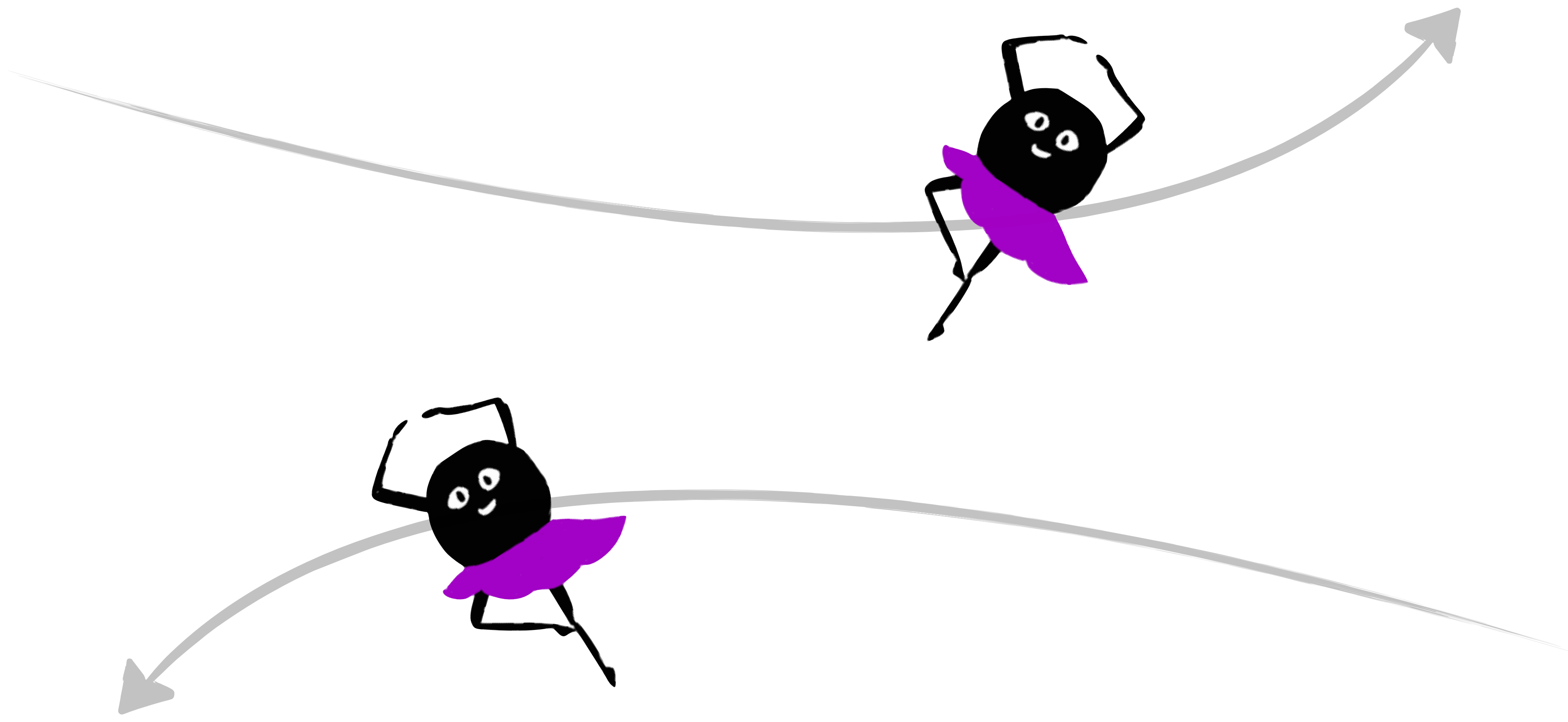
$$V^{(1)}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\mathrm{a}},\boldsymbol{S}_{\mathrm{b}}) = -\frac{Gm_{\mathrm{a}}^2m_{\mathrm{b}}^2\gamma^2}{2E_{\mathrm{a}}E_{\mathrm{b}}}\sum_{\pm}(1\mp v)^2$$
$$\times\sum_{n_1,n_2=0}^{\infty}\frac{C_{\mathrm{a}n_1}C_{\mathrm{b}n_2}}{n_1!n_2!}\left(\pm\frac{1}{m_{\mathrm{a}}}[\hat{\boldsymbol{p}}\times\boldsymbol{S}_{\mathrm{a}}]\cdot\nabla_{\boldsymbol{r}}\right)^{n_1}\left(\pm\frac{1}{m_{\mathrm{b}}}[\hat{\boldsymbol{p}}\times\boldsymbol{S}_{\mathrm{b}}]\cdot\nabla_{\boldsymbol{r}}\right)^{n_2}\frac{1}{|\boldsymbol{r}|}.$$

While for Kerr:

$$V^{(1)}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\mathrm{a}},\boldsymbol{S}_{\mathrm{b}}) = -\frac{Gm_{\mathrm{a}}^2m_{\mathrm{b}}^2\gamma^2}{2E_{\mathrm{a}}E_{\mathrm{b}}}\sum_{\pm}\frac{(1\pm v)^2}{|\boldsymbol{r}\pm\hat{\boldsymbol{p}}\times(\boldsymbol{a}_{\mathrm{a}}+\boldsymbol{a}_{\mathrm{b}})|},$$

In a different gauge than [Chung, Huang, Kim, Lee, '19]

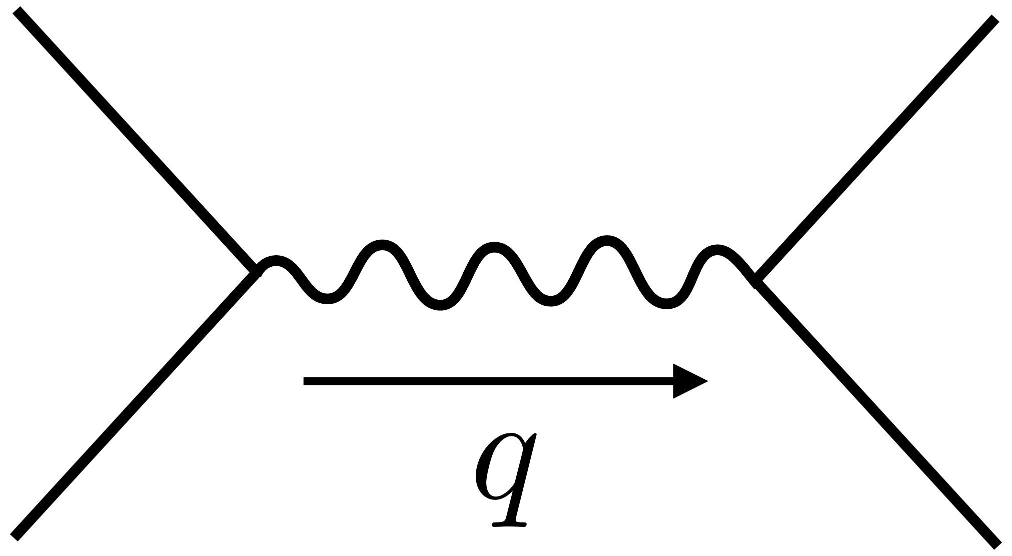
Heavy Classical
scattering at 2PM



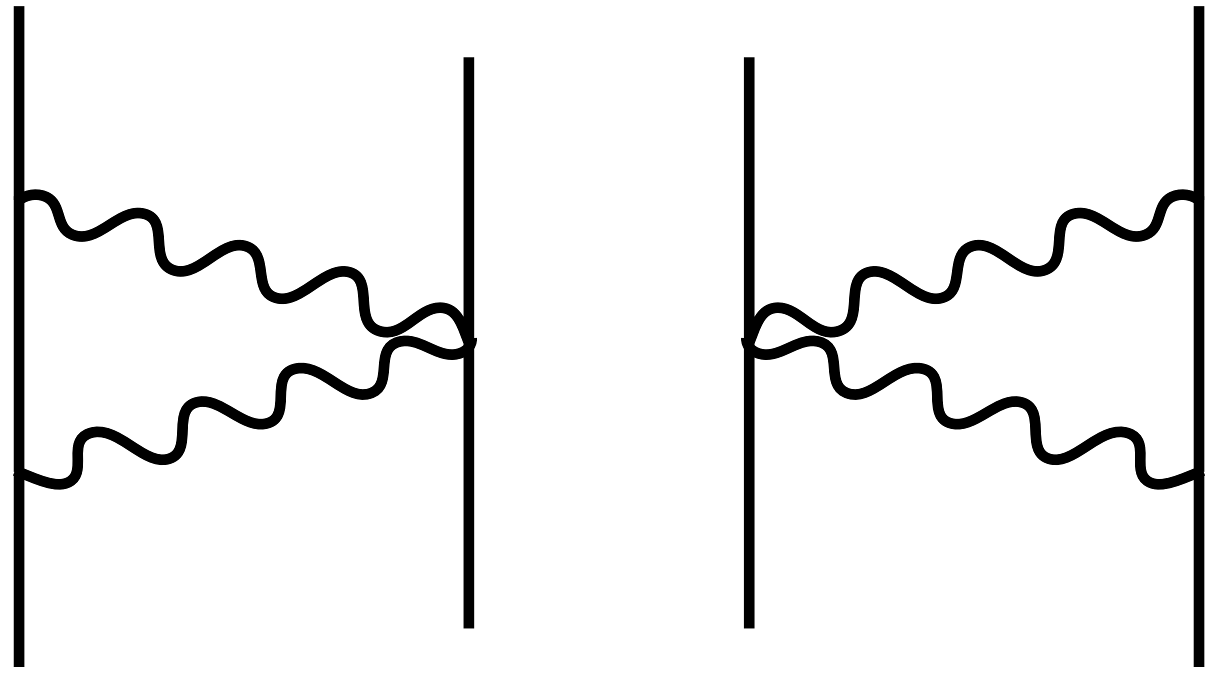
w/ Kays Haddad and Andreas Helset

Conservative tree level four-points

We can use $\mathcal{M}^{\pm}_{\min} = -\frac{\kappa}{2} x^{\pm 2} e^{\mp k \cdot \mathfrak{a}}$ to obtain



But we want to improve to 2PM, i.e.

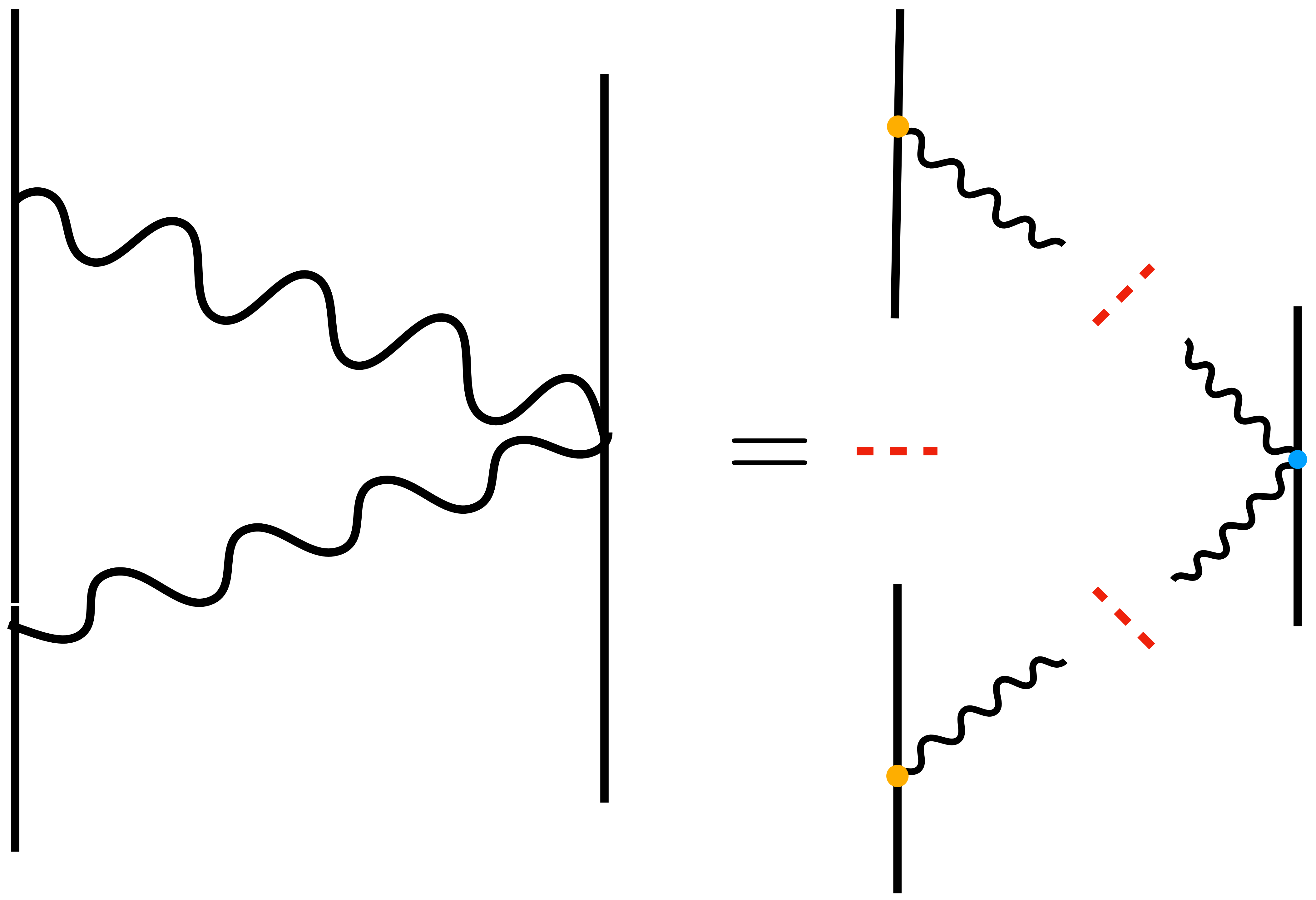


which requires the Compton Amplitude

See also [Bern, Kosmopoulos, Luna, Roiban, Teng, '22]
[Chen, Chung, Huang, Kim, '21]

2PM Amplitudes

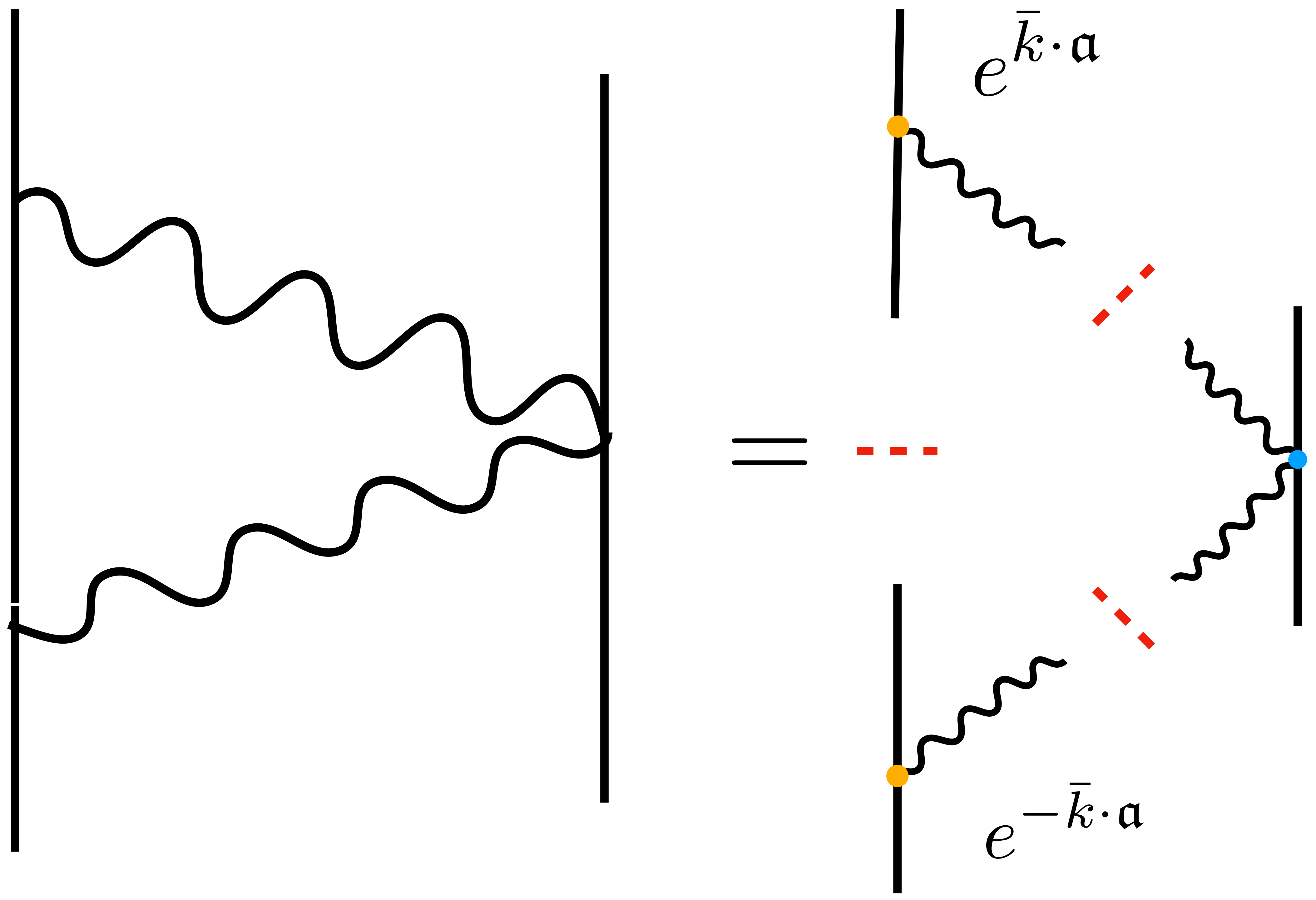
Using unitarity, we construct the relevant 1-loop diagrams



+ other triangle

2PM Amplitudes

Using unitarity, we construct the relevant 1-loop diagrams

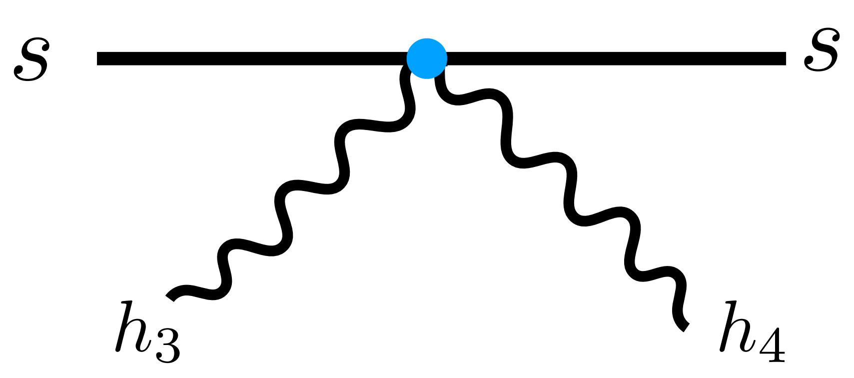


Compton for high-spins ?

+ other triangle

Classical Compton scattering

[Aoude, Haddad, Helset, 22']



$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, 3^-, 4^+) = \frac{y^4}{s_{34}t_{13}t_{14}} \left(\frac{\langle \mathbf{31} \rangle [\mathbf{42}] - \langle \mathbf{32} \rangle [\mathbf{41}]}{y} \right)^{2s}$$

After some rewriting in HPET variables and cl. limit $\quad y \equiv [4|p_1|3\rangle \qquad w^\mu \equiv [4|\bar{\sigma}^\mu|3\rangle/2$

$$\mathcal{A}^s = \mathcal{A}^0 \exp\left\{ \left[q_4 - q_3 + \frac{(t_{14} - t_{13})}{y} w \right] \cdot \mathfrak{a} \right\},$$

Scalar Compton:

$$\mathcal{A}^0_{\mathrm{GR}} = \frac{y^4}{s_{34}t_{13}t_{14}}$$

Matches GR calculations up to $\mathcal{O}(S^4)$ [Saketh, Vines '22, Bautista's thesis '22]

Spurious pole for s>2 in Gravity

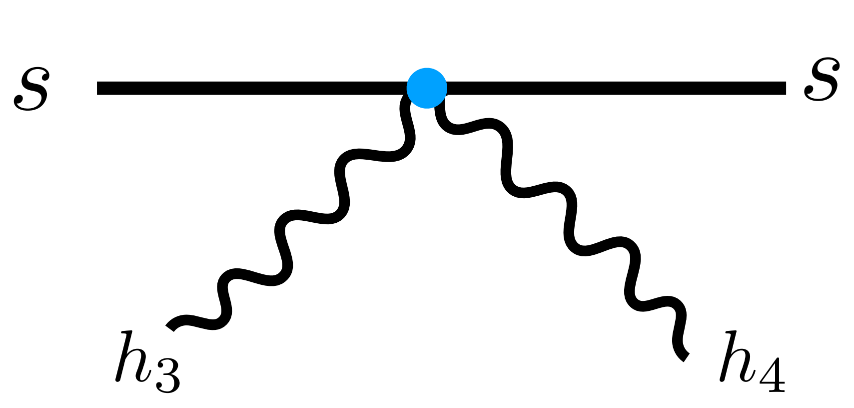
We need to remedy for higher orders.

See also [Bautista, Guevara, Kavanagh and Vines '21, Bautista's thesis '22]

See Bautista's and Kavanagh Talk!

[Aoude, Haddad, Helset, 22']

Classical Compton scattering



$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, 3^-, 4^+) = \frac{y^4}{s_{34} t_{13} t_{14}} \left(\frac{\langle 3\mathbf{1}\rangle[4\mathbf{2}] - \langle 3\mathbf{2}\rangle[4\mathbf{1}]}{y} \right)^{2s}$$

After some rewriting in HPET variables and cl. limit $\quad y \equiv [4|p_1|3\rangle \qquad w^\mu \equiv [4|\bar{\sigma}^\mu|3\rangle/2$

$$\mathcal{A}^s = \mathcal{A}^0 \exp\left\{ \left[q_4 - q_3 + \frac{(t_{14} - t_{13})}{y} w \right] \cdot \mathfrak{a} \right\},$$

To remove the spurious pole:

1. recursive approach (when there is a massless pole)

[Chung, Huang, Kim Lee 19'] [Falkowski, Machado 20']

2. adding four-point contact terms (that respects the black-hole spin-structure assumption/shift symmetry)

See also [Bern, Kosmopoulos, Luna, Roiban, Teng, '22]

Classical Gravity Compton to all orders in spin

For gravity

$$\mathcal{M}^s_{\rm cl} = e^{(q_4-q_3)\cdot\mathfrak{a}} \sum_{n=0}^{2s} \frac{1}{n!} K_n$$

$$K_n \equiv \frac{y^4}{s_{34} t_{13} t_{14}} \left(\frac{t_{14} - t_{13}}{y} w \cdot \mathfrak{a} \right)^n$$

Classical Gravity Compton to all orders in spin

For gravity

$$\mathcal{M}^s_{\rm cl} = e^{(q_4-q_3)\cdot\mathfrak{a}}\sum_{n=0}^{2s}\frac{1}{n!}K_n, \qquad K_n \equiv \frac{y^4}{s_{34}t_{13}t_{14}}\left(\frac{t_{14}-t_{13}}{y}w\cdot\mathfrak{a}\right)^n$$

To remove the spurious pole, we need to use a recursive approach

$$K_n = 16m^2\frac{(t_{14}-t_{13})^{n-4}}{y^{n-4}}(w\cdot\mathfrak{a})^n + 2\mathfrak{s}_1K_{n-1} - \mathfrak{s}_2K_{n-2}.$$

Based on the relation: $(t_{14}-t_{13})^2(w\cdot\mathfrak{a})^2 = -4m^2s_{34}(w\cdot\mathfrak{a})^2 + 2y(t_{14}-t_{13})\mathfrak{s}_1(w\cdot\mathfrak{a}) - y^2\mathfrak{s}_2,$

From the Gram determinant of: $q_4^\mu, \quad q_3^\mu, \quad p_1^\mu, \quad \mathfrak{a}^\mu, \quad w^\mu$

Classical Gravity Compton to all orders in spin

For gravity

$$\mathcal{M}^s_{\rm cl} = e^{(q_4-q_3)\cdot\mathfrak{a}} \sum_{n=0}^{2s} \frac{1}{n!} K_n, \qquad K_n \equiv \frac{y^4}{s_{34} t_{13} t_{14}} \left(\frac{t_{14}-t_{13}}{y} w\cdot\mathfrak{a} \right)^n$$

To remove the spurious pole, we need to use a recursive approach

$$\mathcal{M}^s_{\rm cl} = e^{-\mathfrak{s}_1} \sum_{n=0}^{2s} \frac{1}{n!} \bar{K}_n \qquad \bar{K}_n \equiv \begin{cases} K_n, & n \le 4, \\ K_4 L_{n-4} - K_3 \mathfrak{s}_2 L_{n-5}, & n > 4, \end{cases}$$

$$L_m \equiv \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m+1}{2j+1} \mathfrak{s}_1^{m-2j} (\mathfrak{s}_1^2 - \mathfrak{s}_2)^j$$

$$\mathfrak{s}_1 \equiv (q_3 - q_4)\cdot\mathfrak{a}$$

$$\mathfrak{s}_2 \equiv -4(q_3\cdot\mathfrak{a})(q_4\cdot\mathfrak{a}) + s_{34}\mathfrak{a}^2$$

plus contact terms!

Black hole spin structure assumption

[Aoude, Haddad, Helset 22']

Based on a structure that appears up to fourth order in spin…

$(q \cdot \mathfrak{a}_i)(q \cdot \mathfrak{a}_j)$ and $q^2(\mathfrak{a}_i \cdot \mathfrak{a}_j)$

Only appears in the following combination:

$$(q \cdot \mathfrak{a}_i)(q \cdot \mathfrak{a}_j) - q^2(\mathfrak{a}_i \cdot \mathfrak{a}_j), \qquad i, j = 1, 2,$$

This is broken by tidal effects

Shift symmetry

[Bern, Kosmopoulos, Luna, Roiban, Teng, 22']

$$a_i^\mu \to a_i^\mu + \xi_i q^\mu / q^2, \quad i = 1, 2$$

2PM Spinning-spinless to all orders

This actually resums into hypergeometric functions (just Bessel functions)

$$\mathcal{M}_{2\mathrm{PM}} = \frac{2G^2\pi^2 m_1^2 m_2^2}{\sqrt{-q^2}} \left(\mathcal{M}_{2\mathrm{PM}}^{\mathrm{even}} + i\omega\mathcal{E}_1\mathcal{M}_{2\mathrm{PM}}^{\mathrm{odd}}\right).$$

$$\mathcal{E}_1 \equiv \varepsilon^{\mu\nu\alpha\beta} v_{1\mu} v_{2\nu} q_\alpha \mathfrak{a}_{1\beta}$$

$$Q \equiv (q\cdot\mathfrak{a}_1)^2 - q^2\,\mathfrak{a}_1^2$$

$$V \equiv q^2 (v_2\cdot\mathfrak{a}_1)^2$$

$$\omega \equiv v_1 \cdot v_2$$

For the odd powers….

$$\mathcal{M}_{2\mathrm{PM}}^{\mathrm{odd}} = -m_1\left[4\mathcal{F}_1 + \sum_{k=0}^{\infty}\frac{(2\omega^2-1)}{(\omega^2-1)^{k+1}}\frac{(-1)^k 8\Gamma[k+1]}{\Gamma[2k+1]}\mathcal{F}_k V^k\right]$$
$$-m_2\left[\frac{15\sqrt{\pi}}{2}\mathcal{F}_{1/2} + \sum_{k=0}^{\infty}\frac{\omega^{2k}}{(\omega^2-1)^{k+1}}\frac{4^{1-k}\mathcal{F}_k V^k}{(1)_k(2k-1)} \times \left[{}_2F_1\left(-\frac{1}{2}-k,-k;\frac{3}{2}-k;\frac{1}{\omega^2}\right) - \left(k+\frac{5}{2}\right){}_2F_1\left(\frac{1}{2}-k,-k;\frac{3}{2}-k;\frac{1}{\omega^2}\right)\right]\right]$$

where

$$\mathcal{F}_j \equiv \frac{1}{\Gamma[j+1]}\,{}_0F_1\left(j+1;\frac{Q}{4}\right)$$

similar for even powers
(contact terms contributes)

Thank you for your attention

Worldline effective action vs. three-point

[Porto, Rothstein, 06']
[Porto, Rothstein, 08']
[Levi, Steinhoff, 15']

Expanding the curvature tensor $R_{\lambda\mu\nu\rho}$ in terms of linear grav. pertubation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ in the effective action

$$S_{\text{Int}} = -\frac{m}{2}\int d\tau \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} C_{\text{ES}^{2n}} (a\cdot\partial)^{2n} u^\mu u^\nu h_{\mu\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} C_{\text{BS}^{2n+1}} (a\cdot\partial)^{2n}\, u^\mu \epsilon^{\nu\rho\sigma\tau} u_\rho a_\sigma \partial_\tau h_{\mu\nu} \right]_{x=r(\tau)} + \mathcal{O}(h^2).$$

Kerr BH corresponds: $C_{\text{ES}^{2n}} = -C_{\text{BS}^{2n+1}} = 1$

This WL interaction gives:

$$\mathcal{A}^{\pm}_{\text{gen}}(p,k) = -\kappa(p\cdot\varepsilon_k^{\pm})^2 \left[\sum_{n=0}^{\infty} \frac{C_{\text{ES}^{2n}}}{(2n)!} (\bar{k}\cdot a)^{2n} \pm \sum_{n=0}^{\infty} \frac{C_{\text{BS}^{2n+1}}}{(2n+1)!} (\bar{k}\cdot a)^{2n+1} \right],$$

For Kerr: $$\mathcal{A}^{\pm}_{\text{min}}(p,k) = -\frac{\kappa}{2} m^2 x^{\pm 2} e^{\mp \bar{k}\cdot a}.$$

HPET variables and some notation

$$p^\mu = mv^\mu + k^\mu$$

HPET spinors:

$$\begin{pmatrix} |p_v^I\rangle \\ |p_v^I] \end{pmatrix} = \left(\mathbb{I} - \frac{\not{k}}{2m}\right)\begin{pmatrix} |p^I\rangle \\ |p^I] \end{pmatrix}$$

Velocity spinors

$$|p_v^I\rangle = \sqrt{m_k}|v^I\rangle, \quad |p_{vI}] = \sqrt{m_k}|v_I]$$

Using aux. variables

$$|\boldsymbol{v}\rangle \equiv |v^I\rangle z_{p,I}, \quad |\boldsymbol{v}] \equiv |v^I] z_{p,I}.$$

Pauli-Lubanski with velocity

$$S^\mu = -\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} v_\nu J_{\alpha\beta}$$

In SL(2,C)

$$(S^\mu)_\alpha{}^\beta = \frac{1}{4}\left[(\sigma^\mu)_{\alpha\dot\alpha} v^{\dot\alpha\beta} - v_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\right]$$

Ring-radius

$$\mathfrak{a}^\mu \equiv \frac{\langle \boldsymbol{v}|S^\mu|\boldsymbol{v}\rangle}{m\langle \boldsymbol{v}\boldsymbol{v}\rangle}$$

From Compton groupings to 2PM

Why the groupings $\mathfrak{s}_1 = (q_3 - q_4) \cdot \mathfrak{a}$ and $\mathfrak{s}_2 = -4(q_3 \cdot \mathfrak{a})(q_4 \cdot \mathfrak{a}) + s_{34}\mathfrak{a}^2$ generate BH-SSA at 2PM?

Loop momenta in the classical limit $$l_{\pm}^{\mu}(t) = -\frac{1}{2}q^{\mu} + tr_1^{\mu} + \frac{\alpha}{t}r_2^{\mu}.$$

$$\mathfrak{s}_1 = (q_3 - q_4) \cdot \mathfrak{a} \to (2l + q) \cdot \mathfrak{a}_i = 2\left(tr_1 + \frac{\alpha}{t}r_2\right) \cdot \mathfrak{a}_i.$$

$$\mathfrak{s}_2 = -4(q_3 \cdot \mathfrak{a})(q_4 \cdot \mathfrak{a}) + s_{34}\mathfrak{a}^2 \to \mathfrak{s}_1^2 - (q \cdot \mathfrak{a}_i)^2 + q^2(\mathfrak{a}_i)^2.$$

To have t^0 in the Laurent series: $\alpha(r_1 \cdot t_1)(r_2 \cdot t_2)$ for $t_1, t_2 = p_1, \mathfrak{a}_1, \mathfrak{a}_2$

All these structures will respect the BH-SSA.

QED at higher-spins

Spurious QED

$$\mathcal{A}^{s}_{\text{cl,QED}} = e^{(q_4-q_3)\cdot\mathfrak{a}} \sum_{n=0}^{2s} \frac{1}{n!} I_n$$

with

$$I_n \equiv \frac{1}{t_{13}t_{14}} \frac{(t_{14}-t_{13})^n}{y^{n-2}} (w\cdot\mathfrak{a})^n$$

Using $t_{13} = -t_{14} + \mathcal{O}(\hbar^2)$ we can just remove the n>2 terms

$$I_{n\geq 2} = -4\frac{(t_{14}-t_{13})^{n-2}}{y^{n-2}} (w\cdot\mathfrak{a})^n + \mathcal{O}(\hbar).$$

that contribute only to the unphysical pole

QED at higher-spins

Spurious QED

$$\mathcal{A}^{s}_{\rm cl,QED} = e^{(q_4-q_3)\cdot \mathfrak{a}} \sum_{n=0}^{2s} \frac{1}{n!} I_n \qquad \text{with} \qquad I_n \equiv \frac{1}{t_{13}t_{14}} \frac{(t_{14}-t_{13})^n}{y^{n-2}} (w\cdot \mathfrak{a})^n$$

What type of contact terms can we add?

LG+ hbar scaling $\quad \dfrac{y^2}{m^2}\mathfrak{a}^2, \qquad y(w\cdot\mathfrak{a})\dfrac{(t_{14}-t_{13})}{m^2}\mathfrak{a}^2, \qquad (w\cdot\mathfrak{a})^2.$

To higher-spins,
we can dress with the monomials: $\quad q_3\cdot\mathfrak{a}, \qquad q_4\cdot\mathfrak{a}, \qquad s_{34}\mathfrak{a}^2, \qquad \dfrac{(t_{14}-t_{13})^2}{m^2}\mathfrak{a}^2,$

Should we add all of the possible combinations?

Contact terms

Spurious-pole free + contact

$$\mathcal{M}_{\mathrm{cl}}^{s} = e^{-\mathfrak{s}_1} \sum_{n=0}^{2s} \frac{1}{n!} \bar{K}_n + m^2 (w \cdot \mathfrak{a})^4 \mathcal{C},$$

Contact terms that respects BHSSA/shift symmetry.

$$\mathcal{C} \equiv \sum_{n=0}^{2s-4} \sum_{j=0}^{\lfloor (2s-4-n)/2 \rfloor} d_{n,j} \mathfrak{s}_1^n (\mathfrak{s}_1^2 - \mathfrak{s}_2)^j$$

Contributes to the spin-even amplitude.