All Things Eikonal

QCD meets Gravity 2022

2211.00085 [B. Bellazzini, GI, M.M. Riva]

Giulia Isabella

13 December 2022

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université PARIS-SACLAY
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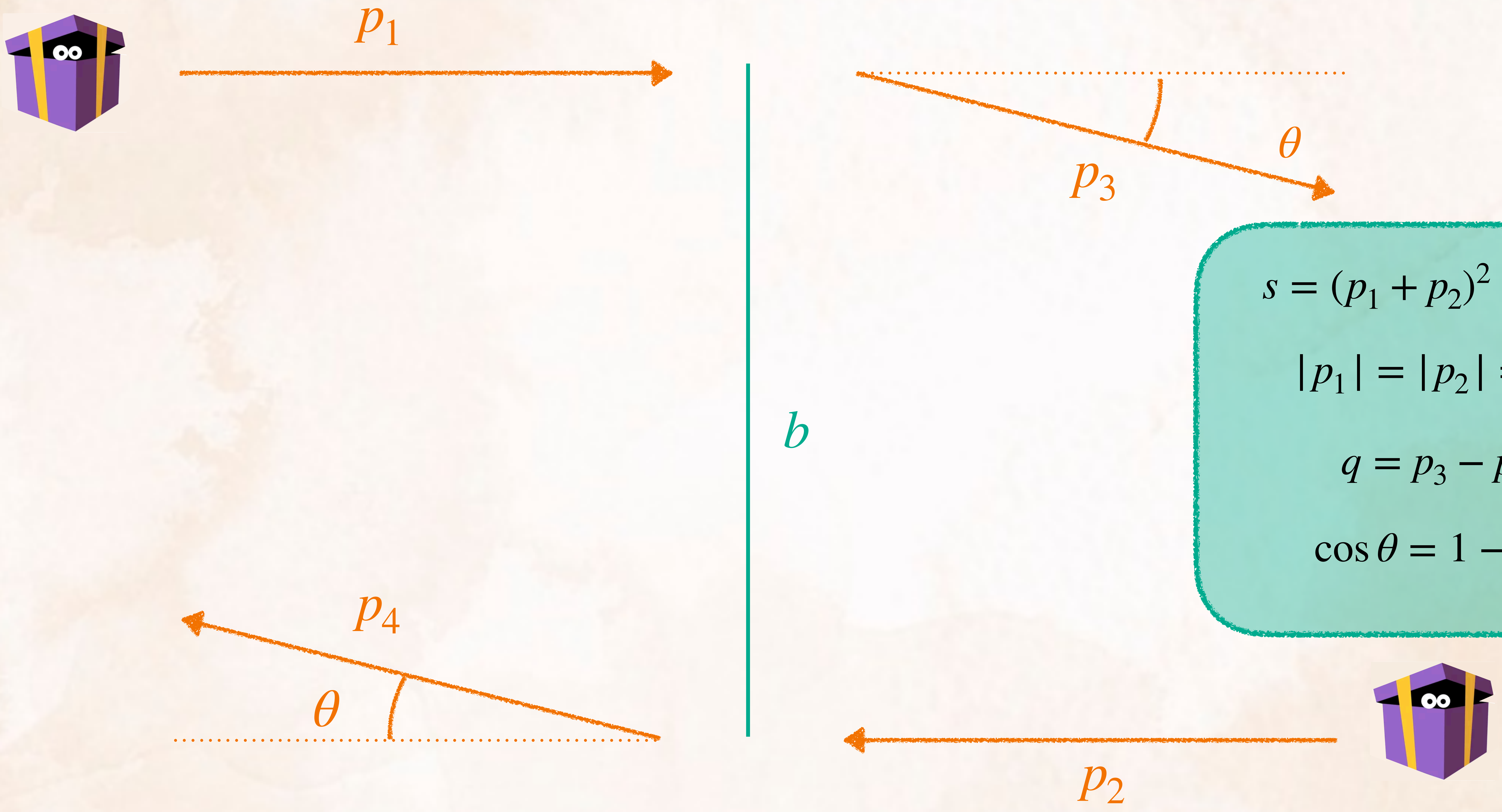
UNIVERSITÉ DE GENÈVE

The setup: Large distance scattering



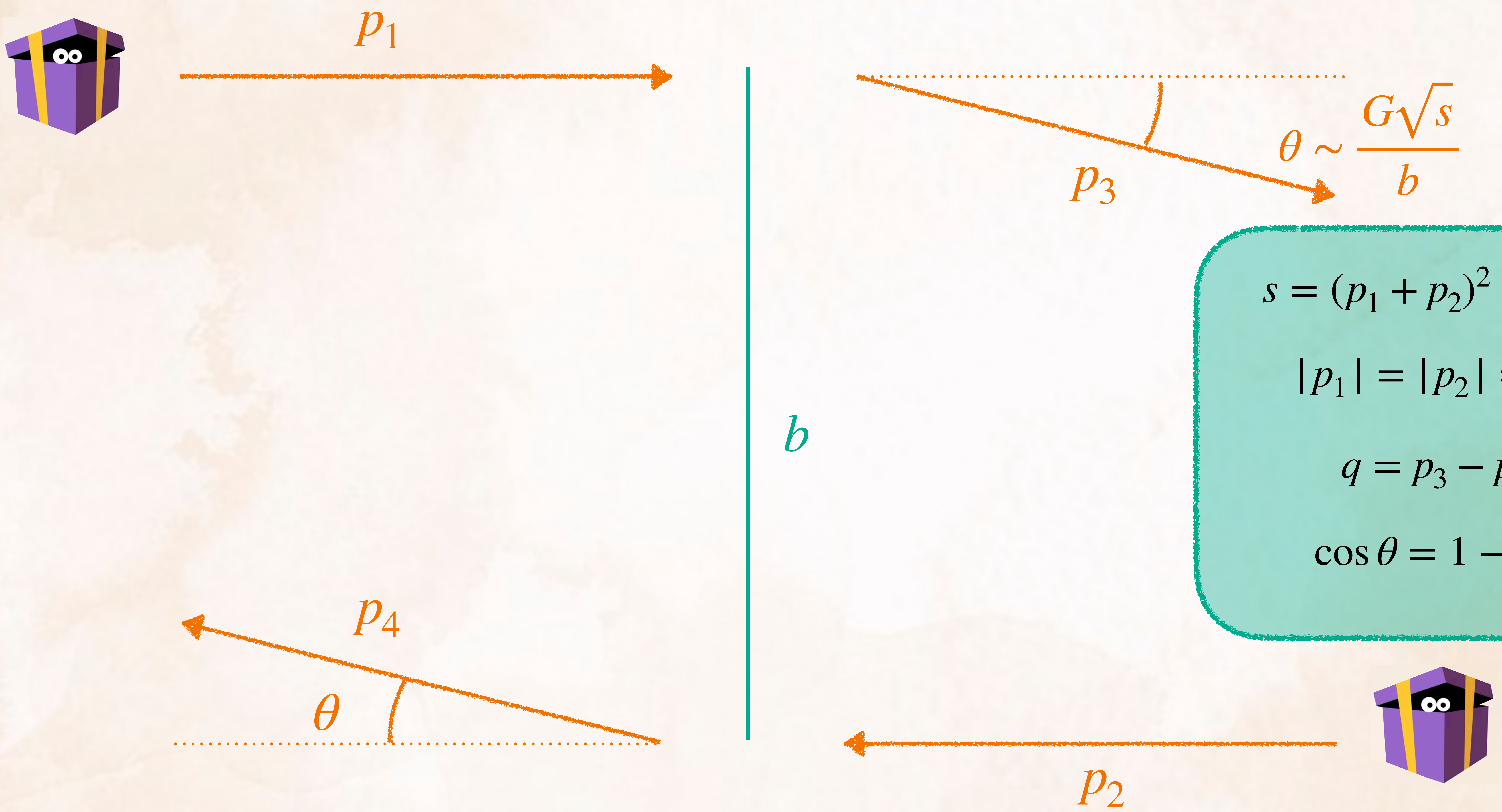
$$s = (p_1 + p_2)^2 = 4E^2$$

$$|p_1| = |p_2| = |p|$$

$$q = p_3 - p_1$$

$$\cos\theta = 1 - \frac{q^2}{2p^2}$$

The setup: Large distance scattering



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The setup: Large distance scattering

The Eikonal limit

$\ell \sim b\,|p| \to \infty \qquad\qquad \theta \to 0$

A tale of gravitational scales

What regime are we looking at?

Kinematic scales

$$\lambda_s \sim \frac{1}{\sqrt{s}} \ , \ \ b$$

Planck length

$$\lambda_{Pl} \sim \sqrt{G}$$

Schwarzschild radius

$$R_s \sim G\sqrt{s}$$

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Gravitational Coupling

$$\alpha_g = R_s \frac{(p_1 \cdot p_2)^2}{s\,|\,p\,|} \sim Gs$$

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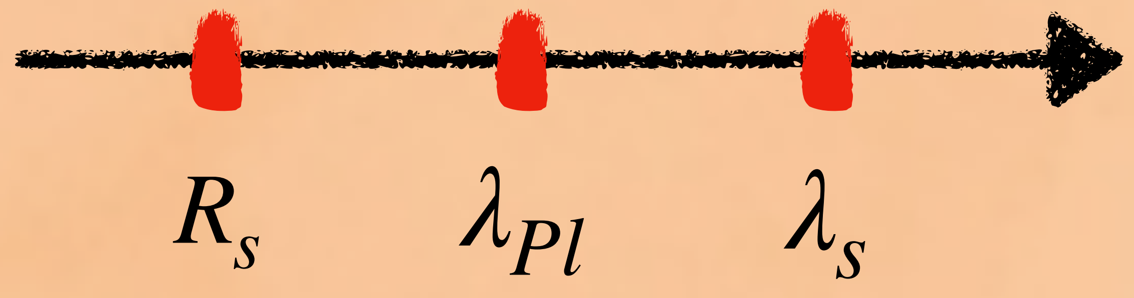
$\ll 1$

Subplanckian



- Dominated by quantum effects
- Perturbative control

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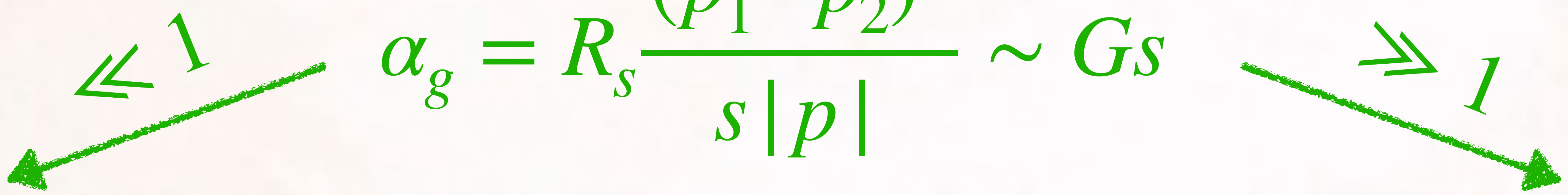

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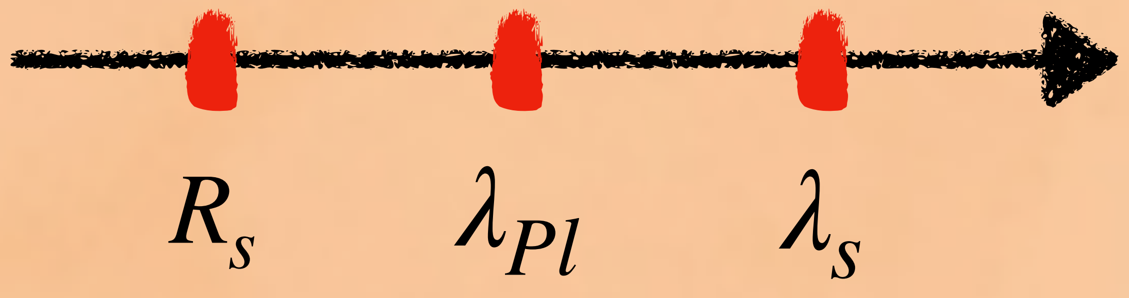

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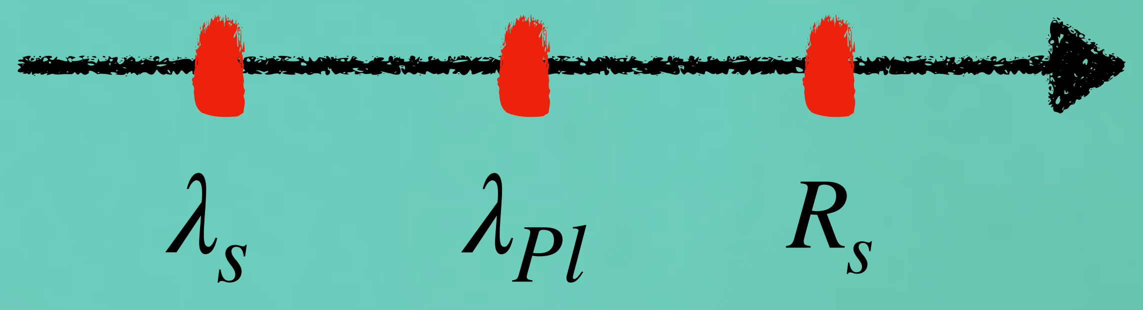
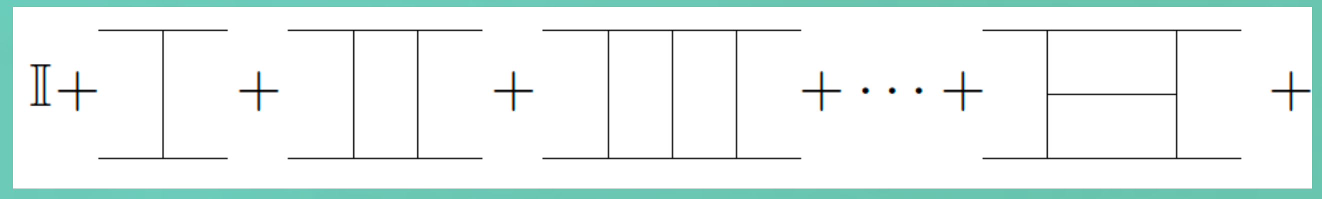
Transplanckian



$e^{i\delta(s,b)} \sim \mathrm{FT}[$  $]$

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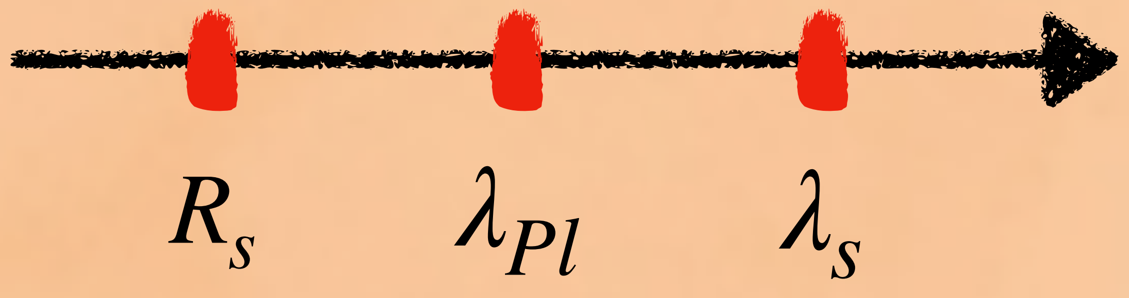
$\ll 1$ $\qquad\qquad$ $\gg 1$

Subplanckian



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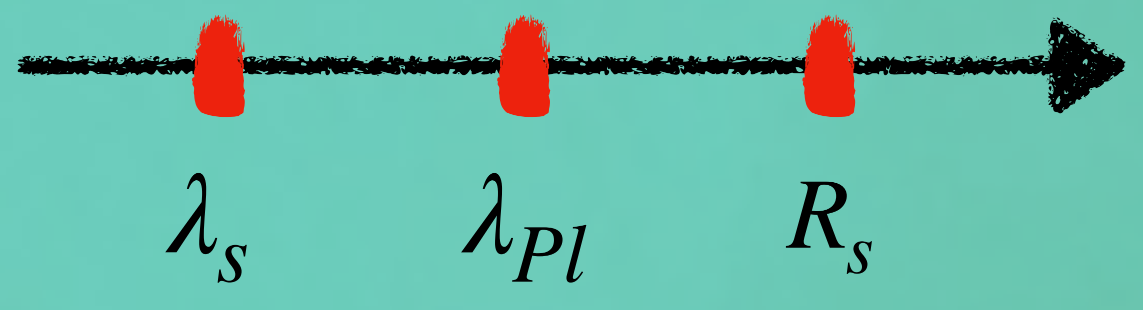
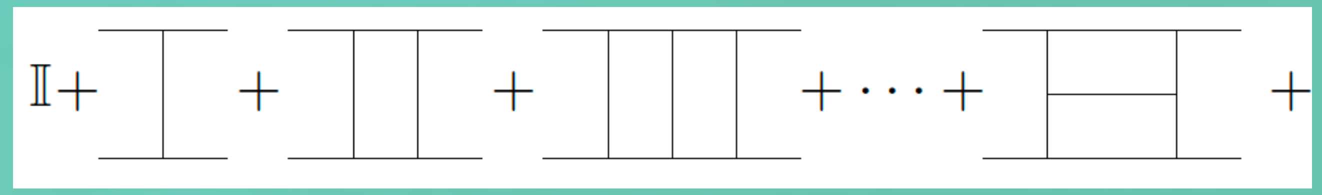


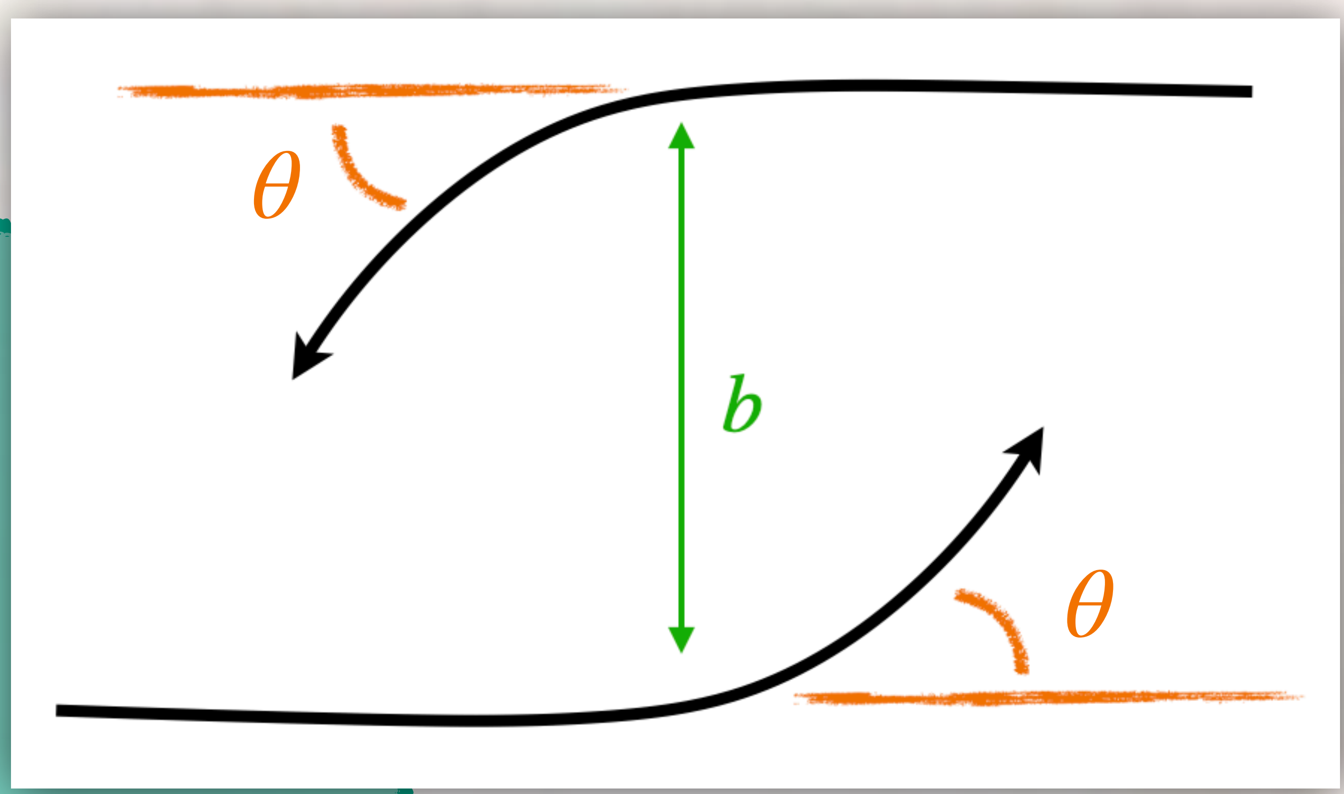
$$\underbrace{e^{i\delta(s,b)}}_{\text{Phase shift}} \sim \mathrm{FT}[\ \ldots\ ]$$

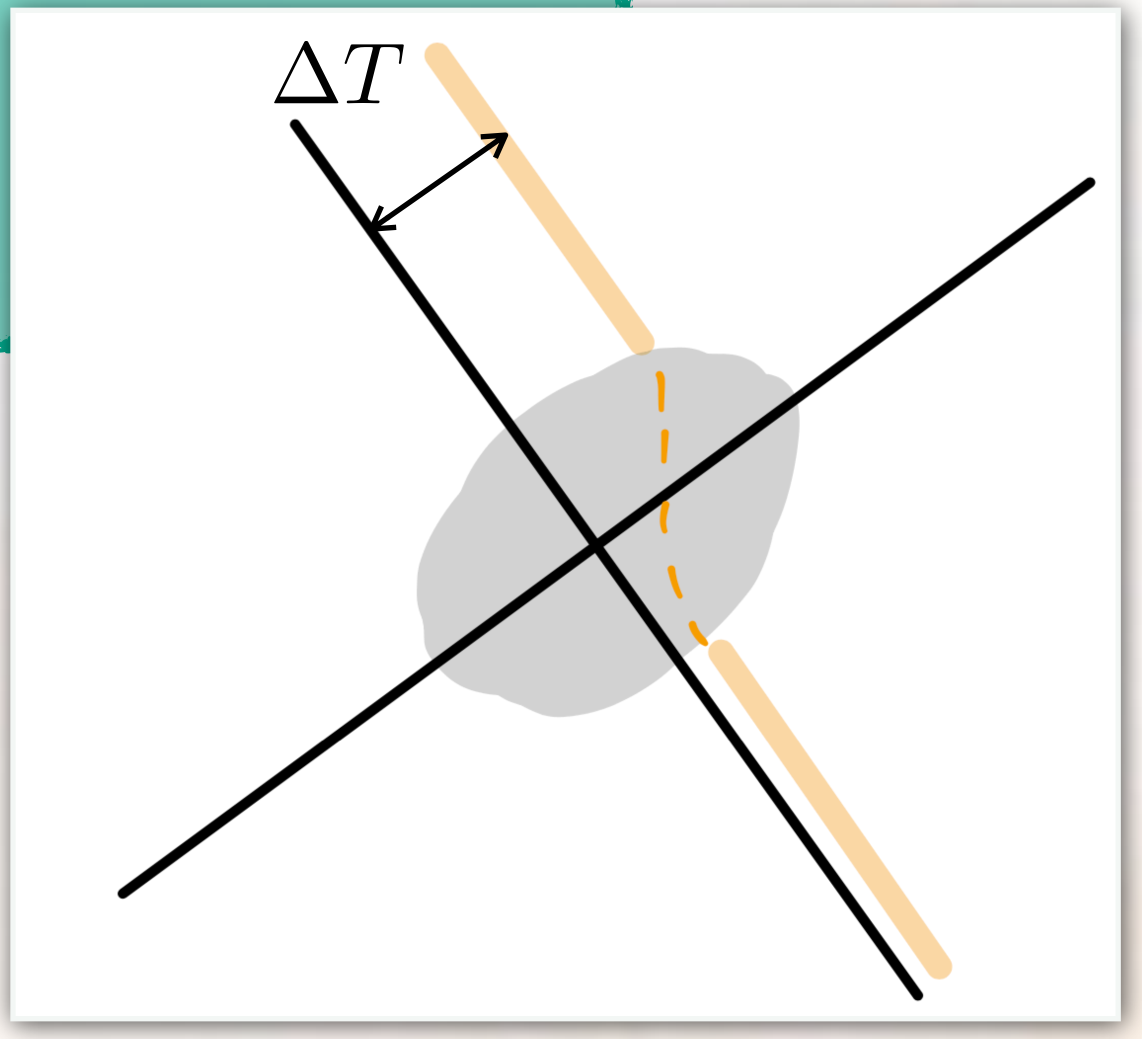


Why do we like the phase shift?

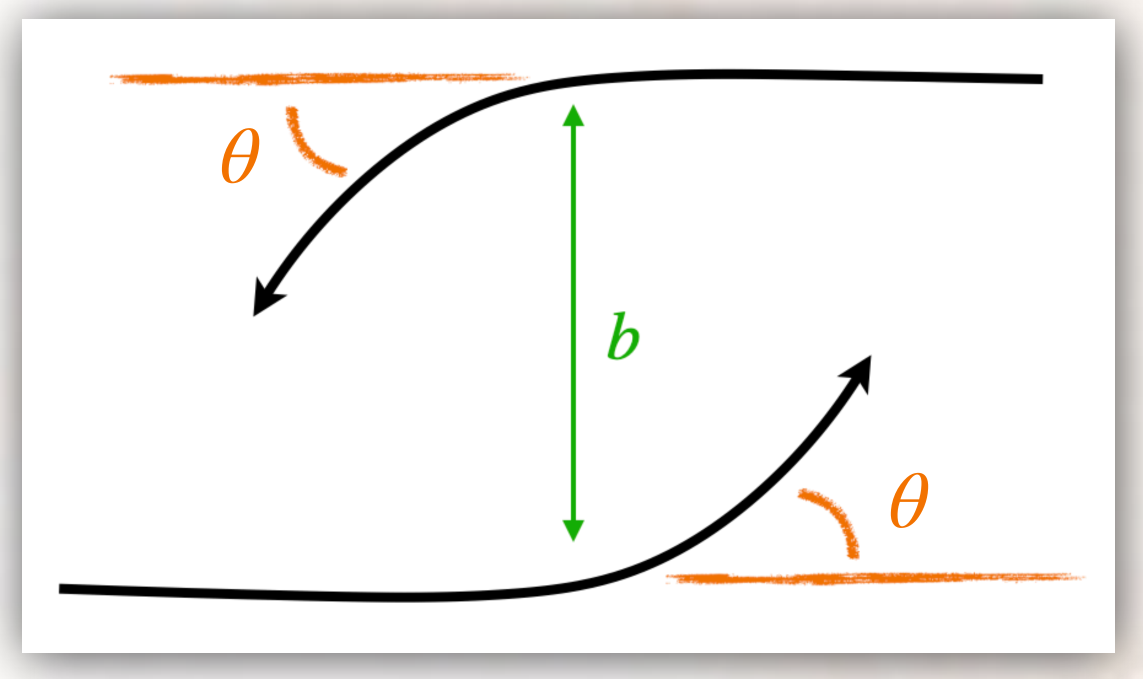
$$\theta = \frac{1}{\sqrt{s}} \frac{\partial \delta(b,s)}{\partial b}$$

$$\Delta T = \frac{\partial \delta(b,s)}{\partial \sqrt{s}}$$





Accessing the Eikonal regime from Amplitudes



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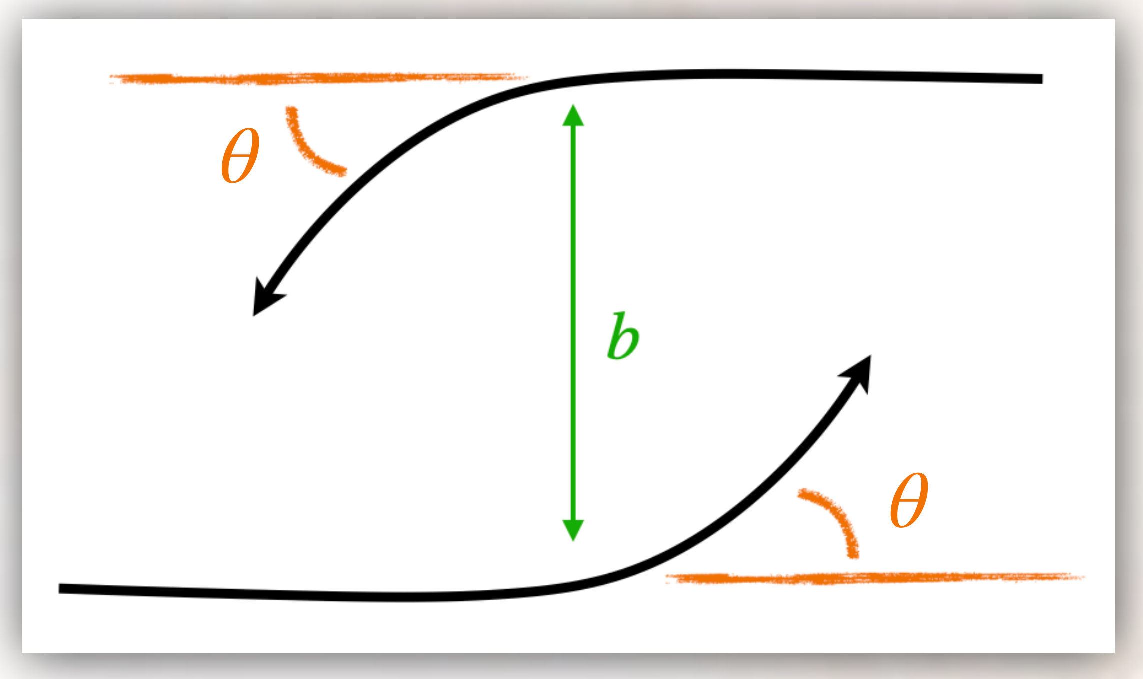

A. The Quantum and Classical Eikonal

- *When can we talk about semi-classical trajectories?*

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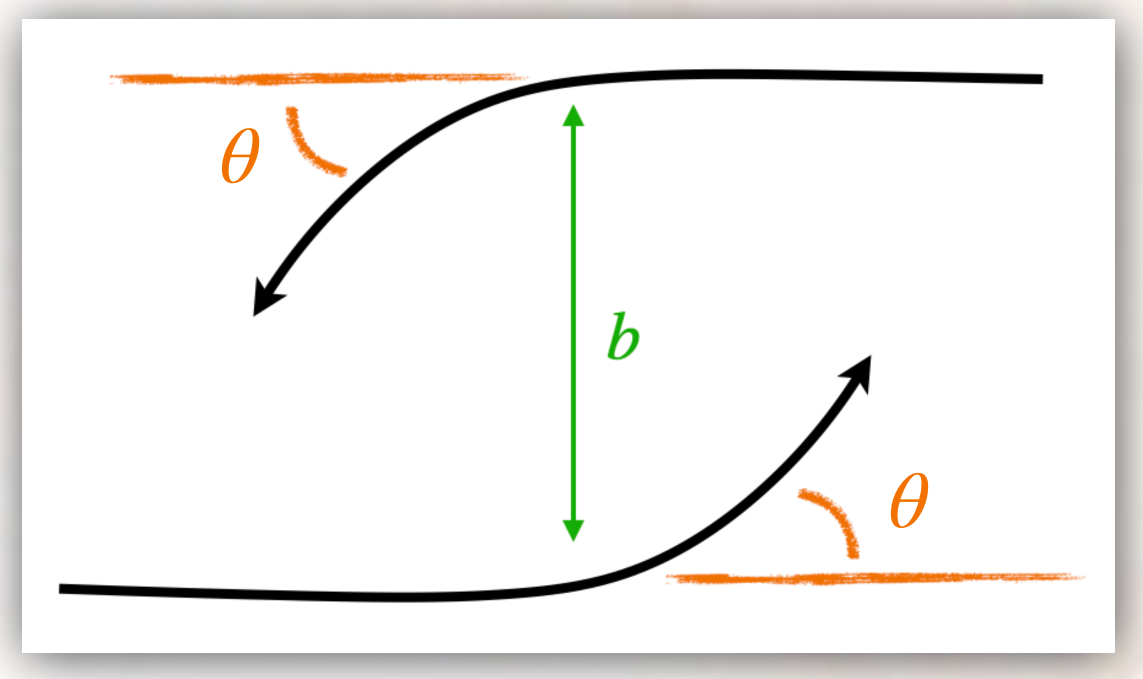

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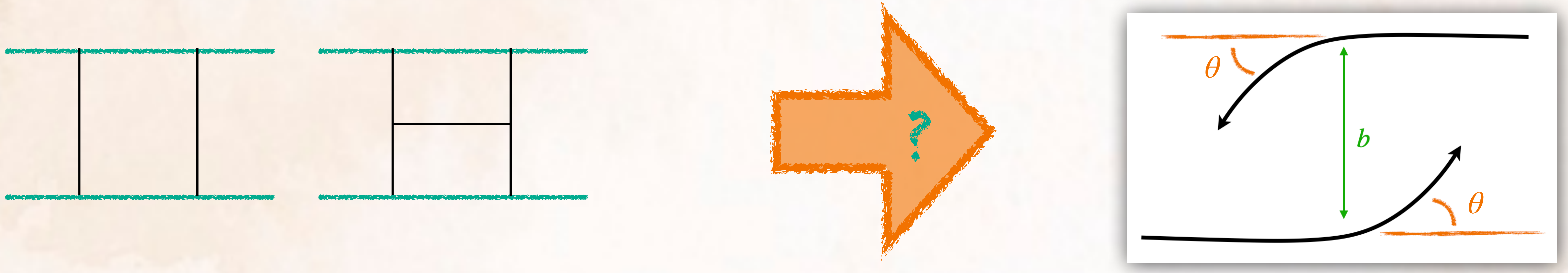

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- *When can we talk about semi-classical trajectories?*
- *What are the subleading corrections to* θ *?*
- *Are those corrections resolvable?*

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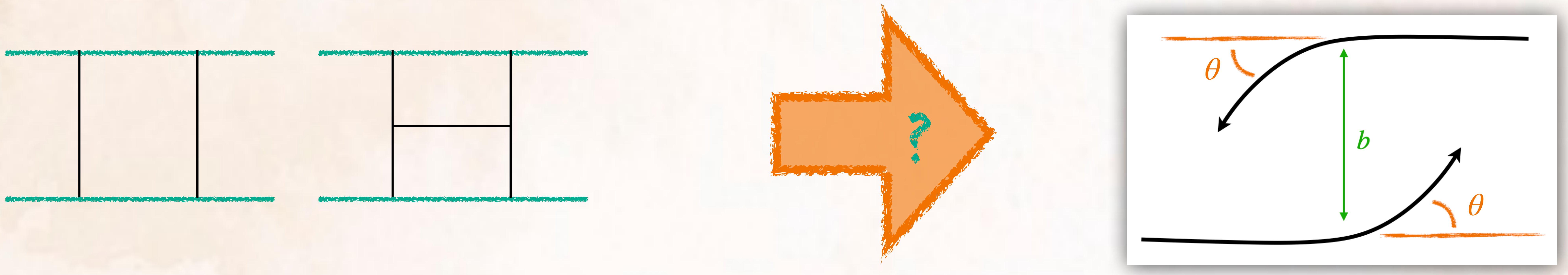
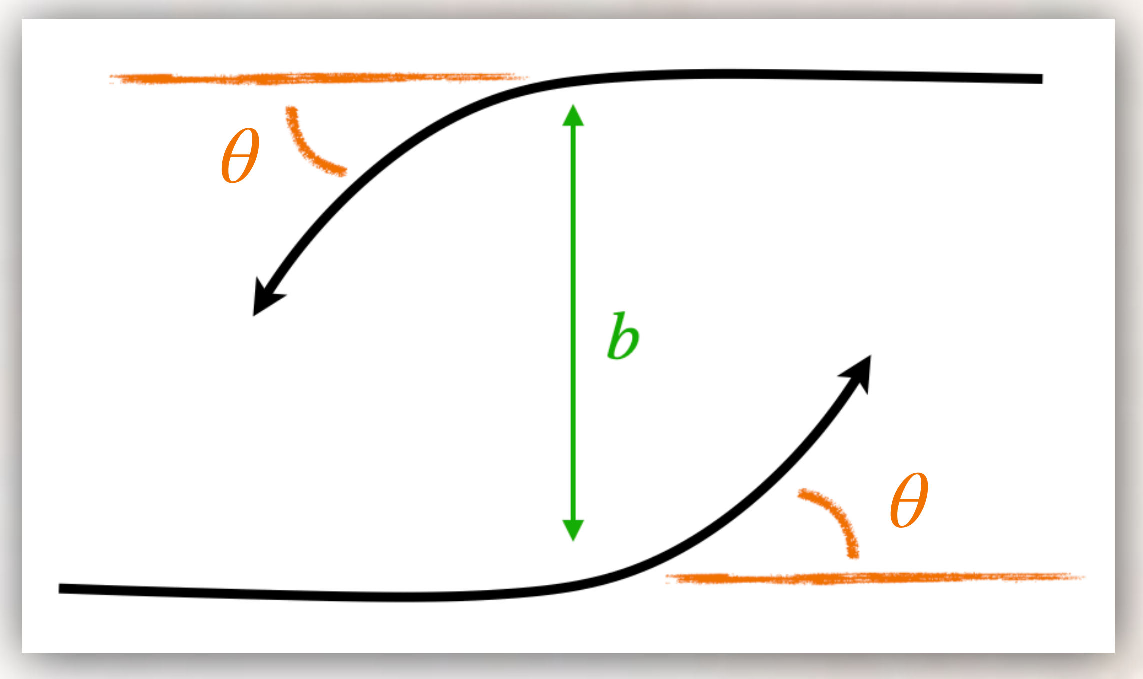


A. The Quantum and Classical Eikonal

B. The Spinning Eikonal Amplitude

- *How does the Eikonal amplitude change including spinning external states and subleading corrections?*

Accessing the Eikonal regime from Amplitudes

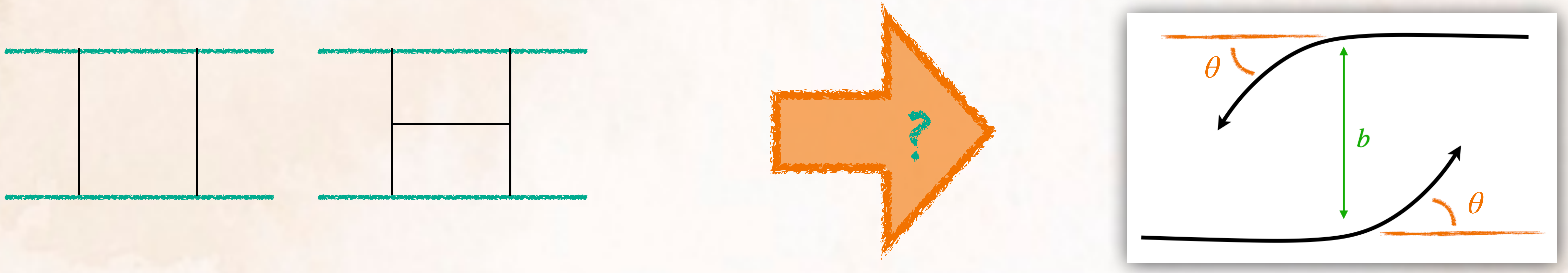


A. The Quantum and Classical Eikonal

B. The Spinning Eikonal Amplitude

- *How does the Eikonal amplitude change including spinning external states and subleading corrections?*
- *How does the continuous classical angular momentum emerge in the* $\ell \to \infty$ *limit?*

Accessing the Eikonal regime from Amplitudes



A. The Quantum and Classical Eikonal

B. The Spinning Eikonal Amplitude

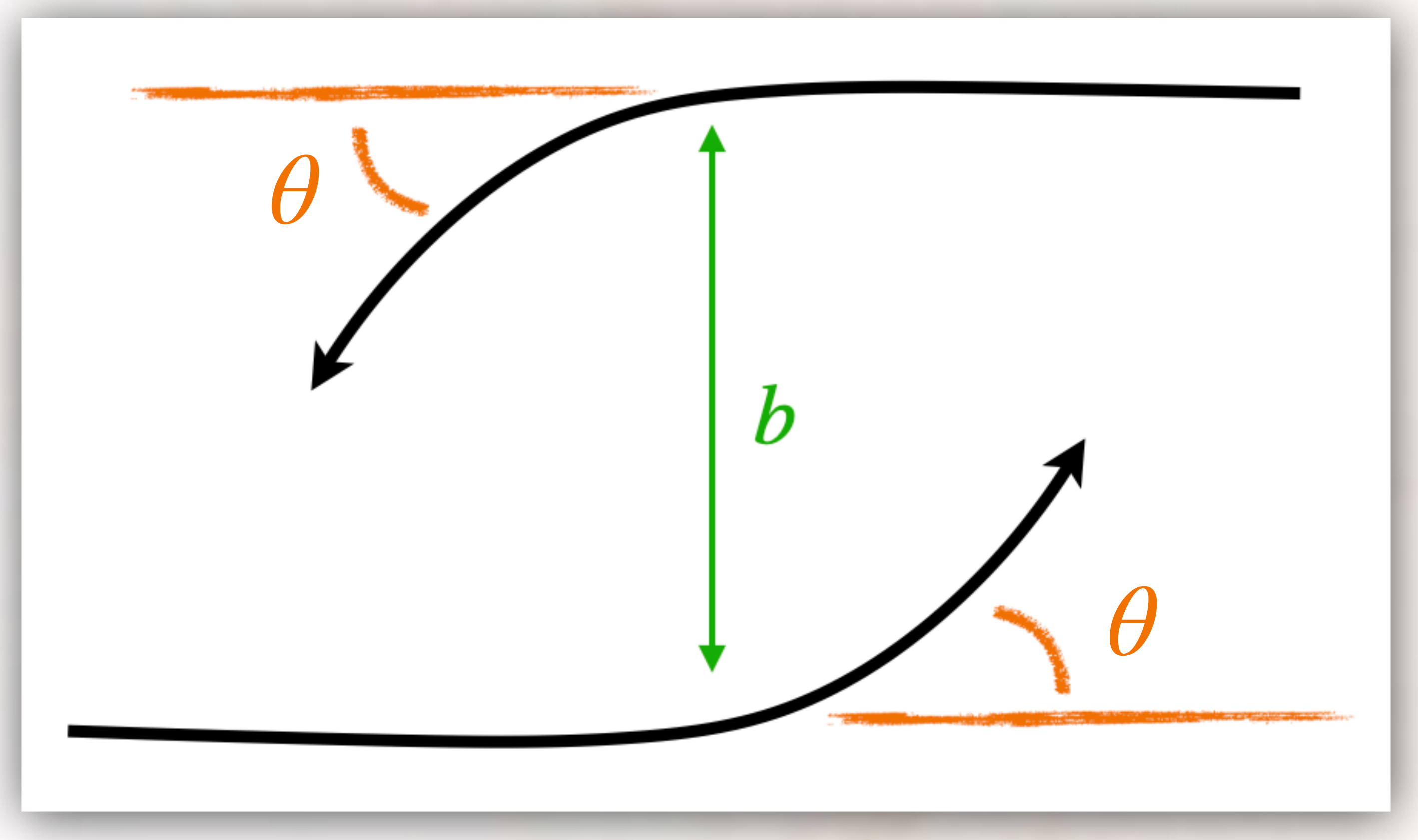
C. Causal structure in Eikonal Amplitudes

- *Can we implement positivity bounds in the Eikonal regime?*

A. The Quantum and Classical Eikonal

Semi-classical trajectories

$$\frac{\Delta\theta}{\theta} \ll 1 \qquad \frac{\Delta b}{b} \ll 1$$



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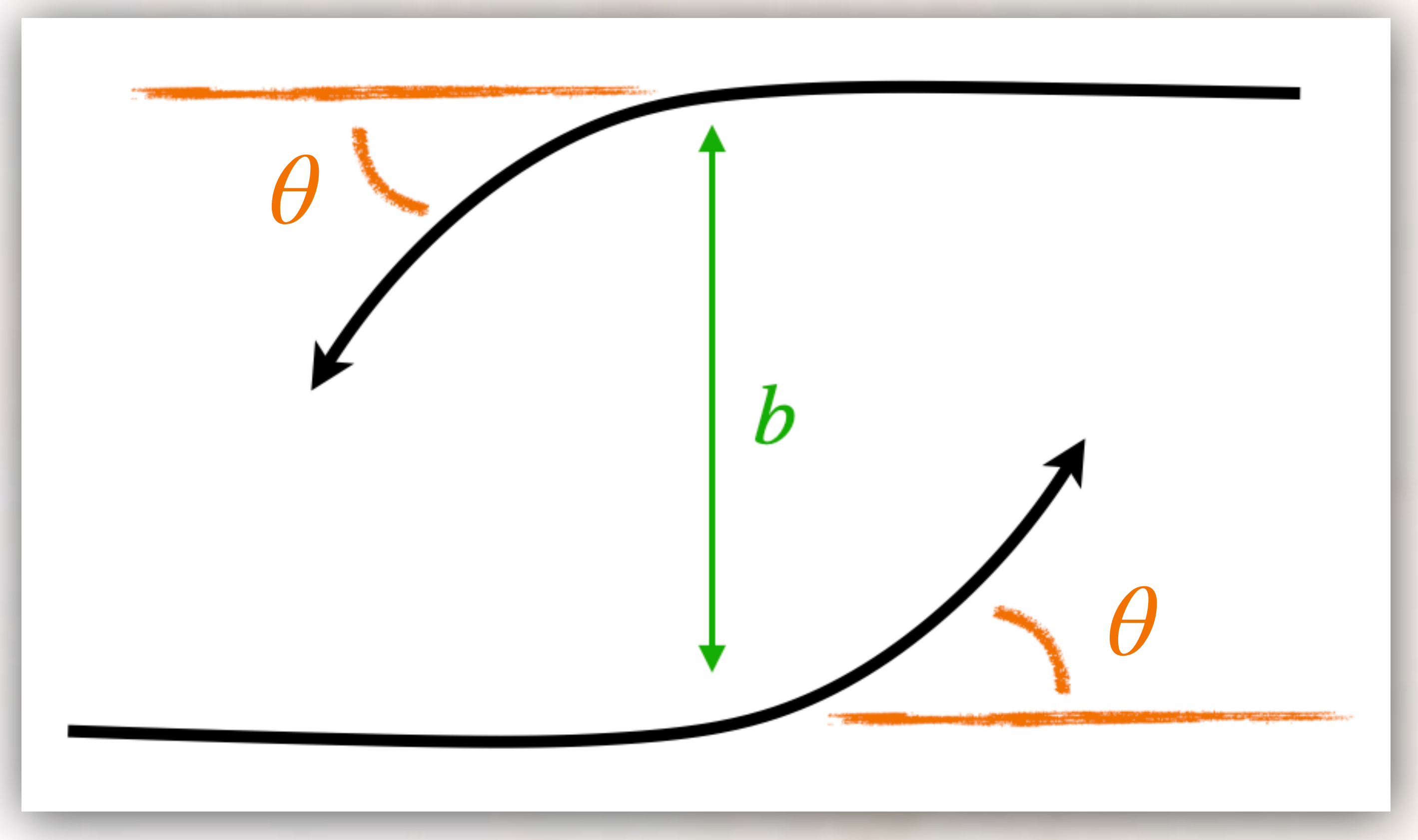

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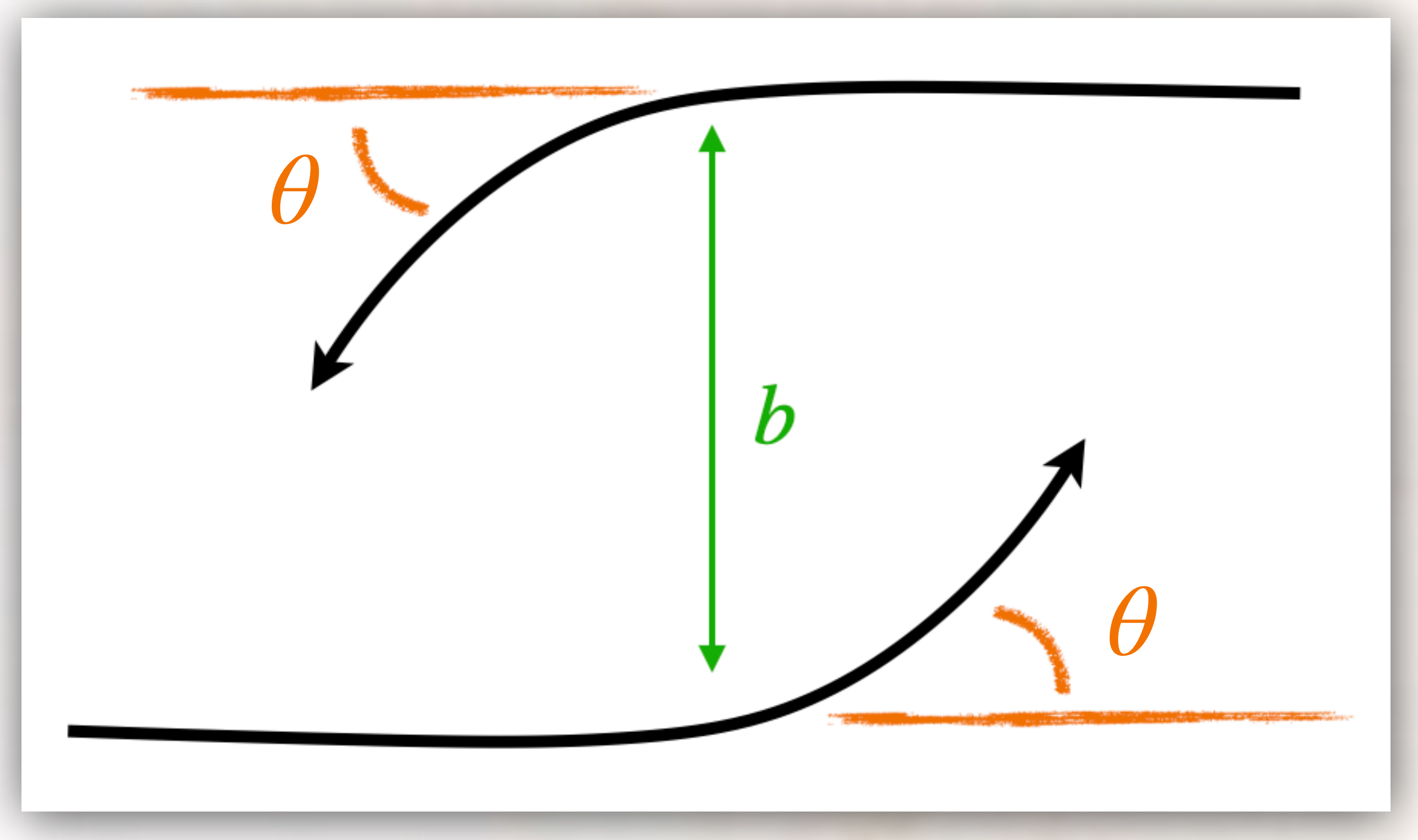

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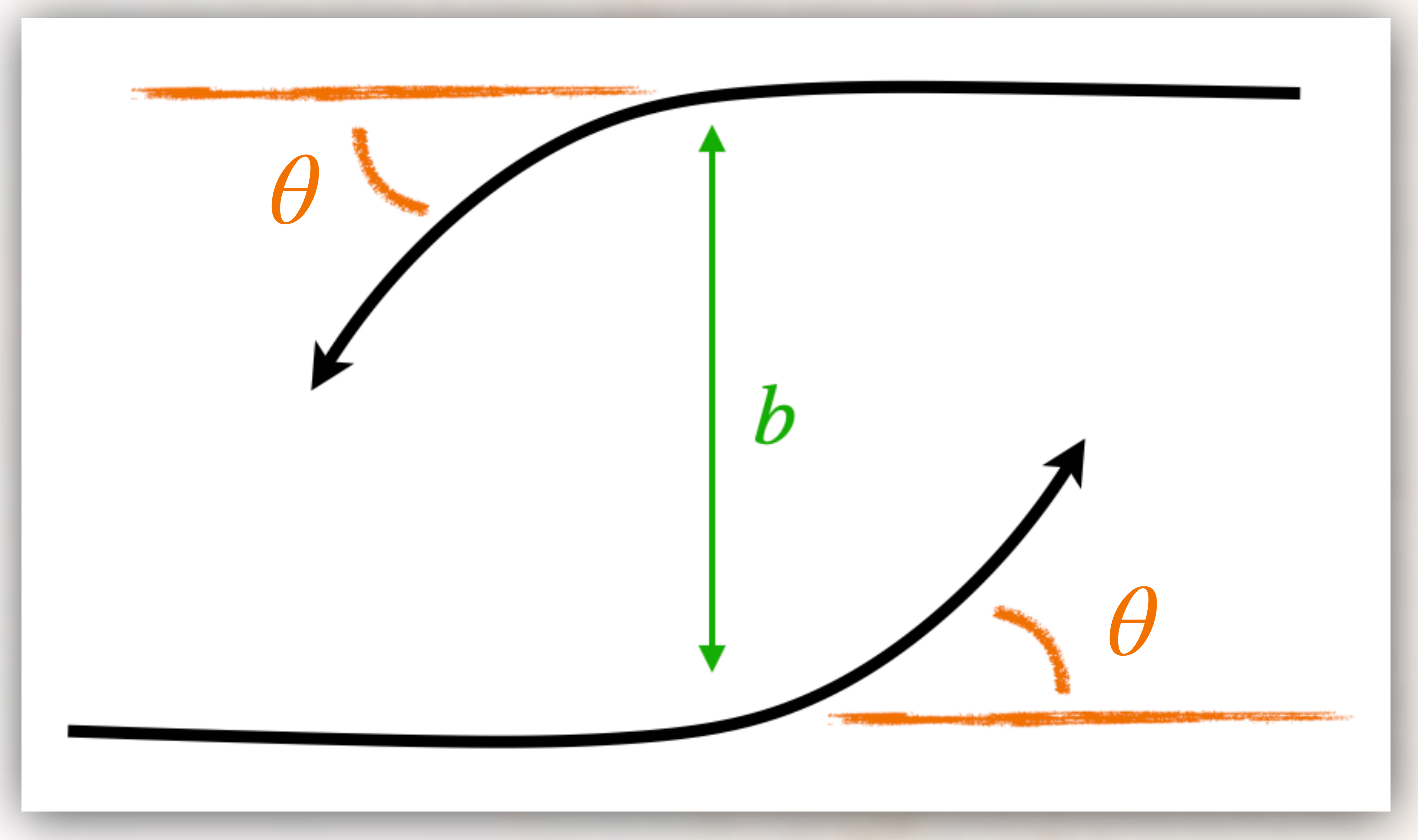

$$\frac{\Delta\theta}{\theta}\frac{\Delta b}{b} \sim \frac{1}{\alpha_g}$$

$$\alpha_g \gg 1$$

Transplanckian ↔ Semi-classicality

Subleading corrections to θ

$$\frac{\delta\theta}{\theta} \sim \begin{cases} (R_s/b)^n & \text{PM,} \\ (\lambda_{\text{Pl}}/b)^{2n} & \text{QG,} \\ \alpha^n \, (\lambdabar/b)^k & \text{Gauge/Tidal} \end{cases}$$

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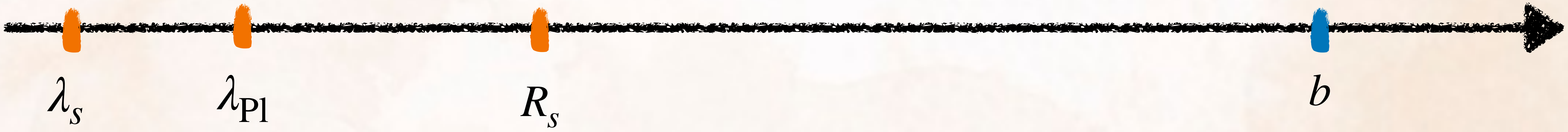

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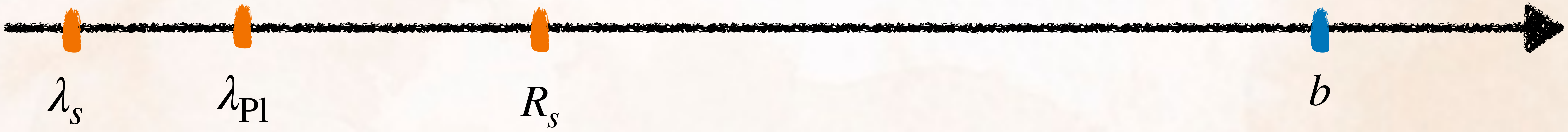

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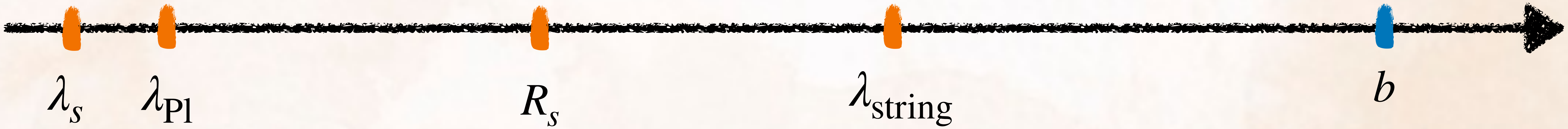
Amati, Ciafaloni, Veneziano
90's



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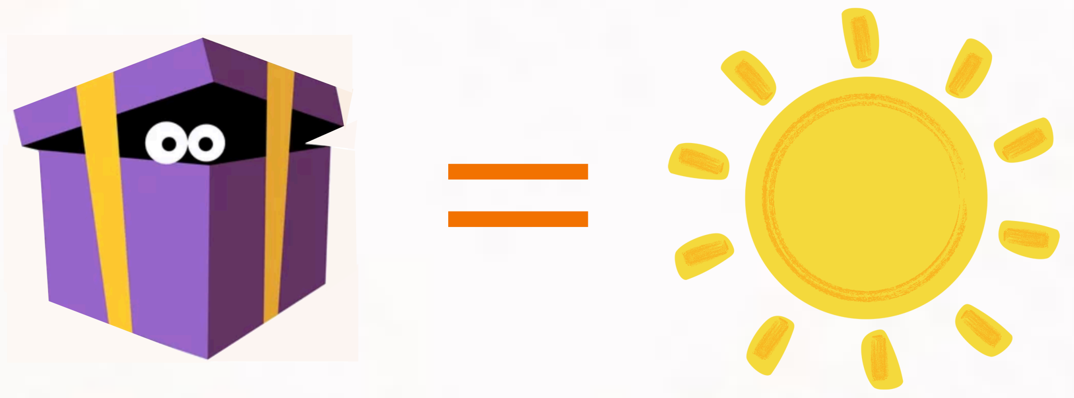

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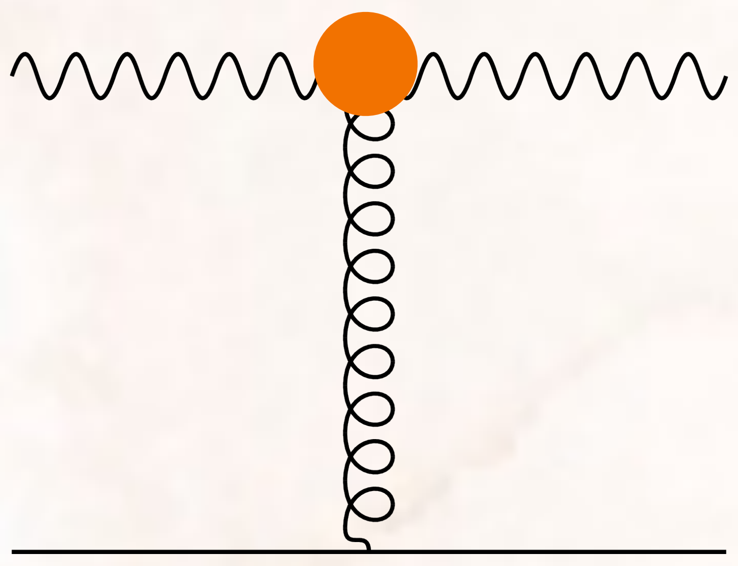
$\lambdabar^2 F_{\mu\nu} F_{\rho\sigma} R^{\mu\nu\rho\sigma}$



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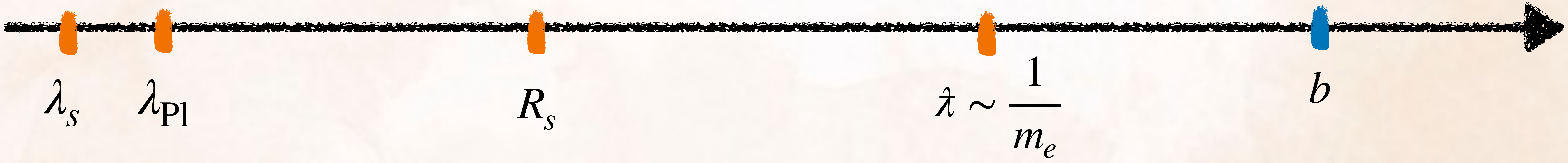

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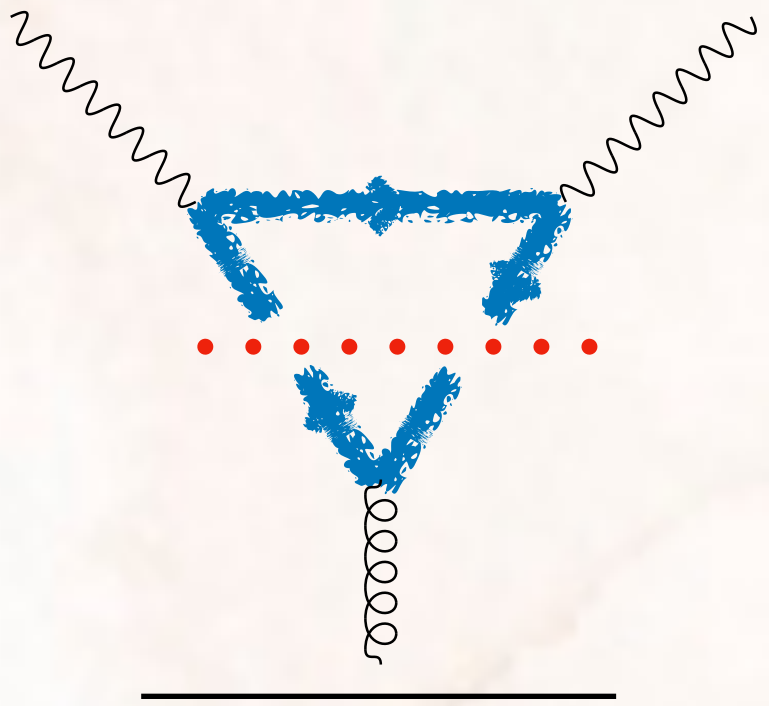
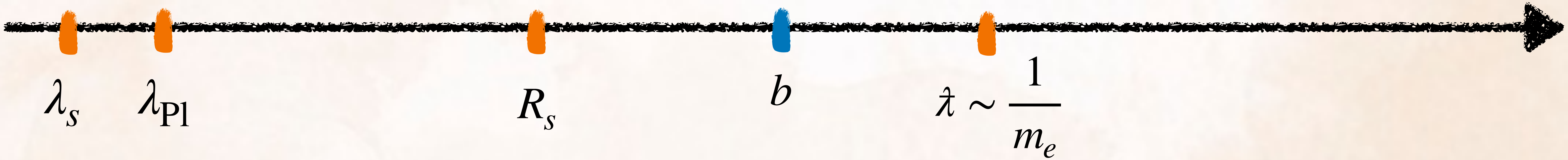


$$\frac{\delta\theta}{\theta} \sim \alpha \log^2 b/\lambda$$



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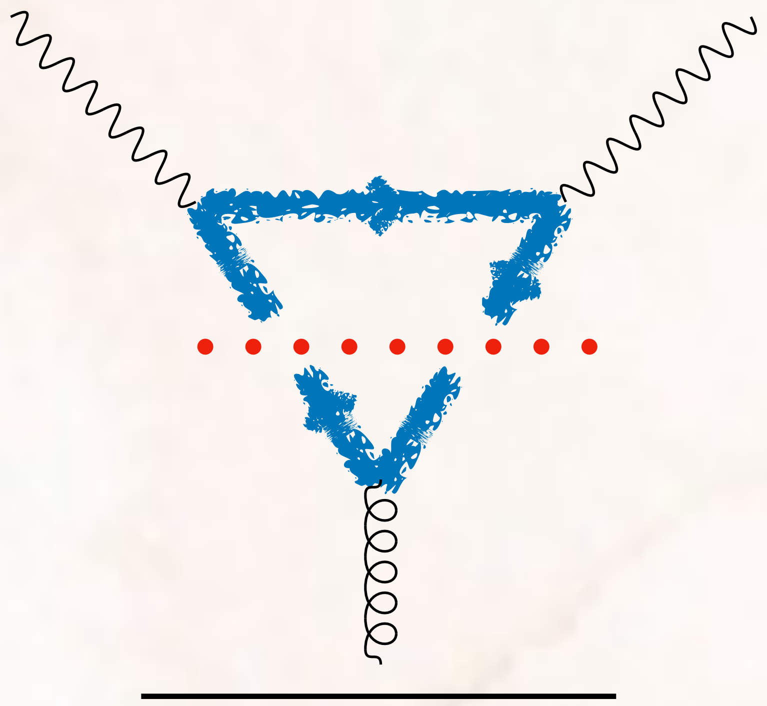

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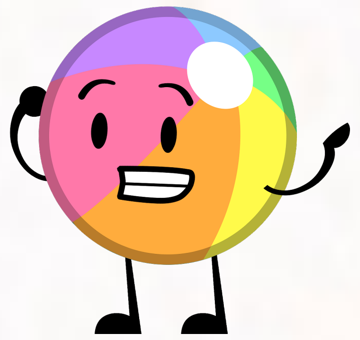

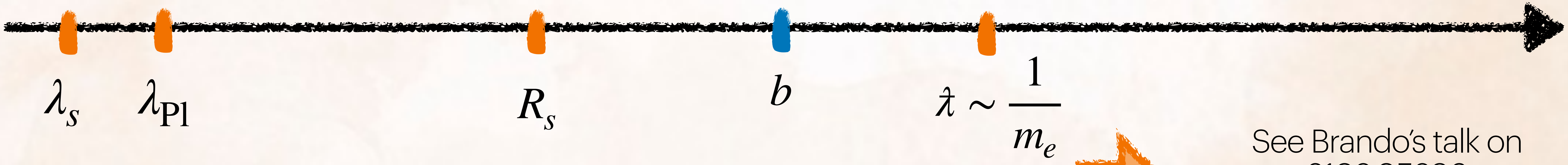

See Brando's talk on
2108.05896
[Bellazzini, GI, Lewandowski, Sgarlata],
QCD meets Gravity 2021

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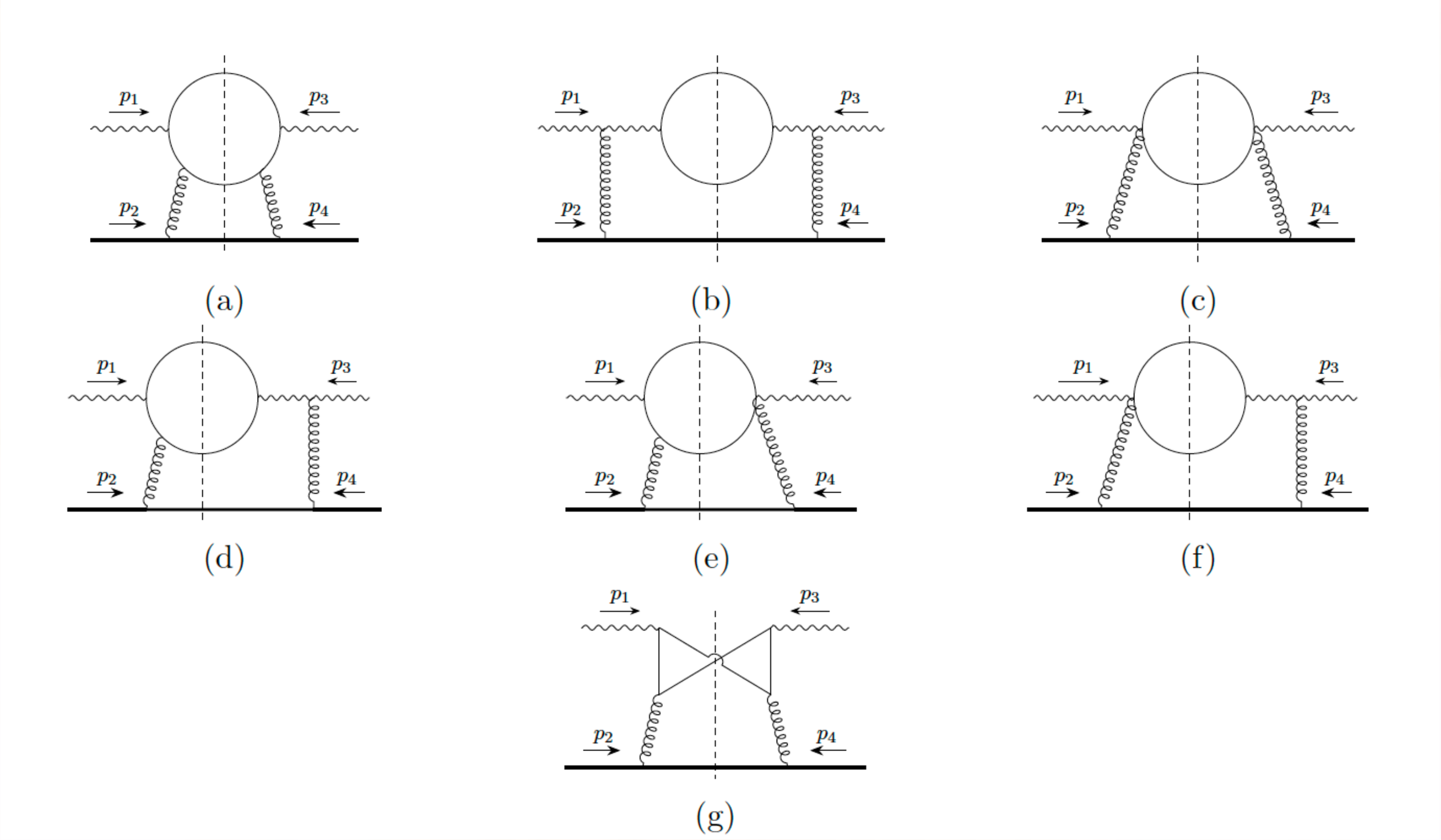

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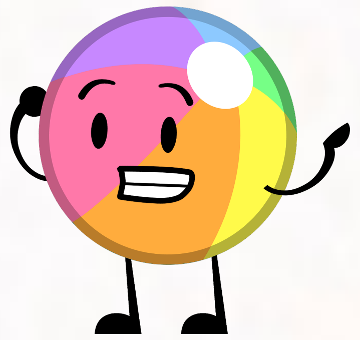
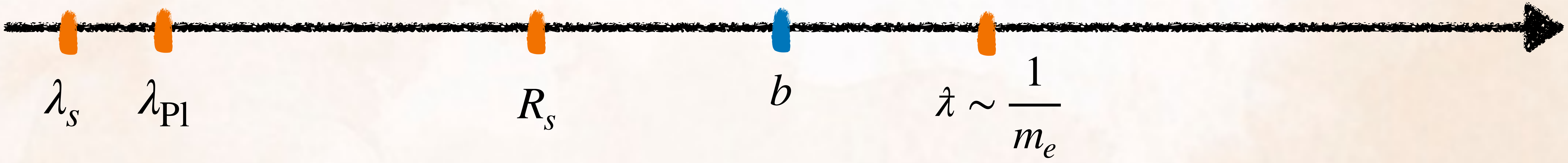
=





Resolvability of subleading corrections

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Post-Minkowskian

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✓

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Quantum Gravity

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✓ ✗

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✔ ✘

Gauge and Tidal

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✓ PM, ✗ QG, ✓ Gauge/Tidal

Gauge and Tidal

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- *When can we talk about semi-classical trajectories?*

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$$\frac{\delta\theta}{\theta} \sim \begin{cases} (R_s/b)^n & \text{PM}, \\ (\lambda_{\text{Pl}}/b)^{2n} & \text{QG}, \\ \alpha^n\,(\lambda\!\!\!^{-}/b)^k & \text{Gauge/Tidal} \end{cases}$$

Summary

A. The Quantum and Classical Eikonal

- *When can we talk about semi-classical trajectories?*

$\alpha_g \gg 1$ Transplanckian ↔ Semi-classicality

- *What are the subleading corrections to* θ *?*

 $= ?$

- *Are those corrections resolvable?*

QFT-like corrections can be resolvable and important

$$\frac{\delta\theta}{\theta} \sim \begin{cases} (R_s/b)^n & \text{PM}, \\ (\lambda_{\rm Pl}/b)^{2n} & \text{QG}, \\ \alpha^n\,(\lambda\!\!\!^{-}/b)^k & \text{Gauge/Tidal} \end{cases}$$

PM ✓, QG ✗, Gauge/Tidal ✓

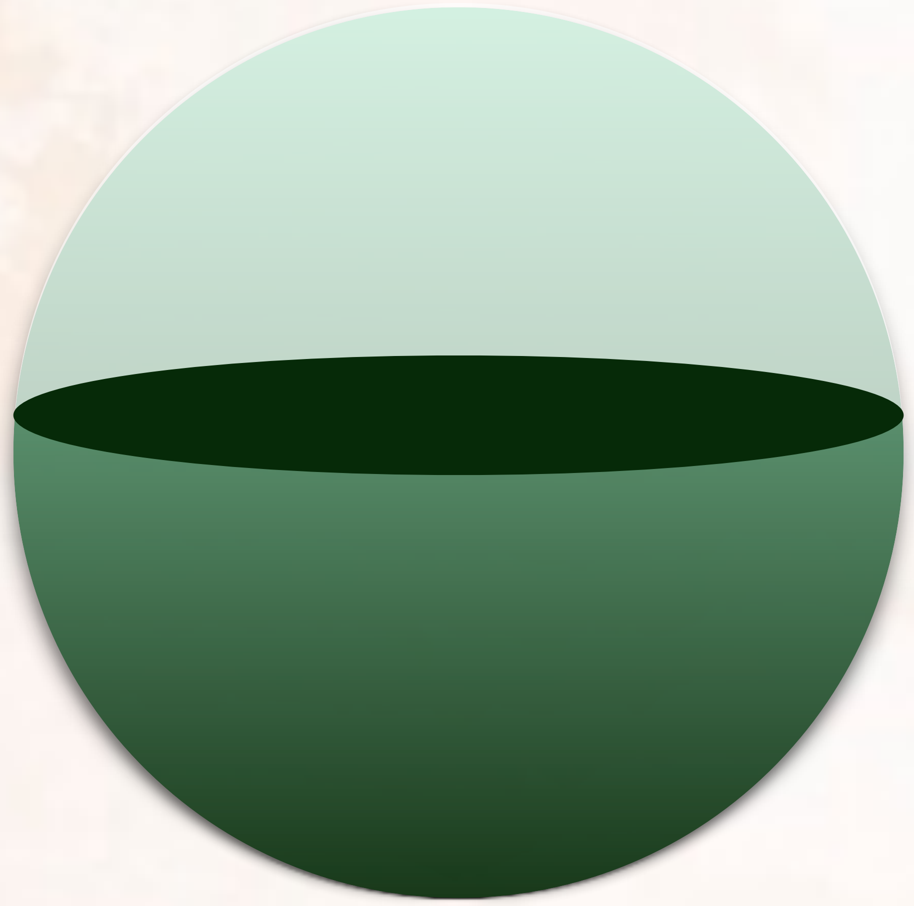
B. The Spinning Eikonal Amplitude

Emergence of a classical ℓ

A geometrical intuition

$$\mathcal{M}_\ell(s)^{\lambda_3\lambda_4}_{\lambda_1\lambda_2} = \frac{|\boldsymbol{p}|}{8\pi\sqrt{s}}\int_{-1}^{1} d\cos\theta\, d^\ell_{\lambda_{12}\lambda_{34}}(\theta)\mathcal{M}^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(p_i)\Big|_{\phi=0}$$

$\lambda_{ij} = \lambda_i - \lambda_j$



$SO(3)$

Compact
Finite dim irreps

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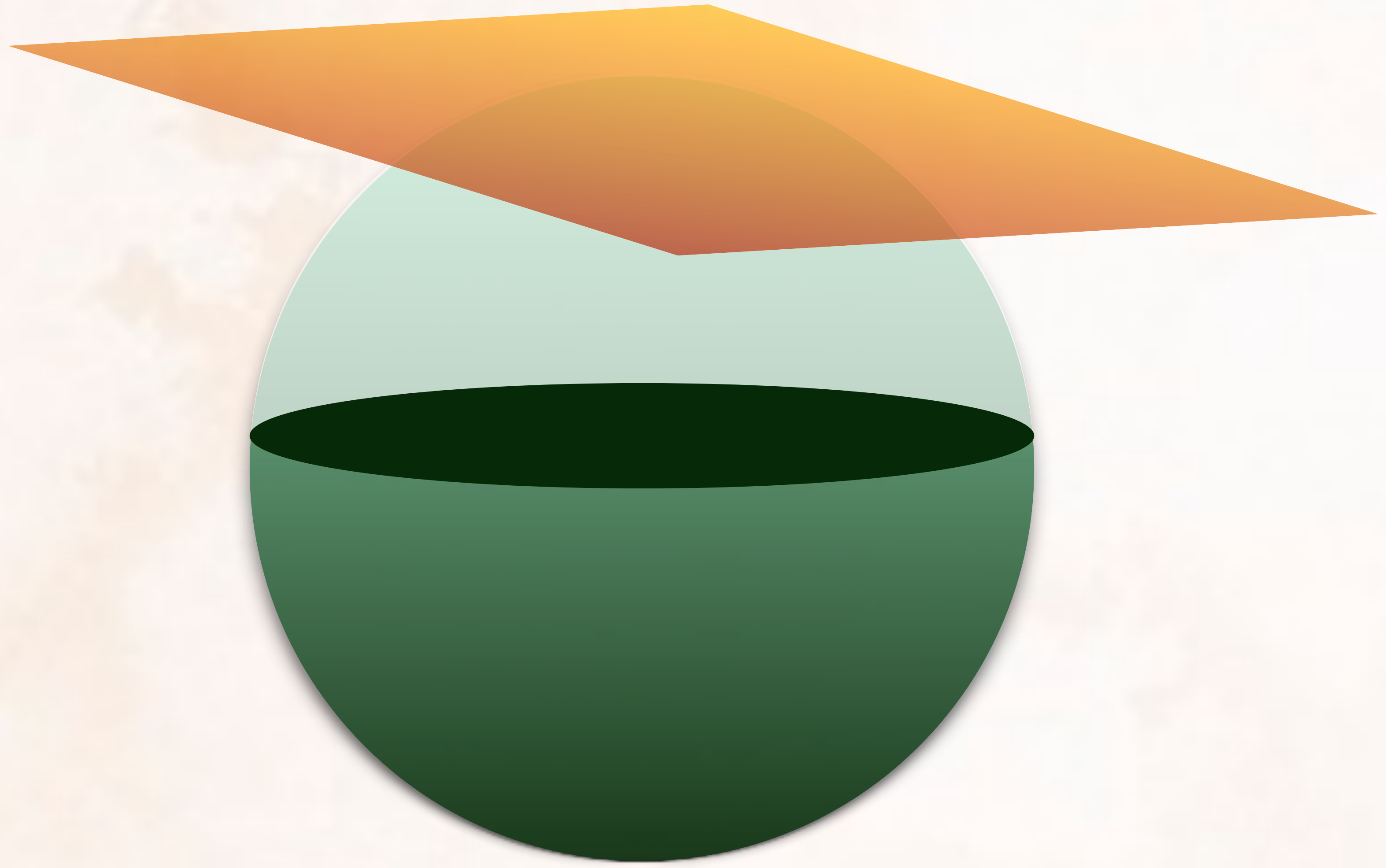

$$SO(3) \underset{\ell\to\infty}{\Rightarrow} ISO(2)$$

Compact	Non-compact
Finite dim irreps	Continuous irreps

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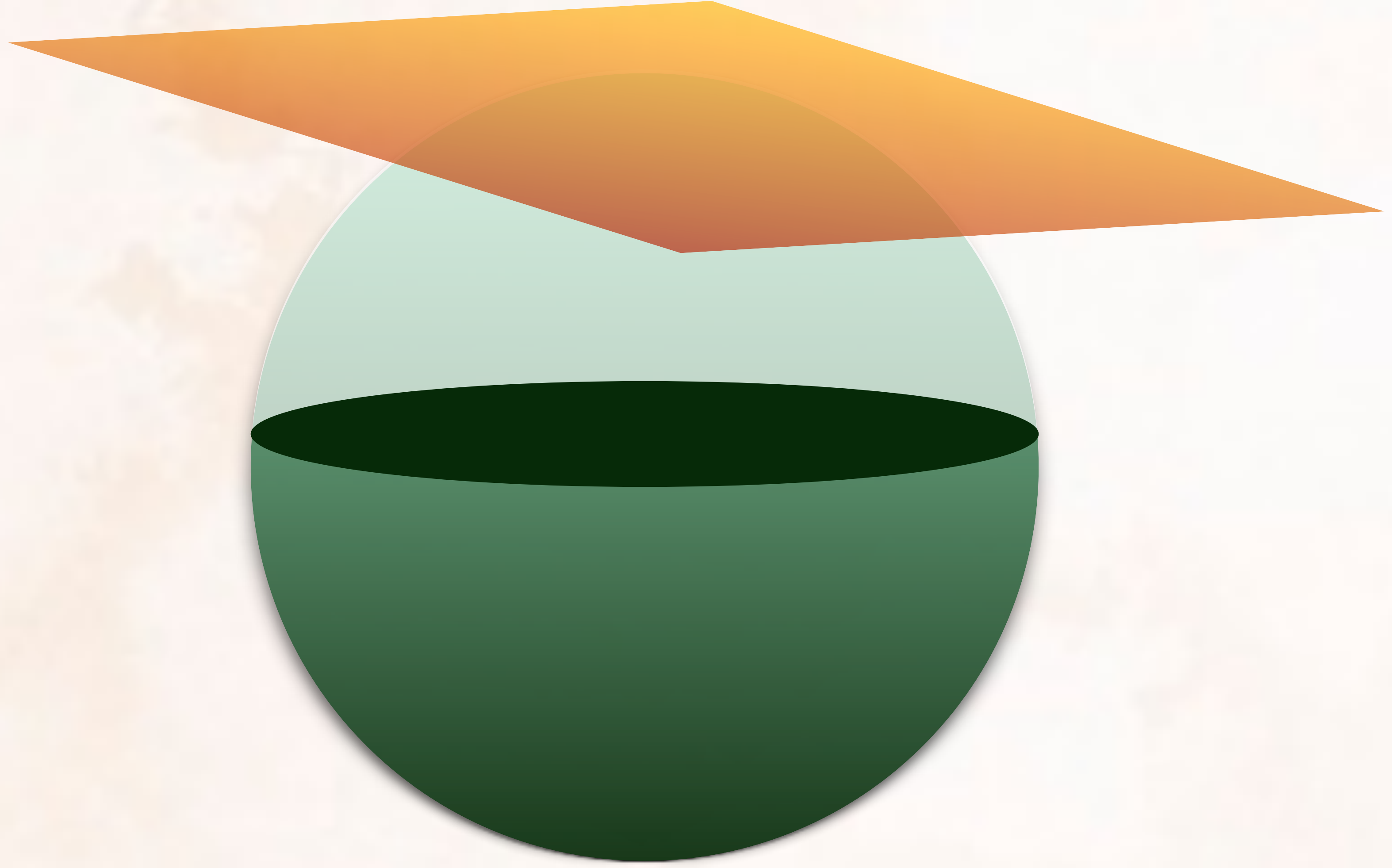

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Large ℓ limit of the Wigner d-Matrix

$$d^{\ell}_{\lambda\lambda'}(\cos\theta) = {}_{\ell}< \lambda' | e^{-\frac{1}{2}(J_+ - J_-)\,\theta} | \lambda >_{\ell}$$

Continuous spin basis: $$|\lambda>_{\ell} = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\lambda\varphi} |\varphi>_{\ell}$$

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Eigenstates of $J_{\pm}$

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$$d^{\ell}_{\lambda'\lambda}(\theta) \xrightarrow[\ell\to\infty]{} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i(\lambda'-\lambda)\varphi} e^{i\theta\ell\sin\varphi} = J_{\lambda-\lambda'}(\ell\theta)$$

The Eikonal amplitude

Emergence of the 2D Fourier Transform

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The Eikonal amplitude

Emergence of the 2D Fourier Transform

$$S_\ell(s) = e^{i\delta_\ell(s)} = \mathbb{1} + i\mathcal{M}_\ell(s)$$

$$\delta_{\ell(b)}(s) \equiv \delta(b, s)$$

$$e^{2i\delta(s,\boldsymbol{b})} - \mathbb{1} \equiv \frac{i}{4|\boldsymbol{p}|\sqrt{s}} \int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{i\mathbf{b}\mathbf{q}}\, \mathcal{M}(\boldsymbol{p},\boldsymbol{q})\Big|_{\substack{\boldsymbol{q} = (\mathbf{q},0) \\ \mathbf{q}^2 \ll \boldsymbol{p}^2}}$$

$$\mathcal{M}(\boldsymbol{p},\boldsymbol{q})\Big|_{\text{eik}} = -4i|\boldsymbol{p}|\sqrt{s} \int d^2\mathbf{b}\; e^{-i\mathbf{q}\mathbf{b}} \left(e^{2i\delta(s,\mathbf{b})} - \mathbb{1}\right)$$

The Eikonal amplitude

Emergence of the 2D Fourier Transform

$$S_\ell(s) = e^{i\delta_\ell(s)} = \mathbb{1} + i\mathcal{M}_\ell(s)$$

$$d^\ell_{\lambda'\lambda}(\theta) = N_{\lambda',\lambda,\ell}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_{\lambda-\lambda'}((\ell+\frac{1}{2})\theta) + \sqrt{\theta}O(1/\ell^{3/2})$$

$$e^{2i\delta(s,\boldsymbol{b})} - \mathbb{1} \equiv \frac{i}{4|\boldsymbol{p}|\sqrt{s}}\int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{i\mathbf{bq}}\,\mathcal{M}(\boldsymbol{p},\boldsymbol{q})\Big|_{\substack{\boldsymbol{q}=(\mathbf{q},0)\\ \mathbf{q}^2 \ll \boldsymbol{p}^2}}$$

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All-Orders Eikonal

$$\mathcal{M}(\boldsymbol{p},\boldsymbol{q})\Big|_{\text{eik}} = -i4|\boldsymbol{p}|\sqrt{s}\,\mathcal{N}(\theta)\int d^2\mathbf{b}_e\; e^{-i\mathbf{b}_e\mathbf{q}}\left(e^{2i\delta(s,\mathbf{b}(\mathbf{b}_e))} - \mathbb{1}\right)$$

$$\mathcal{N}(\theta) = \left[\left(\frac{\theta}{\sin\theta}\right)^{1/2}\left(\frac{\sin\theta/2}{\theta/2}\right)^2\right] = 1 + O(\theta^4)$$

$$\mathbf{b} = \left(\frac{\sin\theta/2}{\theta/2}\right)\mathbf{b}_e$$

Summary

B. The Spinning Eikonal Amplitude

- *How does the Eikonal amplitude change including spinning external states and subleading corrections?*

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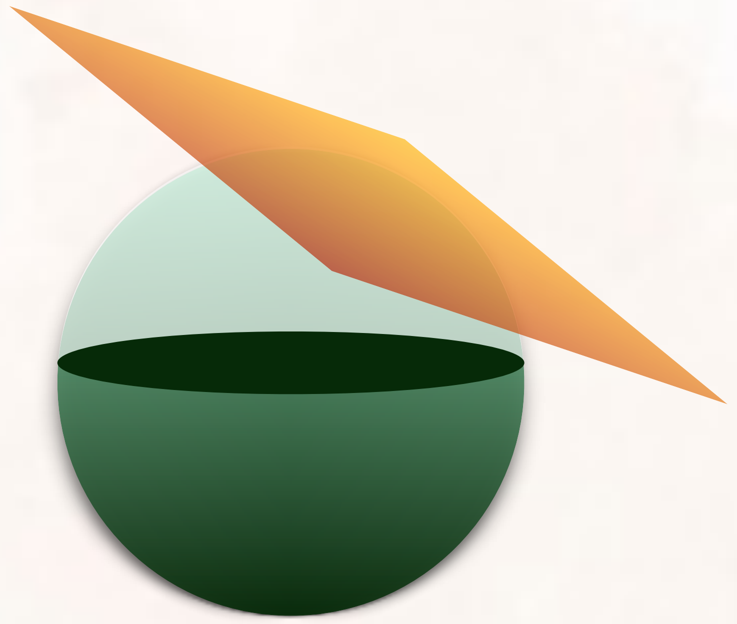
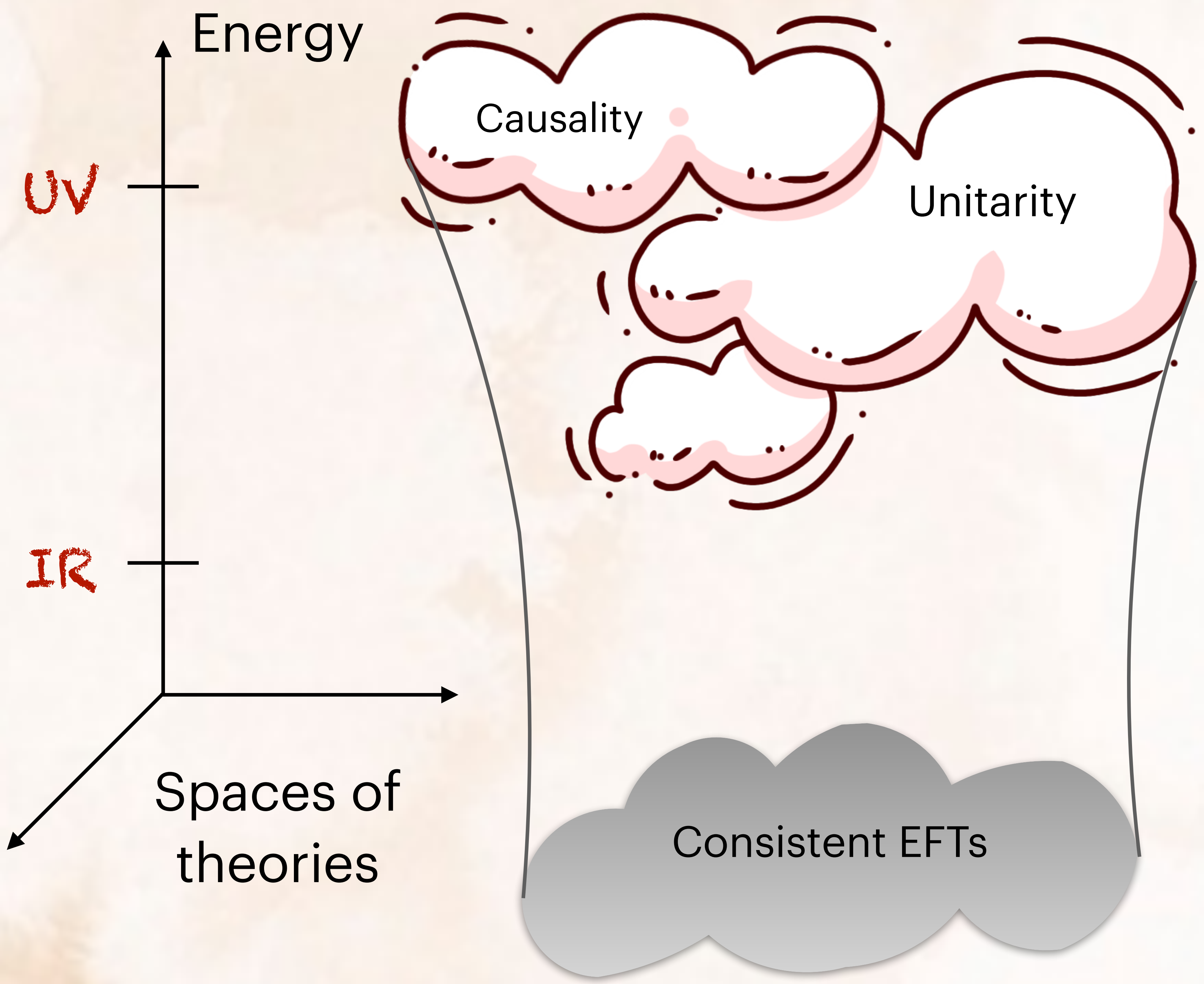
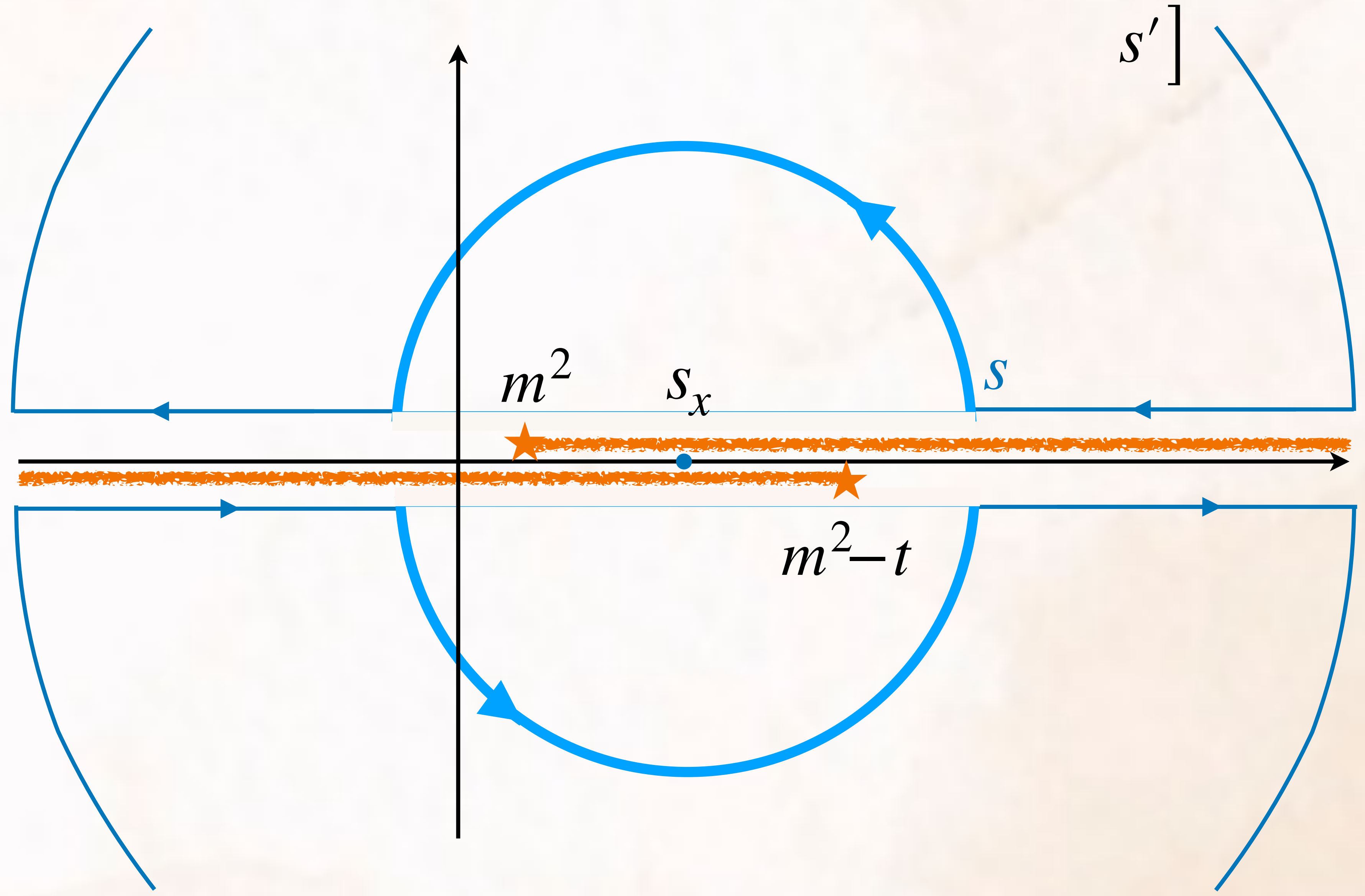


$$SO(3) \underset{\ell \to \infty}{\Rightarrow} ISO(2)$$

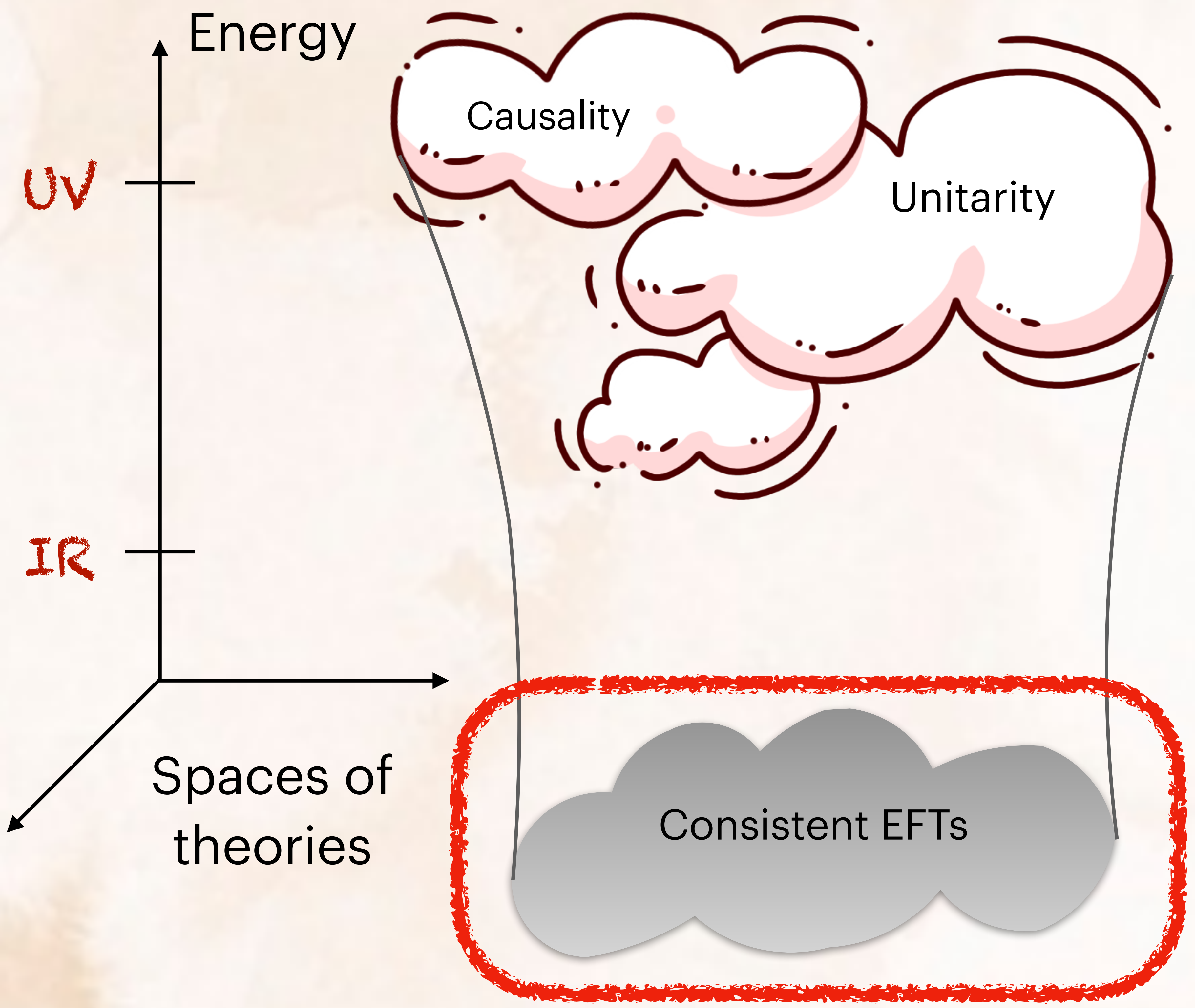
C. Causal structure in Eikonal Amplitudes

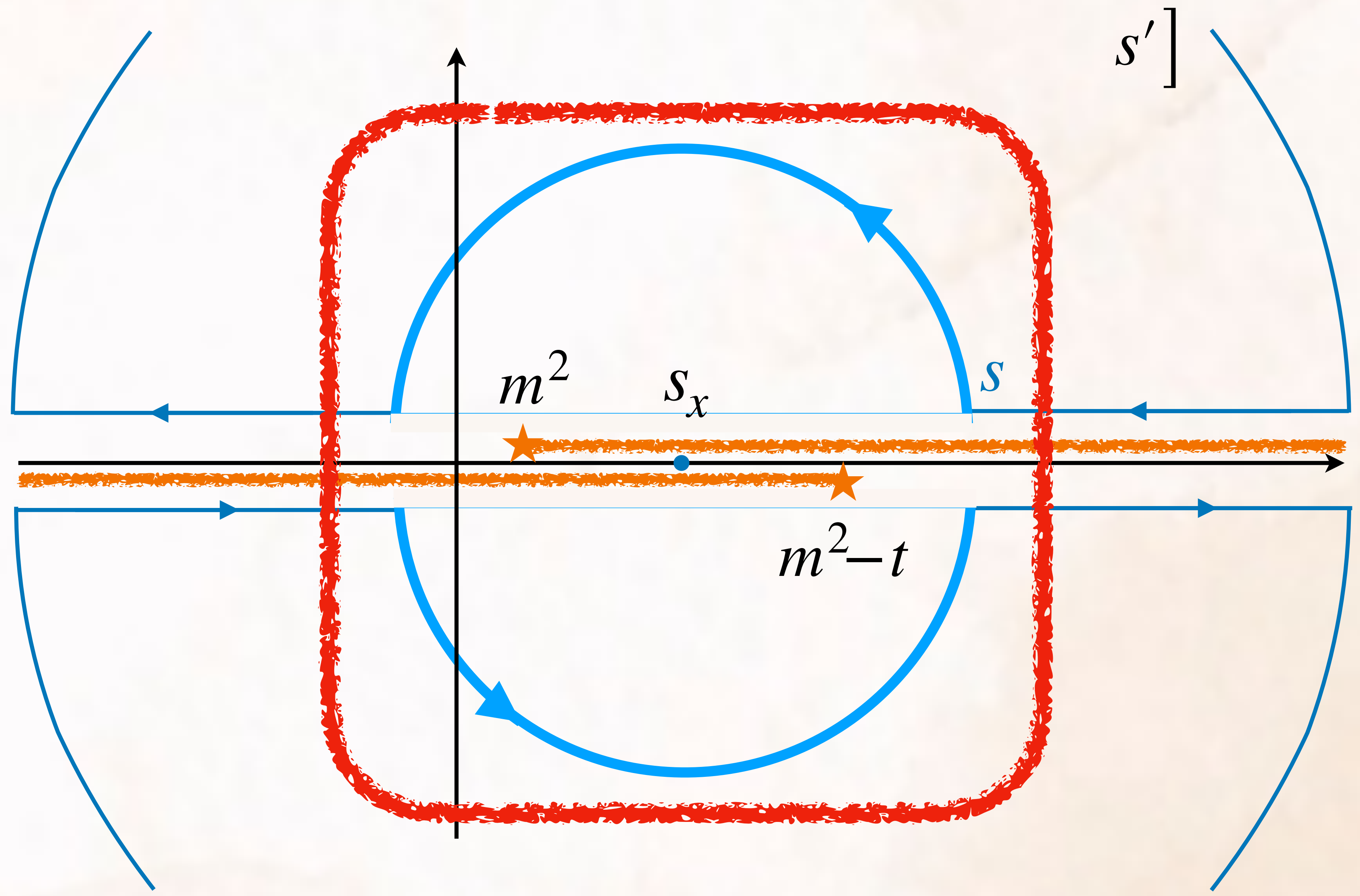
Constraining EFTs from fundamental principles





Constraining EFTs from fundamental principles





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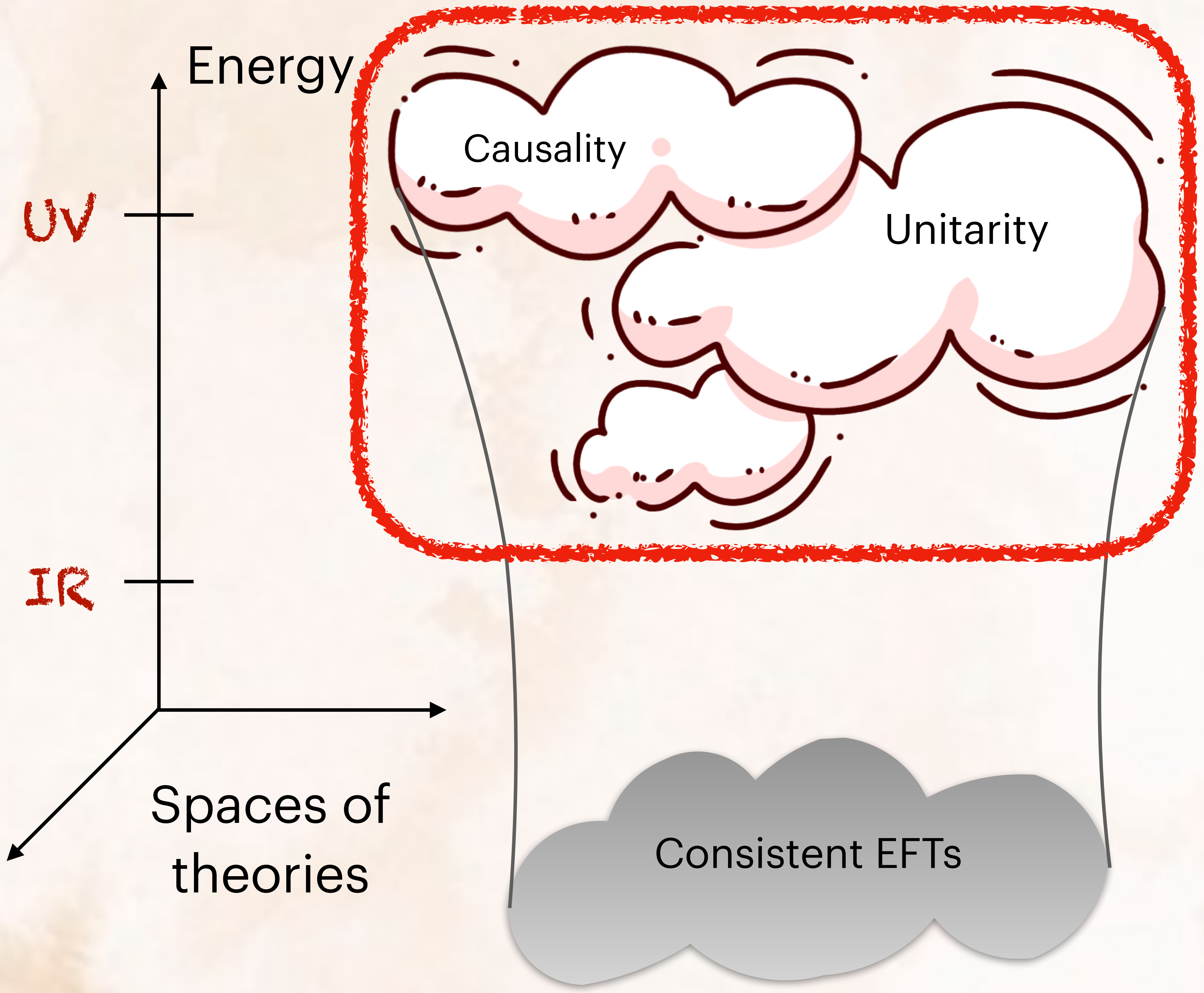

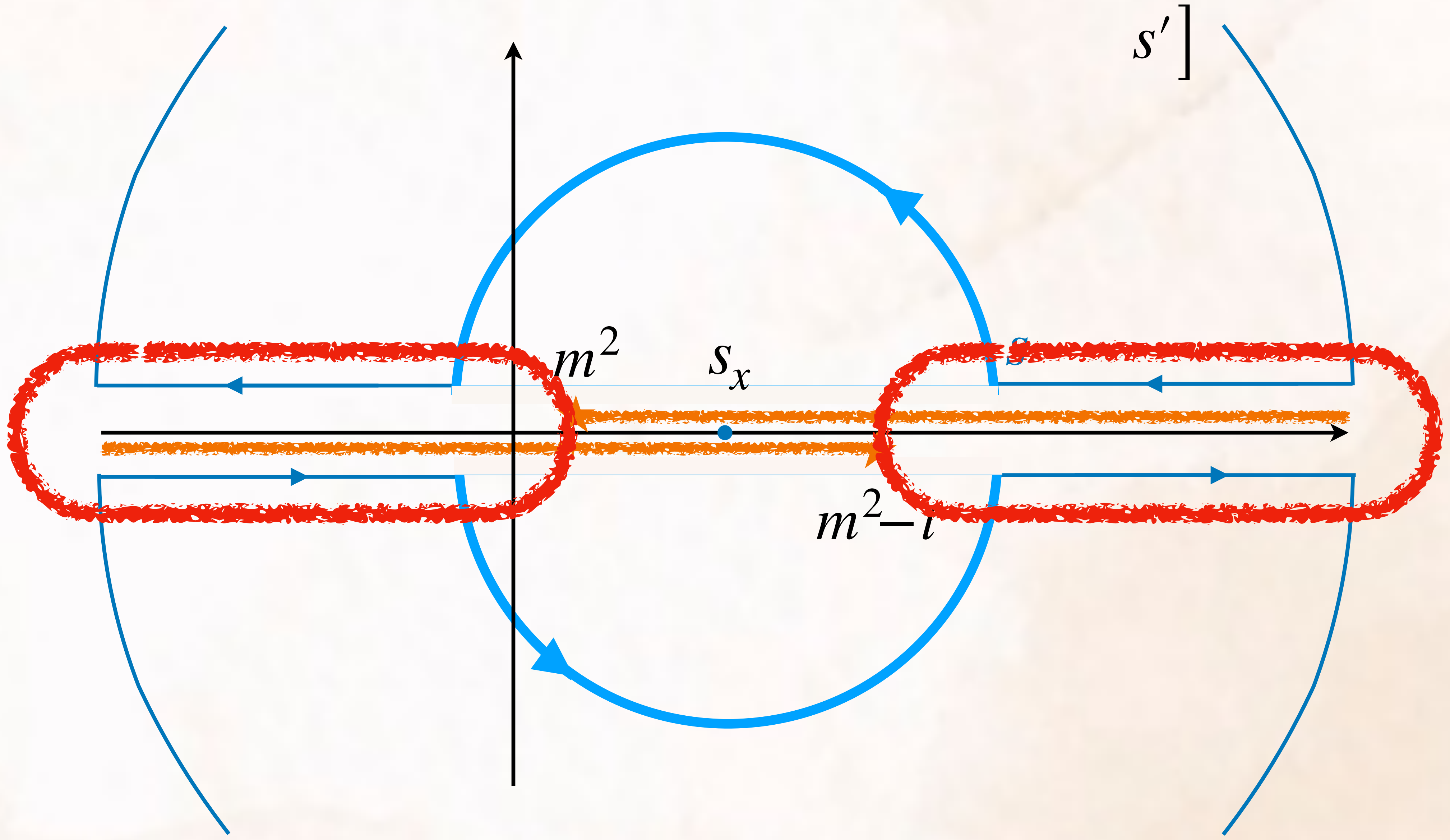

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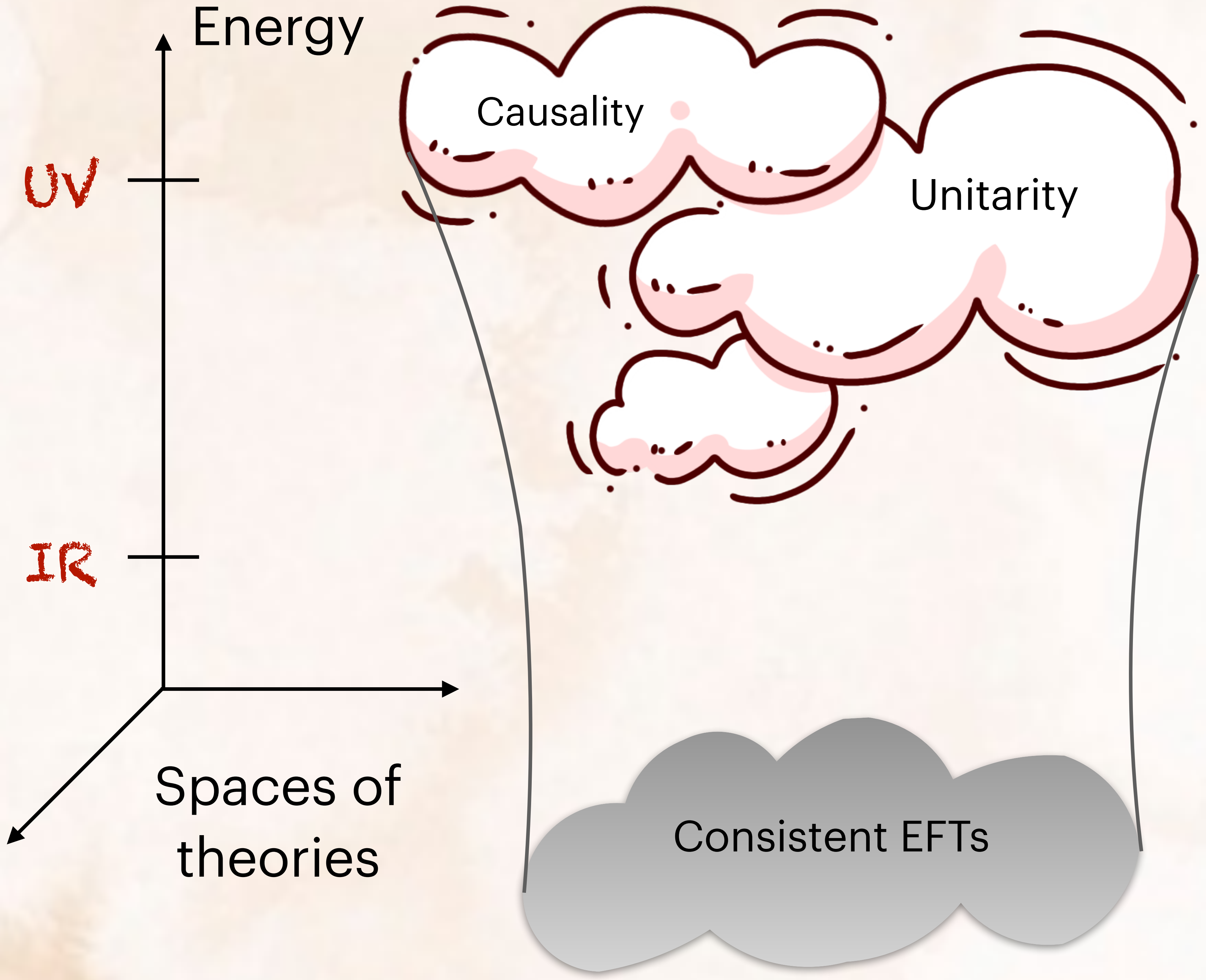
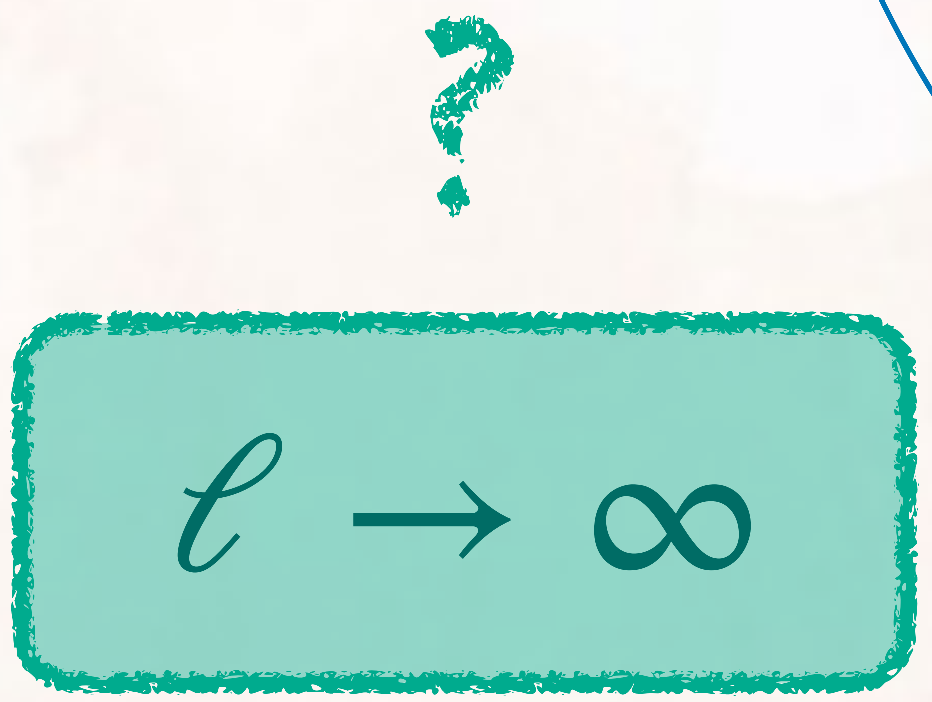
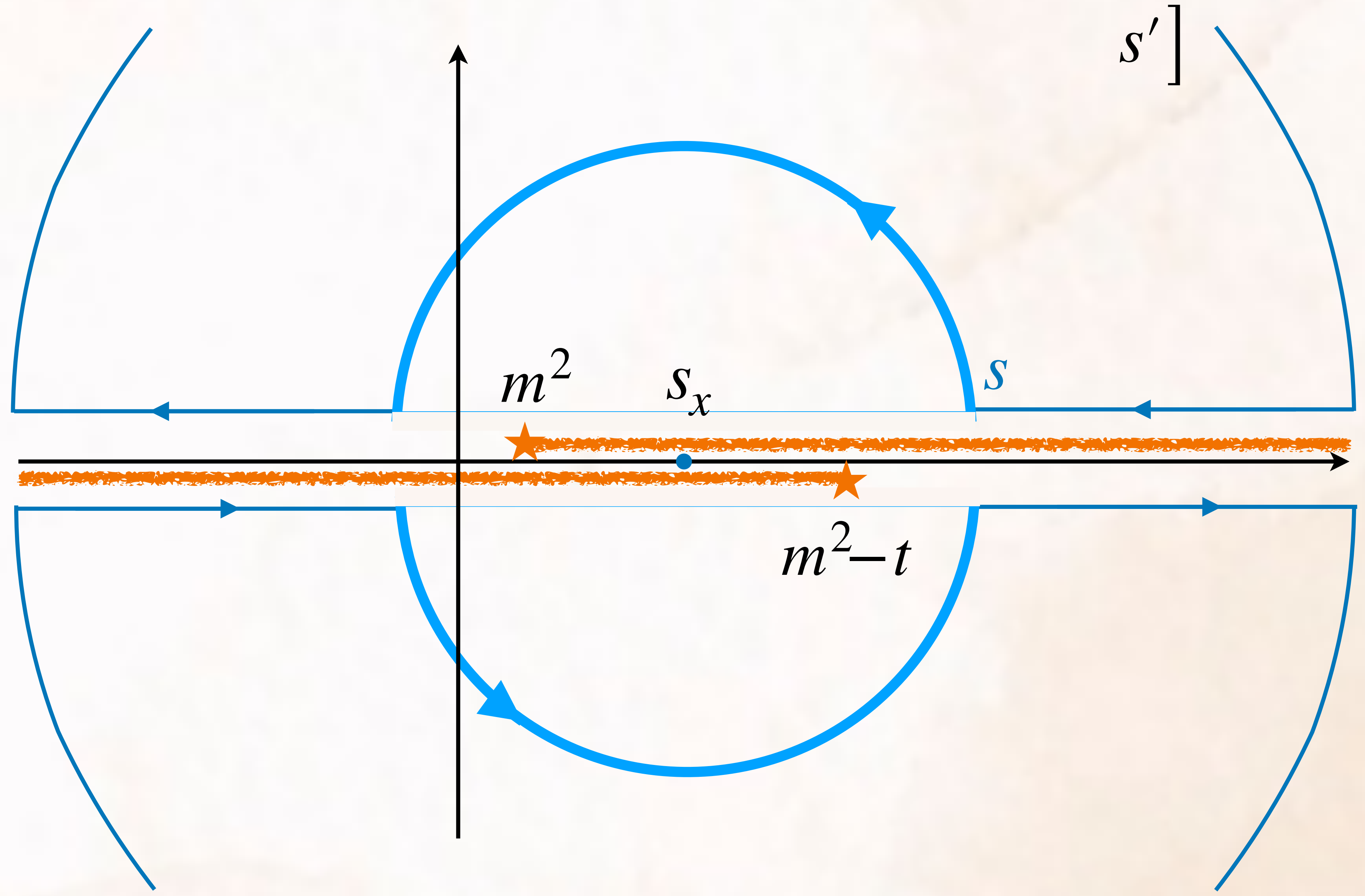
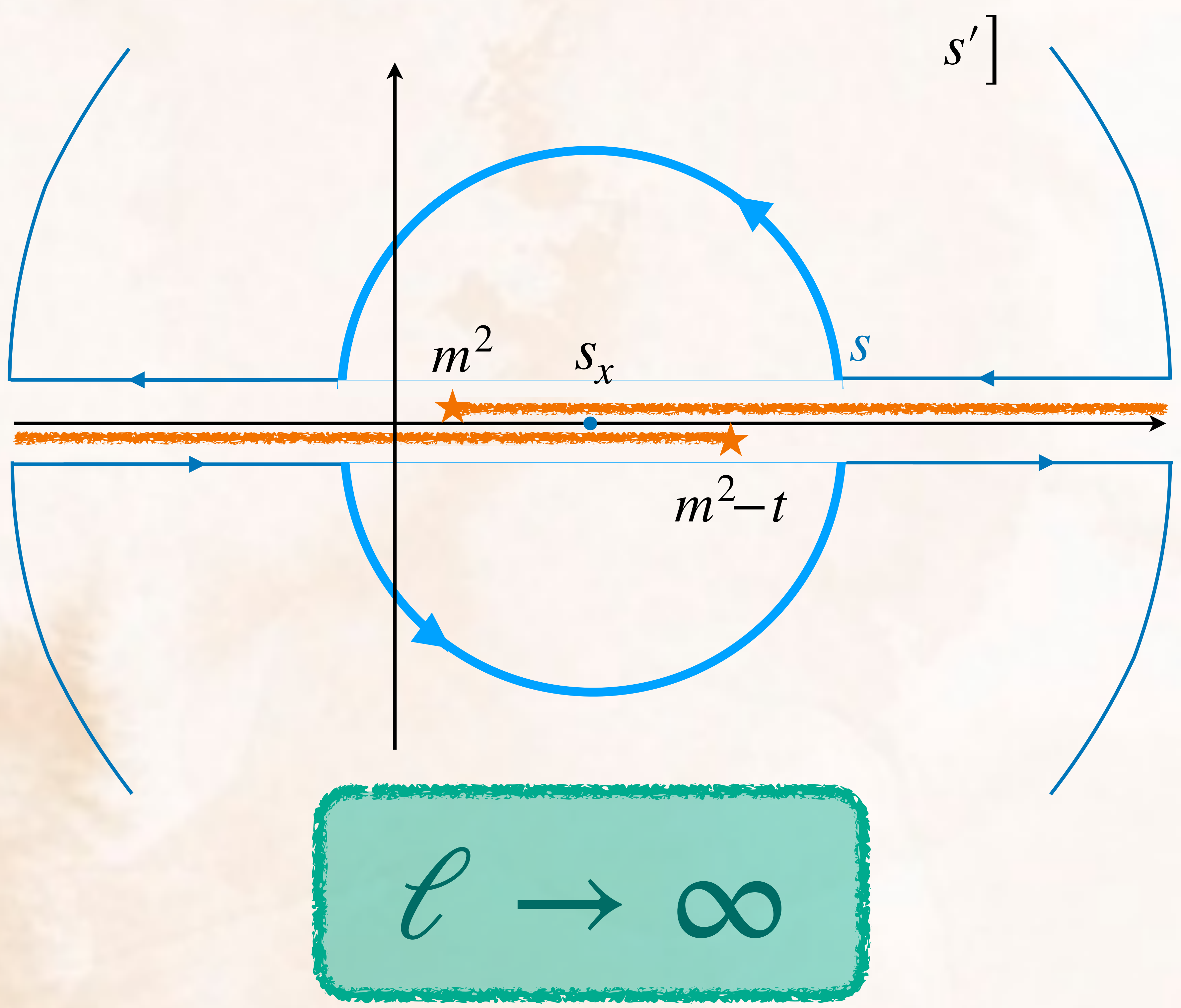






Graviton-Scalar scattering

- *What can we learn from Eikonal arcs?*



$$a^{(n)}_{\lambda_1\lambda_3}(\omega, \mathbf{b}) \equiv \oint \frac{d\omega}{2\pi i} \frac{\mathcal{M}_{\lambda_1\lambda_3}(\omega, \mathbf{b})}{\omega^{2n+2}} \succ 0$$

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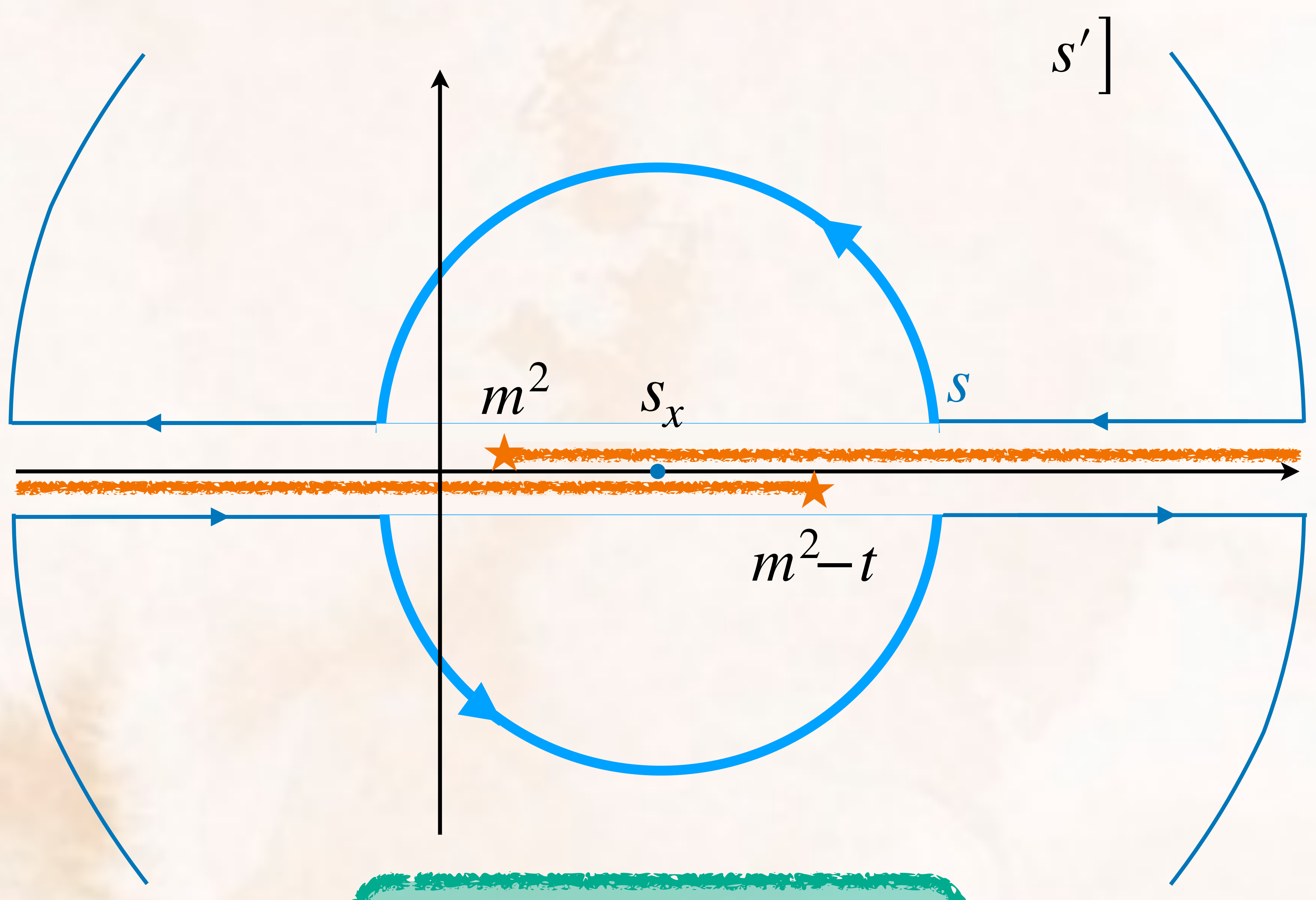

$$\ell \to \infty$$

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- $n = 0$

$$\mathcal{T}_{\lambda_1\lambda_3}(\omega,\mathbf{b}) \equiv 2\frac{\partial}{\partial\omega}\delta_{\lambda_1\lambda_3}(s,\mathbf{b})\Big|_{\omega=0} \succ 0$$

S-Matrix principles → Asymptotic Causality

Analyticity + Unitarity

Graviton-Scalar scattering

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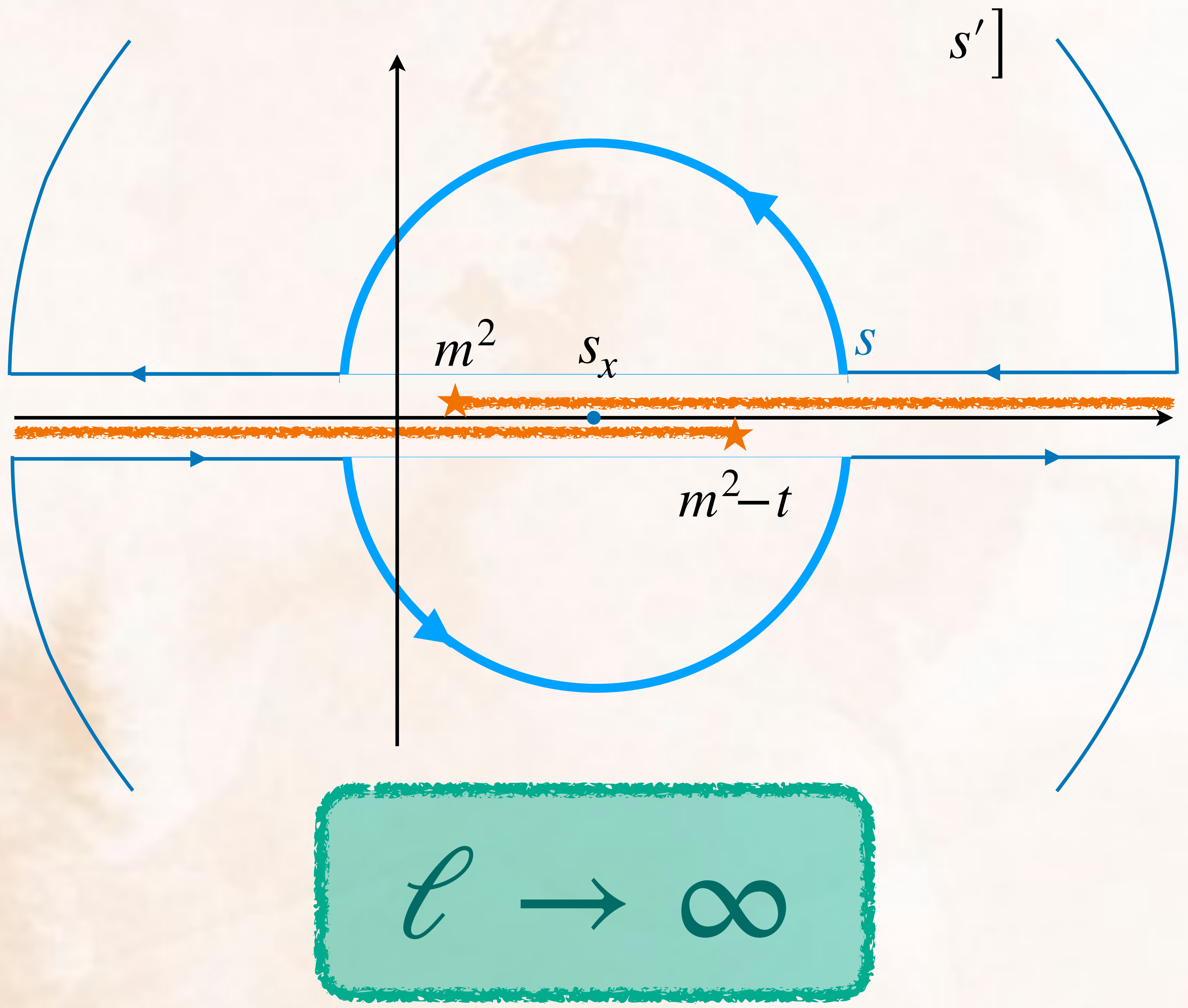

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- $n \geq 1$

Conclusions

- The Eikonal regime is a powerful setup to perturbatively extract observables in different theories.

- The exponentiation takes place at all orders and for any spin.

- In the Eikonal limit, we recover an infinite set of positivity constraints of which $\mathscr{T} > 0$ is the simplest.

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Backup

Large ℓ limit of Wigner d-matrix

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$$[J^3, J_\pm] = \pm J_\mp$$

$$[J_+, J_-] = 2J^3$$

$$J^3 | \lambda >_\ell = \lambda | \lambda >_\ell$$

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$$\sim \sqrt{\ell(\ell+1)}\, | \lambda \pm 1 >_\ell + \mathcal{O}(\lambda/\ell)$$

$\ell/\lambda \gg 1$ →

$ISO(2)$

$j_\pm \equiv J_\pm / \sqrt{\ell(\ell+1)}$ $\quad j_3 \equiv J_3$

$$[j^3, j_\pm] = \pm j_\mp$$

$$[j_+, j_-] = 0$$

$$j^3 | \lambda >_\ell = \lambda | \lambda >_\ell$$

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The continuous spin basis

$$d^{\ell}_{\lambda,\lambda'}(\cos\theta) = {}_{\ell}< \lambda' | e^{-iJ_2\theta} | \lambda >_{\ell} = {}_{\ell}< \lambda' | e^{-\frac{\sqrt{\ell(\ell+1)}}{2}(j_+ - j_-)\theta} | \lambda >_{\ell}$$

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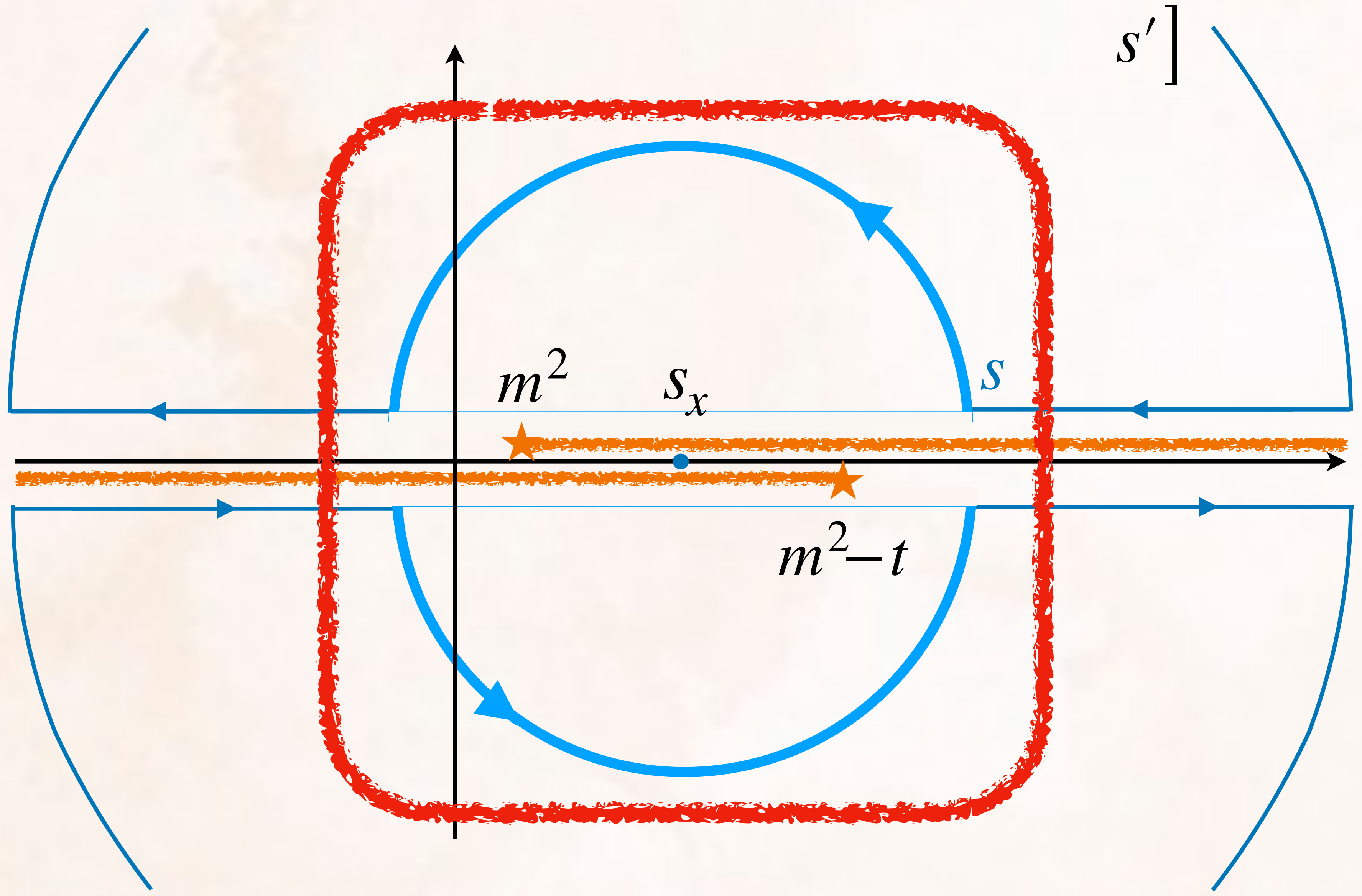
Dispersive arcs

Graviton-Scalar scattering

Spinning dispersive arcs

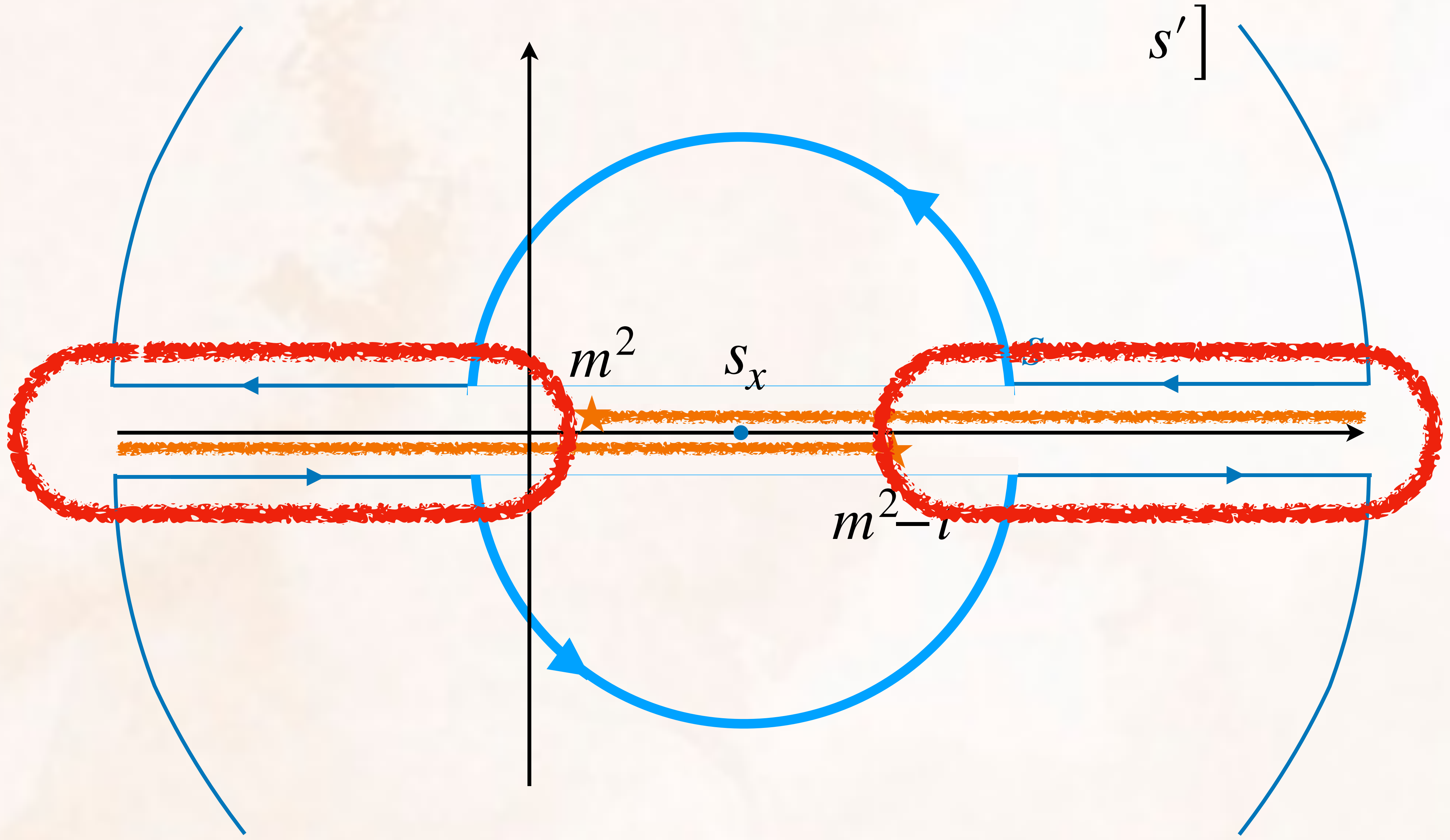


$$a^{(n)}_{\lambda_1\lambda_3}(s,t) \equiv \oint \frac{ds'}{2\pi i} \frac{\mathcal{M}_{\lambda_1\lambda_3}(s',t)}{(s'-s_\times)^{2n+3}}$$

$$s_\times = m^2 - \frac{t}{2}$$

Graviton-Scalar scattering

Spinning dispersive arcs



$$a^{(n)}_{\lambda_1\lambda_3}(s,t) = \frac{1}{2\pi i}\int_s^\infty ds' \frac{\mathrm{Disc}\mathcal{M}_{\lambda_1\lambda_3}(s',t) + \mathrm{Disc}\mathcal{M}_{\lambda_3\lambda_1}(s',t)}{(s'-s_\times)^{2n+3}}$$

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Graviton-Scalar scattering

Spinning dispersive arcs

$\langle 3^{\lambda_3} 4 | \mathcal{M} - \mathcal{M}^\dagger | 1^{\lambda_1} 2 \rangle =$
$i \langle 3^{\lambda_3} 4 | \mathcal{M}^\dagger \mathcal{M} | 1^{\lambda_1} 2 \rangle$

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$$= \frac{2\sqrt{s}}{|\boldsymbol{p}|}\sum_{\ell\geq 2}(2\ell+1)\int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^\ell_{\lambda_1,-\lambda_3}(\theta(t,s'))\mathcal{I}^\ell_{\lambda_1\lambda_3}(s')$$

$$\mathcal{I}^\ell_{\lambda_1\lambda_3}(s') = \langle \ell\lambda_3 | \mathcal{M}^\dagger\mathcal{M} | \ell\lambda_1\rangle + \langle \ell\lambda_1 | \mathcal{M}^\dagger\mathcal{M} | \ell\lambda_3\rangle \succ 0$$

Graviton-Scalar scattering

The large ℓ limit of dispersive arcs

$$a^{(n)}_{\lambda_1\lambda_3}(s,t) = \frac{2\sqrt{s}}{|\boldsymbol{p}|}\sum_{\ell'\geq 2}(2\ell'+1)\int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^{\ell'}_{\lambda_1,-\lambda_3}(\theta(t,s'))\mathcal{I}^{\ell'}_{\lambda_1\lambda_3}(s')$$

Graviton-Scalar scattering

The large ℓ limit of dispersive arcs

$$\int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\; a^{(n)}_{\lambda_1\lambda_3}(s,t) = \int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\; \frac{2\sqrt{s}}{|\boldsymbol{p}|}\sum_{\ell'\geq 2}(2\ell'+1)\int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^{\ell'}_{\lambda_1,-\lambda_3}(\theta(t,s'))\mathcal{I}^{\ell'}_{\lambda_1\lambda_3}(s')$$

Graviton-Scalar scattering

The large ℓ limit of dispersive arcs

$\ell \to \infty$

$$\int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\, a^{(n)}_{\lambda_1\lambda_3}(s,t) = \int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta) \; \frac{2\sqrt{s}}{|\boldsymbol{p}|} \sum_{\ell'\geq 2} (2\ell'+1) \int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^{\ell'}_{\lambda_1,-\lambda_3}(\theta(t,s')) \mathcal{F}^{\ell'}_{\lambda_1\lambda_3}(s')$$

Graviton-Scalar scattering

The large ℓ limit of dispersive arcs

$\ell \to \infty$

$$\int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\; a^{(n)}_{\lambda_1\lambda_3}(s,t) = \int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\; \frac{2\sqrt{s}}{|\boldsymbol{p}|}\sum_{\ell'\geq 2}(2\ell'+1)\int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^{\ell'}_{\lambda_1,-\lambda_3}(\theta(t,s'))\mathcal{F}^{\ell'}_{\lambda_1\lambda_3}(s')$$

$$a^{(n)}_{\lambda_1\lambda_3}(s,\mathbf{b}) \equiv \oint \frac{ds'}{2\pi i}\frac{\mathcal{M}_{\lambda_1\lambda_3}(\boldsymbol{p},\mathbf{b})}{(s-m^2)^{2n+2}}$$

Graviton-Scalar scattering

The large ℓ limit of dispersive arcs

$$\ell \to \infty$$

$$\int_0^\infty dq^2 d^\ell_{\lambda_1,-\lambda_3}(\theta)\; a^{(n)}_{\lambda_1\lambda_3}(s,t) = \int_0^\infty dq^2 d^\ell_{\lambda_1,-\lambda_3}(\theta)\; \frac{2\sqrt{s}}{|\boldsymbol{p}|}\sum_{\ell'\geq 2}(2\ell'+1)\int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^{\ell'}_{\lambda_1,-\lambda_3}(\theta(t,s'))\mathcal{I}^{\ell'}_{\lambda_1\lambda_3}(s')$$

$$a^{(n)}_{\lambda_1\lambda_3}(s,\mathbf{b}) \equiv \oint \frac{ds'}{2\pi i}\frac{\mathcal{M}_{\lambda_1\lambda_3}(\boldsymbol{p},\mathbf{b})}{(s-m^2)^{2n+2}}$$

$$\propto \frac{1}{\pi}\int_s^\infty \frac{ds'}{(s'-m^2)^{2n+2}}\mathcal{I}_{\lambda_1\lambda_3}(b',s') > 0$$

Graviton-Scalar scattering

The large ℓ limit of dispersive arcs

$\ell \to \infty$

$$\int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\, a^{(n)}_{\lambda_1\lambda_3}(s,t) = \int_0^\infty dq^2 d^{\ell}_{\lambda_1,-\lambda_3}(\theta)\, \frac{2\sqrt{s}}{|\boldsymbol{p}|}\sum_{\ell'\geq 2}(2\ell'+1)\int_s^\infty \frac{ds'}{(s'-s_\times(t))^{2n+3}} d^{\ell'}_{\lambda_1,-\lambda_3}(\theta(t,s'))\mathcal{I}^{\ell'}_{\lambda_1\lambda_3}(s')$$

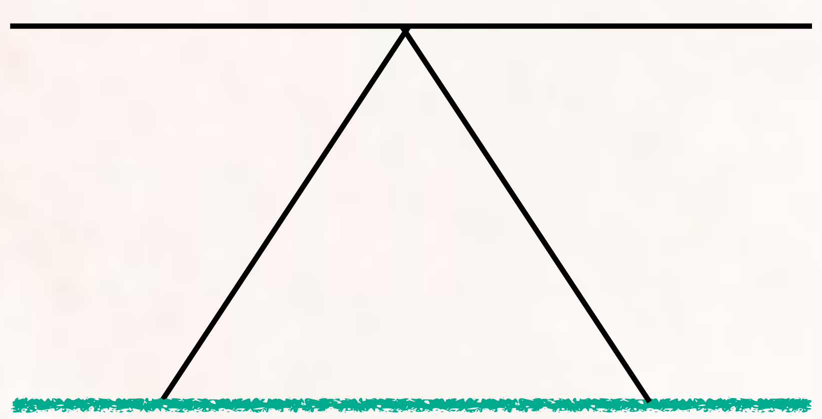
$$a^{(n)}_{\lambda_1\lambda_3}(s,\mathbf{b}) \equiv \oint \frac{ds'}{2\pi i}\frac{\mathcal{M}_{\lambda_1\lambda_3}(\boldsymbol{p},\mathbf{b})}{(s-m^2)^{2n+2}} \succ 0$$

$$\propto \frac{1}{\pi}\int_s^\infty \frac{ds'}{(s'-m^2)^{2n+2}}\mathcal{I}_{\lambda_1\lambda_3}(b',s') \succ 0$$

Bounds on R^4

Bounding R^4 with Eikonal arcs

$$\mathcal{S} \supset \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-R+\beta_1(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})^2+\beta_3(R_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta}_{\gamma\delta}R^{\gamma\delta\mu\nu})^2\right]$$



$$\Delta\delta_{\lambda_1\lambda_3}(s,\boldsymbol{b}) = \frac{315\pi}{16}\frac{G^2m^2\omega^3}{b}\begin{pmatrix}\frac{\tilde\beta}{b^4} & \frac{\beta}{16b_+^4}\\ \frac{\beta}{16b_-^4} & \frac{\tilde\beta}{b^4}\end{pmatrix}$$

- $n = 1$

$$a^{(1)}_{\lambda_1\lambda_3}(\omega,\boldsymbol{b}) = \frac{\partial^3}{\partial\omega^3}\left(e^{2i\delta_{\lambda_1\lambda_3}(\omega,\boldsymbol{b})} - \mathbb{I}\right)$$

$$b_\pm = (b_1 \pm ib_2)/2$$

$$\tilde\beta = 4(\beta_1+\beta_3)$$

$$\beta = 4(\beta_1-\beta_3)$$

$\beta > 0$ $\quad$ $\tilde\beta > 0$ $\quad$ Up to $\mathcal{O}(\Lambda/M_{\rm Pl} \times M/M_{\rm Pl})$ corrections

Accettulli Huber, Brandhuber, De Angelis, Travaglini

Bounds in the "weak rigid" limit