Anomaly and double copy in quantum self-dual Yang-Mills and gravity

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Self-dual Yang-Mills (SDYM)

- The SDYM action is

$$S_{\mathsf{SDYM}}(B, A) = \mathrm{tr} \int B \wedge F_{\mathsf{ASD}} \tag{1}$$

- Go to light-cone coordinates $x^\mu = \{u, v, w, \bar{w}\}$ with Minkowski metric

$$ds^2 = 2(-dudv + dwd\bar{w}) \tag{2}$$

- Adopt the light-cone gauge and integrate out two components of B to enforce

$$A_u = 0, \quad A_v = \frac{1}{2}\partial_w \Psi, \quad A_w = 0, \quad A_{\bar{w}} = \frac{1}{2}\partial_u \Psi \tag{3}$$

- The result is [Chalmers, Siegel 96]

$$S_{\mathsf{SDYM}}(\Psi, \bar{\Psi}) = \mathrm{tr} \int d^4x \, \bar{\Psi} \left(\Box \Psi + i[\partial_u \Psi, \partial_w \Psi] \right) \tag{4}$$

Self-dual gravity (SDG)

- Following a similar procedure we find the metric

$$ds^2 = 2(-dudv + dwd\bar{w}) + \partial_w^2\phi dv^2 + \partial_u^2\phi d\bar{w}^2 + 2\partial_u\partial_w\phi dudw \tag{5}$$

and an action

$$S_{\mathsf{SDG}}(\phi,\bar{\phi}) = \int d^4x\,\bar{\phi}\,(\Box\phi + \{\partial_u\phi,\partial_w\phi\}) \tag{6}$$

with the Poisson bracket

$$\{f,g\} = \partial_u f\partial_w g - \partial_w f\partial_u g \tag{7}$$

Feynman rules

	SDYM	SDG
Propagator $(+-)$	$\frac{1}{k^2}\delta^{a_1 a_2}$	$\frac{1}{k^2}$
Vertex $(++-)$	$X(k_1, k_2) f^{a_1 a_2 a_3}$	$X(k_1, k_2)^2$
External $p_i^{\pm}$	$\langle \eta i \rangle^{\mp 2}$	$\langle \eta i \rangle^{\mp 4}$

- $X(k_i, k_j) = k_{iw} k_{ju} - k_{iu} k_{jw} = -X(k_j, k_i)$
- η is an on-shell reference vector

Scattering in SDYM and SDG

- The only possible diagrams are

$$\text{Tree-level } (- + + \cdots +)$$

$$\text{One-loop } (+ + + \cdots +)$$

- The tree-level amplitudes vanish to all n

$$A_n^{\text{tree}}(1^- 2^+ \cdots n^+) = 0 \tag{8}$$

- At 1-loop they are simple rational functions of external momenta, e.g. in SDYM

$$A_4^{\text{1-loop}}(1^+ 2^+ 3^+ 4^+) \sim \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \tag{9}$$

The double copy in the self-dual sector

- The double copy is well understood in the self-dual sector

- Colour-kinematics duality is manifest in the vertices

$$V_{\mathsf{SDYM}} = X(k_1, k_2) f^{a_1 a_2 a_3}, \quad V_{\mathsf{SDG}} = X(k_1, k_2)^2 \tag{10}$$

- $f^{a_1 a_2 a_3}$ $\rightarrow$ structure constants of the colour Lie algebra SU(N)

- $X(k_1, k_2)$ $\rightarrow$ structure constants of the kinematic Lie algebra

- In the SD sector the kinematic algebra is that of area-preserving diffeomorphisms in the u-w plane [Monteiro, O'Connell 11]

Classical integrability of SDYM

- Classical equations of motion

$$0 = \Box\,\Psi + i[\partial_u \Psi, \partial_w \Psi] \tag{11}$$
$$0 = \Box\,\bar{\Psi} + i[\partial_u \bar{\Psi}, \partial_w \Psi] + i[\partial_u \Psi, \partial_w \bar{\Psi}] \tag{12}$$

- The second coincides with a linearised deformation of the first $\Psi \to \Psi + \epsilon\bar{\Psi}$

- It can be expressed as the conservation of a current, $\partial_\mu J^\mu = 0$, where

$$J = \left(\partial_{\bar{w}} \bar{\Psi} - \tfrac{i}{2}[\bar{\Psi}, \partial_u \Psi]\right) \partial_w - \left(\partial_v \bar{\Psi} - \tfrac{i}{2}[\bar{\Psi}, \partial_w \Psi]\right) \partial_u \tag{13}$$

Classical integrability of SDYM

- We can build an infinite tower of conserved currents

$$\partial_\mu J_r^\mu = 0, \quad r = 0, 1, 2, ... \tag{14}$$

- Correspondingly, an infinite tower of solutions $\{\Lambda_r\}$ from $\Psi \to \Psi + \epsilon\Lambda_r$

- Can set up recursion relations to build this tower

- [Bardeen 96, Cangemi 97] showed that

$$\text{Classical integrability} \implies A_n^{\text{tree}}(1^-2^+ \cdots n^+) = 0 \tag{15}$$

The quantum-corrected formalism

- To include quantum effects let us write

$$S_{\text{q.c.SDYM}}(\Psi, \bar{\Psi}) = \operatorname{tr} \int d^4x \, \Big(\bar{\Psi} \big(\Box \Psi + i[\partial_u \Psi, \partial_w \Psi] \big) + V_{\text{1-loop}}[\Psi] \Big) \tag{16}$$

- $V_{\text{1-loop}}[\Psi]$ contains an infinite number of one-loop effective vertices

$$V_{\text{1-loop}}[\Psi] = \sum_{m=2}^{\infty} V_{\text{1-loop}}^{(m)}[\Psi] \,, \quad V_{\text{1-loop}}^{(m)}[\Psi] \sim \Psi^m \tag{17}$$

- With $S_{\text{q.c.SDYM}}$ we only need to work at tree-level

Anomalous classical symmetries

- Quantum-corrected equations of motion

$$0 = \Box\,\Psi + i[\partial_u\Psi, \partial_w\Psi]\,, \tag{18}$$

$$0 = \Box\,\bar{\Psi} + i[\partial_u\bar{\Psi}, \partial_w\Psi] + i[\partial_u\Psi, \partial_w\bar{\Psi}] + \frac{\delta V_{\text{1-loop}}[\Psi]}{\delta\Psi} \tag{19}$$

- The second no-longer coincides with a linearised deformation of the first

- Can write a quantum-corrected current such that

$$\partial^\mu J^{\text{q.c.}}_\mu = \text{EoM 2} \tag{20}$$

- However the classical symmetries $\Psi \to \Psi + \epsilon\Lambda_r$ are now anomalous

$$\partial^\mu J^{\text{q.c.}}_{r\ \mu} = \frac{1}{2}\,\frac{\delta V_{\text{1-loop}}[\Psi]}{\delta\Psi}\,, \quad \forall r \tag{21}$$

A route to $V_{\text{1-loop}}$

- The one-loop vertices are off-shell and thus non-unique

$$\implies \text{Many possible approaches to construction}$$

- We will take influence from recent work in twistor space
  [Costello 21; Costello, Paquette 22; Bittleston, Sharma, Skinner 22]

- Introduce an axion-like scalar field ρ with
  - Quartic propagator $\frac{1}{k^4}$
  - Coupling to the gauge field/graviton via $\rho F \wedge F$, $\rho R \wedge R$

- In SDYM this only works for restricted gauge groups

A route to $V_{\text{1-loop}}$

- Our approach:
  1. Integrate out the axion to obtain non-local effective vertices
  2. Flip the sign of the vertices

- Along the way we will
  - Extend the effective vertices to SU(N)
  - Find a novel loop-level double copy *after* integration

- Let's see how this works in practice...

The ρ-SDYM action

- Consider the action [Costello 21; Costello, Paquette 22;]

$$S_{\rho-\mathsf{SDYM}}(B, A, \rho) = \int \mathrm{tr}\left(B \wedge F_{\mathsf{ASD}} + d^4x\, \frac{1}{2}(\Box\rho)^2 + \tilde{a}\,\rho\, F \wedge F\right) \tag{22}$$

- Cancellation occurs for SU(2), SU(3), SO(8) or an exceptional group

A 4-point one-loop vertex

- Restrict the gauge group and integrate out the axion

$$S'_{\mathsf{q.c.SDYM}}(B,A) = \int \mathrm{tr}(B \wedge F_{\mathsf{ASD}}) + d^4x\, \frac{\tilde{a}^2}{2}\Big(\frac{1}{\Box}\,\mathrm{tr}(\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda})\Big)^2 \tag{23}$$

$$\Big\downarrow A_u = 0$$

$$S'_{\mathsf{q.c.SDYM}}(\Psi,\bar{\Psi}) = \int d^4x\ \mathrm{tr}\left(\bar{\Psi}\big(\Box\Psi + i[\partial_u\Psi,\partial_w\Psi]\big)\right) + a\left(\frac{1}{\Box}\,\mathrm{tr}(\Psi\overleftrightarrow{P}^2\Psi)\right)^2 \tag{24}$$

where

$$\overleftrightarrow{P} = \overleftarrow{\partial}_u\overrightarrow{\partial}_w - \overleftarrow{\partial}_w\overrightarrow{\partial}_u \tag{25}$$

- In momentum space

$$\Psi(x)\overleftrightarrow{P}\Psi(x) \quad \rightarrow \quad X(k_1,k_2)\Psi(k_1)\Psi(k_2) \tag{26}$$

A 4-point one-loop vertex

- We now have a 4-point one-loop vertex

$$\frac{X(1,2)^2 X(3,4)^2}{s_{12}^2}\delta^{a_1 a_2}\delta^{a_3 a_4}, \qquad s_{12} = (k_1 + k_2)^2 \tag{27}$$

- The colour-ordered 4-point one-loop all-plus amplitude is then simply

$$\left(\prod_{i=1}^{4}\langle \eta i\rangle^{-2}\right)\frac{X(1,2)^2 X(3,4)^2}{s_{12}^2} = \frac{[12]^2[34]^2}{s_{12}^2} = \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \tag{28}$$

- Continuing beyond four-points we encounter an issue for SU(N)

$\rightarrow$ This is where the restriction of the gauge group acts

Quantum Feynman rules for SU(N) SDYM

- [Bern, Chalmers, Dixon, Kosower 94] observed that the colour-ordered amplitude splits as

$$A^{(1)}_{\mathsf{SDYM}}(123\cdots n) = M_n \, \frac{E(123\cdots n) + O(123\cdots n)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{29}$$

  where

$$E(123\cdots n) = \sum_{1\le i_1<i_2<i_3<i_4\le n} \langle i_1 i_2\rangle[i_2 i_3]\langle i_3 i_4\rangle[i_4 i_1] + [i_1 i_2]\langle i_2 i_3\rangle[i_3 i_4]\langle i_4 i_1\rangle \tag{30}$$

$$O(123\cdots n) = -\sum_{1\le i_1<i_2<i_3<i_4\le n} \varepsilon(i_1, i_2, i_3, i_4) \tag{31}$$

- Note that $O(1234) = 0$

Quantum Feynman rules for SU(N) SDYM

- We work with the colour-dressed amplitudes

$$\mathcal{A}^{(1)}_{n\,\mathsf{SDYM}} = \sum_{\sigma\in S_{n-1}} c^{a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}\cdots a_{\sigma(n)}}\, A^{(1)}_{\mathsf{SDYM}}\big(\sigma(1)\sigma(2)\sigma(3)\cdots\sigma(n)\big) \tag{32}$$

  where

$$c^{a_1a_2a_3\cdots a_n} = f^{b_1a_1b_2} f^{b_2a_2b_3} f^{b_3a_3b_4} \cdots f^{b_na_nb_1} \tag{33}$$

- We will also need

$$\begin{aligned} c^{(a_1a_2)a_3\cdots a_n} &= c^{a_1a_2a_3\cdots a_n} + c^{a_2a_1a_3\cdots a_n} \\ c^{[a_1a_2]a_3\cdots a_n} &= c^{a_1a_2a_3\cdots a_n} - c^{a_2a_1a_3\cdots a_n} \\ c^{[(a_1a_2)a_3]a_4\cdots a_n} &= c^{(a_1a_2)a_3a_4\cdots a_n} - c^{a_3(a_1a_2)a_4\cdots a_n} \end{aligned} \tag{34}$$

Quantum Feynman rules for SU(N) SDYM

- The following set of m-point one-loop vertices reproduce the E-part of the amplitude on-shell

$$\sum_{i=2}^{m-2} X(k_1, k_2)^2 \left(\prod_{j=2}^{i-1} \frac{X(k_{1,\cdots,j}\,, k_{j+1})}{s_{1\cdots j}} \right) \frac{1}{s_{1\cdots i}^2} \left(\prod_{l=i+1}^{m-2} \frac{X(k_{1,\cdots,l-1}\,, k_l)}{s_{1\cdots l}} \right) X(k_{m-1}, k_m)^2$$
$$\cdot\, c^{[[\cdots[(a_1 a_2)a_3]\cdots]a_i][a_{i+1}[\cdots[a_{m-2}(a_{m-1}a_m)]\cdots]]} \quad (35)$$

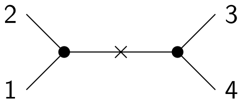
- Checked numerically up to 7-points

- Let's see some examples...

Quantum Feynman rules for SU(N) SDYM

- The four-point vertex is

$$\frac{X(1,2)^2 X(3,4)^2}{s_{12}^2} c^{(a_1 a_2)(a_3 a_4)} = \quad (36)$$

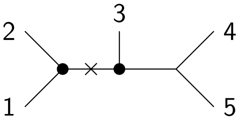


- A contribution of this vertex at 5-points is

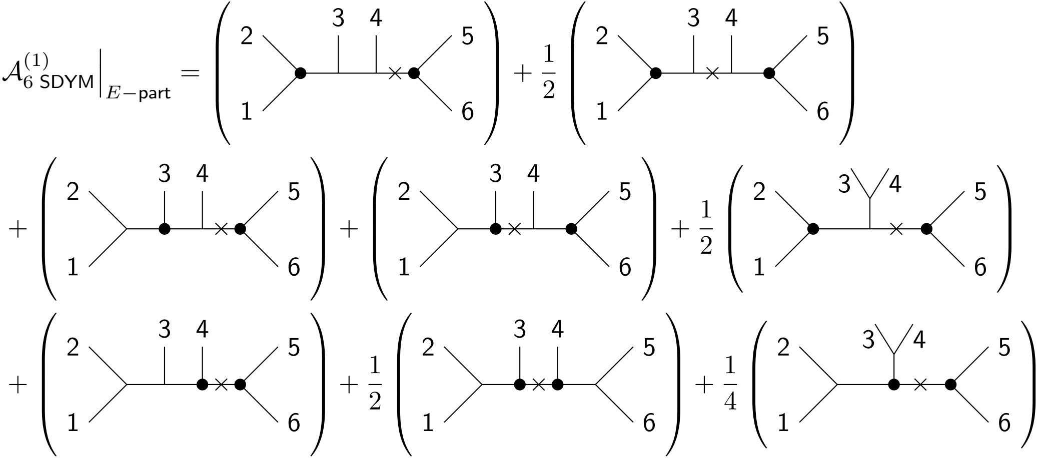
$$\frac{X(1,2)^2 X(3,4+5)^2}{s_{12}^2} c^{(a_1 a_2)(a_3 b)} \cdot \frac{X(4,5)}{s_{45}} f^{b a_4 a_5}$$

$$= \frac{X(1,2)^2 X(3,4+5)^2 X(4,5)}{s_{12}^2 s_{45}} c^{(a_1 a_2)(a_3 [a_4 a_5])}$$

$$= \quad (37)$$



A 6-point one-loop example



$+$ permutations$\{123456\}$

Quantum Feynman rules for SU(N) SDYM

- These vertices can be encoded in the action via

$$\left((\Psi \overleftrightarrow{P}^2 \Psi) \frac{1}{\mathbb{1}\overleftarrow{\Box} - \overleftarrow{\mathrm{ad}}_{\Psi \overleftrightarrow{P}}} \right)^{bc} \left(\frac{1}{\mathbb{1}\overrightarrow{\Box} - \overrightarrow{\mathrm{ad}}_{\Psi \overleftrightarrow{P}}} (\Psi \overleftrightarrow{P}^2 \Psi) \right)^{cb} \tag{38}$$

where $\mathrm{ad}_X Y = [X, Y]$ and we have the geometric series

$$\begin{aligned} \left(\frac{1}{\mathbb{1}\overrightarrow{\Box} - \overrightarrow{\mathrm{ad}}_{\Psi \overleftrightarrow{P}}} (\Psi^{a_1} \overleftrightarrow{P}^2 \Psi^{a_2}) T^{a_1} T^{a_2} \right)^{bc} &= \frac{1}{\overrightarrow{\Box}} f^{b(a_1|e} f^{e|a_2)c} (\Psi^{a_1} \overleftrightarrow{P}^2 \Psi^{a_2}) \\ &+ (f^{ba_3d} f^{d(a_1|e} f^{e|a_2)c} - f^{b(a_1|e} f^{e|a_2)d} f^{da_3c}) \frac{1}{\overrightarrow{\Box}} (\Psi^{a_3} \overleftrightarrow{P} \frac{1}{\overrightarrow{\Box}} (\Psi^{a_1} \overleftrightarrow{P}^2 \Psi^{a_2})) \\ &+ \mathcal{O}\left(ffff \frac{1}{\Box} (\Psi P \frac{1}{\Box} (\Psi P \frac{1}{\Box} (\Psi P \Psi))) \right) \end{aligned} \tag{39}$$

The O-part

- The O-part of the amplitude can be written off-shell e.g. at 5-points

$$\frac{X(1,2)}{s_{12}}\frac{X(2,3)}{s_{23}}\frac{X(3,4)}{s_{34}}\frac{X(4,5)}{s_{45}}\frac{X(5,1)}{s_{51}}\,\varepsilon(1,2,3,4)\;c^{a_1a_2a_3a_4a_5} \tag{40}$$

- We were unable to find an all-order expression for this part of the vertices

- Fortunately, it is the E-part that connects to SDG...

Quantum Feynman rules for SDG

- Take a tree-like double copy approach

- The SDG one-loop vertices are then

$$\sum_{i=2}^{m-2} X(k_1,k_2)^4 \left(\prod_{j=2}^{i-1} \frac{X(k_{1,\cdots,j}\,,k_{j+1})^2}{s_{1\cdots j}} \right) \frac{1}{s_{1\cdots i}^2} \left(\prod_{l=i+1}^{m-2} \frac{X(k_{1,\cdots,l-1}\,,k_l)^2}{s_{1\cdots l}} \right) X(k_{m-1},k_m)^4 \quad (41)$$

- Checked numerically up to 6-points

- This is unexpected! At loop-level the double copy usually applies at the level of the integrand

Quantum Feynman rules for SDG

- The quantum-corrected action is

$$S_{\text{q.c.SDG}}(\phi,\bar\phi) = \int d^4x \,\Big(\bar\phi\big(\Box\phi + \{\partial_u\phi, \partial_w\phi\}\big) + V_{\text{1-loop}}[\phi] \Big) \tag{42}$$

- Our effective vertices follow straightforwardly from the inclusion of

$$V_{\text{1-loop}}[\phi] = \Big(\frac{1}{\Box_\phi} (\phi \overset{\leftrightarrow}{P}{}^4 \phi) \Big)^2 = \left(\frac{1}{\Box - (\phi \overset{\leftrightarrow}{P}{}^2 \,\cdot)} (\phi \overset{\leftrightarrow}{P}{}^4 \phi) \right)^2 \tag{43}$$

where $\Box_\phi$ is the wave operator on the background of the self-dual metric

- This action corresponds precisely to that presented in [Bittleston, Sharma, Skinner 22] after integrating out the axion and flipping the sign of the resulting interaction term

Summary and future directions

- We have studied the form of quantum-corrected actions for SDYM and SDG in light-cone gauge
- This led to quantum-corrected integrability currents that exhibit the anomaly suggested by W. Bardeen
- Our one-loop effective vertices feature an unexpected loop-level double copy after integration
- Future directions:
    - Is there an all-order expression for the O-part of the vertices?
    - Can the anomaly be connected to other known properties of self-dual amplitudes?
    - Does our construction generalise in any way to the full theories?

Thanks!