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Rational Terms at Two Loops

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ArXiv: [2001.11388] & [2007.03713]

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Motivation: towards two-loop numerical calculation

- Aim to $\mathcal{O}(1\%)$ precision for LHC processes $\Rightarrow$ **Automation of two-loop calculation**
- Higher-order calculations are usually performed in D dimension to regularise divergences in Feynman integrals, but D-dim vector cannot be implemented in a numerical program.
- Automated numerical tools construct the numerator of loop integrand in 4-dim, e.g. OpenLoops, Recola, MadLoop at one-loop level.
- **Rational terms** is the ingredient, which reconstructs the missing terms originated from $(D-4)$-dim part of loop numerator, that enables the automated methods.

  $\Rightarrow$ one loop: rational counterterms of type R_2 [Ossola, Papadopoulos, Pittau, Garzelli et al., 08', 09']

  $\Rightarrow$ in this talk: **two-loop UV rational counterterms**

Outline

I. Introduction to one-loop rational terms and tadpole decomposition

II. Structure of two-loop rational terms

III. Proof and recipe to compute two-loop rational terms

IV. Two-loop rational terms in Yang-Mills theories in a generic renormalisation scheme

Introduction to one-loop rational terms

Amplitude of amputated one-loop diagram γ in $D = 4 - 2\varepsilon$ dimension

$$\bar{\mathcal{A}}_{1,\gamma} \;=\; \mu^{2\varepsilon} \int \mathrm{d}\bar{q}_1 \, \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1) \cdots D_{N-1}(\bar{q}_1)}\,, \qquad \text{with} \quad D_k(\bar{q}_1) = (\bar{q}_1 + p_k)^2 - m_k^2$$

Rational term emerges by splitting numerator into 4-dim and ε-dim parts

$$\bar{\mathcal{N}}(\bar{q}_1) \;=\; \mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1)\,, \quad \text{with} \quad \begin{cases} \bar{q} & = \; q + \tilde{q} \\ \bar{\gamma}^{\bar{\mu}} & = \; \gamma^{\mu} + \tilde{\gamma}^{\tilde{\mu}} \\ \bar{g}^{\bar{\mu}\bar{\nu}} & = \; g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}} \end{cases}$$

leads to

$$\bar{\mathcal{A}}_{1,\gamma} \;=\; \underbrace{\mathcal{A}_{1,\gamma}}_{\substack{\text{compute} \\ \text{numerically}}} \;+\; \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\substack{\text{compute} \\ \text{analytically}}}$$

- $\delta\mathcal{R}_{1,\gamma}$ from interplay between ε-dim $\tilde{\mathcal{N}}$ and $\frac{1}{\varepsilon}$ UV pole. $\Rightarrow$ requires technique to extract UV pole

Tadpole decomposition [Chetyrkin, Misiak, Münz, 98', Zoller, 14']

The UV divergence can be captured by **massive tadpole decomposition** of denominators

$$\frac{1}{D_k(\bar{q}_1)} = \underbrace{\frac{1}{\bar{q}_1^2 - M^2}}_{\substack{\text{leading UV tadpole} \\ \mathcal{O}(1/\bar{q}_1^2)}} + \underbrace{\frac{\Delta_k(\bar{q}_1, p_k)}{\bar{q}_1^2 - M^2}\frac{1}{D_k(\bar{q}_1)}}_{\substack{\text{subleading UV term} \\ \mathcal{O}(1/\bar{q}_1^3)}}$$

with

$$\Delta_k(\bar{q}_1, p_k) = -p_k^2 - 2\,\bar{q}_1 \cdot p_k + m_k^2 - M^2$$

Apply recursively to obtain **tadpole expansion ($\mathbf{S}_X$)** up to order $(1/\bar{q}_1)^{X+2}$

$$\frac{1}{D_k(\bar{q}_1)} = \underbrace{\sum_{\sigma=0}^{X} \text{UV-div tadpoles}}_{\mathbf{S}_X(1/D_k)} + \text{UV-finite remainder}$$

Rational terms from UV singularities

- Use tadpole expansions $\mathbf{S}_X$ to fully **isolate UV divergent part** (of degree X)

$$\bar{\mathcal{A}}_{1,\gamma}\big|_{\text{UV div}} = \mathbf{S}_X \int \mathrm{d}\bar{q}_1 \frac{\mathcal{N}(q_1)+\tilde{\mathcal{N}}(\tilde{q}_1)}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)} = \underbrace{\int \mathrm{d}\bar{q}_1 \sum_{\sigma=0}^{X} \frac{(\mathcal{N}(q_1)+\tilde{\mathcal{N}}(\tilde{q}_1))\Delta^{(\sigma)}}{\left(\bar{q}_1^2-M^2\right)^{N+\sigma}}}_{\text{UV-divergent tadpole integrals}}$$

- Define $\bar{\mathbf{K}}$ operator extracts **full UV pole contribution**, with splitting of numerator into 4-dim and ε-dim

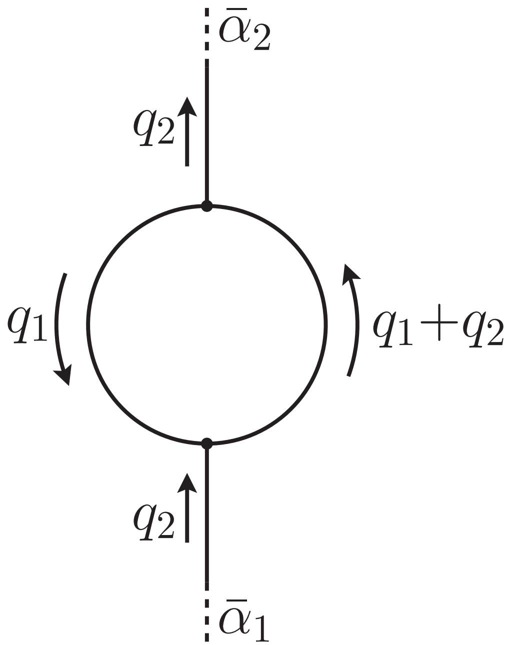
$$\bar{\mathbf{K}}\,\bar{\mathcal{A}}_{1,\gamma} = \underbrace{-\delta Z_{1,\gamma}}_{\frac{1}{\varepsilon}\,\mathrm{MS}\text{ pole}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\substack{\text{finite}\\\text{rational term}}}$$

- $\delta\mathcal{R}_{1,\gamma}$ and $\delta Z_{1,\gamma}$ from same UV singularity $\Rightarrow$ $\delta\mathcal{R}_{1,\gamma}$ local counterterm like $\delta Z_{1,\gamma}$

- $\delta\mathcal{R}_{1,\gamma}$ does not correspond to a finite renormalisation of fields and couplings in bare Lagrangian, e.g. there is a rational term of 4-photon vertice.

First object in two-loop diagram: subdivergence

- **Subdivergence** originates from the UV divergent **one-loop subdiagram**
  $\Rightarrow$ needs to be firstly **subtracted** in renormalisation procedure
- **Subdiagram** has D-dim external loop momenta



One-loop diagram with 4-dim q_2:

$$D_k(\bar{q}_1, q_2) = (\bar{q}_1 + q_2)^2 = \bar{q}_1^2 + \underbrace{2\,\bar{q}_1 \cdot q_2 + q_2^2}_{4\text{-dim}}$$

One-loop subdiagram with D-dim $\bar{q}_2 = q_2 + \tilde{q}_2$:

$$D_k(\bar{q}_1, \bar{q}_2) = D_k(\bar{q}_1, q_2) + \underbrace{(2\,\bar{q}_1 \cdot \tilde{q}_2 + \tilde{q}_2^2)}_{\varepsilon\text{-dim}}$$

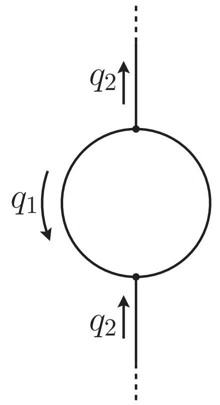
$\Rightarrow$ **extra pole term** $\propto \tilde{q}_2^2/\varepsilon$ can show up
$\Rightarrow$ **pole structure changes** in 4-dim numerator case

Subdiagram with D-dim external momentum $\bar{q}_2$ and 4-dim numerator

Tadpole expansion

$$\mathbf{S}_X \frac{1}{(\bar{q}_1 + q_2 + \tilde{q}_2)^2} = \frac{1}{\bar{q}_1^2 - M^2} + \frac{-(q_2 + \tilde{q}_2)^2 - 2\,\bar{q}_1 \cdot (q_2 + \tilde{q}_2) - M^2}{(\bar{q}_1^2 - M^2)^2} + \ldots$$

Contribution to UV pole



$$\bar{\mathbf{K}}\, \mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2) = \bar{\mathbf{K}} \int \mathrm{d}\bar{q}_1 \sum_{\sigma=0}^{X} \frac{\mathcal{N}(q_1, q_2)\, \Delta^{(\sigma)}(\bar{q}_1,\, q_2 + \tilde{q}_2)}{\left(\bar{q}_1^2 - M^2\right)^{N+\sigma}}$$

$$= \underbrace{-\,\delta Z^{\alpha}_{1,\gamma}(q_2)}_{\frac{1}{\varepsilon}\ \mathrm{MS}\ \text{pole}} \quad \underbrace{-\,\delta \tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)}_{\textbf{extra pole of } \mathcal{O}(1)}$$

- $\delta \tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)$ is **non-vanishing only in quadratic divergent subdiagrams**, and has the form

$$\delta \tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2) \propto \frac{\tilde{q}_2^2}{\varepsilon} = \mathcal{O}(1)$$

Renormalised one-loop subdiagrams

Subtract poles and rational terms in both D- and 4-dim, we can **identify amplitudes** with

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim full subtraction}} \;=\; \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2)}_{4\text{-dim full subtraction}} + \mathcal{O}(\varepsilon,\tilde{q})$$

Recall

$$\begin{aligned}
\bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) &= -\delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) + \delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon) \\
\bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2) &= -\delta Z^{\alpha}_{1,\gamma}(q_2) - \delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)
\end{aligned}$$

$\Rightarrow$ **Renormalised one-loop sub-amplitude**

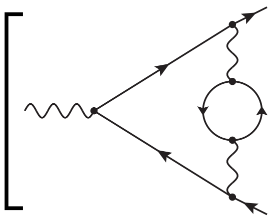
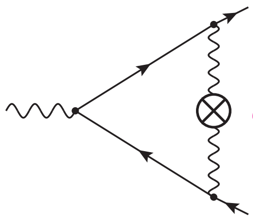
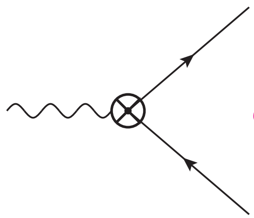
$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) + \delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim renormalisation}} \;=\; \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(q_2) + \delta Z^{\alpha}_{1,\gamma}(q_2) + \delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)}_{\substack{4\text{-dim renormalisation}\\ \text{compute numerically}}} + \underbrace{\delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2)}_{\text{rational parts}} + \mathcal{O}(\varepsilon,\tilde{q})$$

Renormalisation of irreducible two-loop diagrams

Renormalisation of D-dim amplitude of diagram Γ with $\mathbf{R}$ operation [Caswell and Kennedy, 82']

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i} \underbrace{\delta Z_{1,\gamma_i}}_{\text{sub-div}} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i} + \underbrace{\delta Z_{2,\Gamma}}_{\text{local two-loop divergence}}$$

Example: QED vertex ($D_{\mathsf{n}} \in \{D, 4\}$ be the numerator dimension)

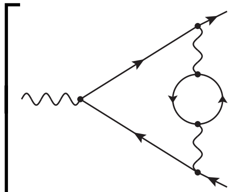
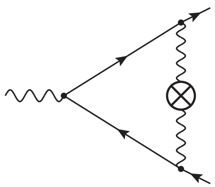
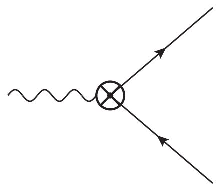
$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \Big[$  $+$  $\delta Z_{1,\gamma} +$  $\delta Z_{2,\Gamma} \Big]_{D_{\mathsf{n}} = D}$

Structure of two-loop UV rational terms [Pozzorini, **HZ**, Zoller, 20']

Relation between renormalised amplitude in $D_\mathsf{n} = D$ and $D_\mathsf{n} = 4$:

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \Bigg[\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i} \underbrace{\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma_i}\right)}_{\text{subdivergences}} \cdot \mathcal{A}_{1,\Gamma/\gamma_i} + \underbrace{\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)}_{\substack{\text{local two-loop}\\ \text{divergence}}}\Bigg]_{D_\mathsf{n}=4} + \mathcal{O}(\varepsilon)$$

Example: QED vertex

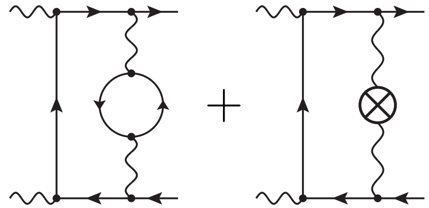
$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \Big[$  $+$  $\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma_i}\right) +$  $\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)\Big]_{D_\mathsf{n}=4} + \mathcal{O}(\varepsilon)$

Two-loop diagrams without global divergence (Proof)

Goal: to show there is no $\delta\mathcal{R}_{2,\Gamma}$:

No global divergence $\Rightarrow$ at most one subdivergence to be subtracted

$$\mathbf{R}\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{\left(\bar{\mathcal{A}}_{1,\gamma_i} + \delta Z_{1,\gamma_i}\right)}_{\text{(a) UV pole subtracted}} \cdot \underbrace{\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}}_{\text{(b) no divergence}} \quad \Leftarrow \quad \text{e.g.}$$



$\delta Z_{1,\gamma_i}$

$$= \underbrace{\left(\mathcal{A}_{1,\gamma_i} + \delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma_i}}_{\text{with 4-dim numerator}} + \mathcal{O}(\varepsilon)$$

$$= \mathcal{A}_{2,\Gamma} + \left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma_i} + \mathcal{O}(\varepsilon)$$

This implies that

$$\text{two-loop} \quad \delta\mathcal{R}_{2,\Gamma} = 0 \quad \text{and} \quad \delta Z_{2,\Gamma} = 0$$

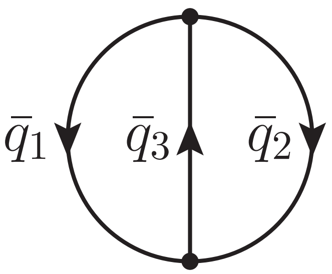
$\Rightarrow$ only globally divergent two-loop diagrams contribute to $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$

$\Rightarrow$ **finite set of $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$ counterterms** in any renormalisable theories

Two-loop diagrams with global divergence (Proof)

Goal: to show $\delta\mathcal{R}_{2,\Gamma}$ is indeed a local counterterm:

- **Isolates all divergences** from three chains of loop momenta $\bar{q}_i$ into tadpoles by tadpole expansion $\mathbf{S}^{(i)}_{X_i}$ on each chain



$\bar{q}_3 = \bar{q}_1 + \bar{q}_2$

$$\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\,\bar{\mathcal{A}}_{2,\Gamma}}_{\text{global divergence}} + \underbrace{\text{subdivergent terms}}_{\text{subtracted in subdiagram}} + \text{finite terms}$$

$$= \underbrace{\int \mathrm{d}\bar{q}_1 \mathrm{d}\bar{q}_2 \sum_{\sigma_1,\sigma_2} \frac{\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2)\,\Delta^{(\sigma_1,\sigma_2)}}{\left(\bar{q}_1^2 - M^2\right)^{N_1+\sigma_1}\left(\bar{q}_2^2 - M^2\right)^{N_2+\sigma_2}}}_{\text{globally divergent tadpoles } \bar{\mathcal{A}}_{2,\Gamma_{\text{tad}}}}$$

$$+ \text{ irrelevant terms}$$

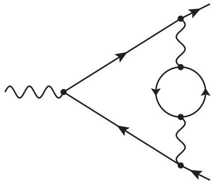
- **Only "simple" tadpoles** $\mathcal{A}_{2,\Gamma_{\text{tad}}}$ **contributes to two-loop** $\delta\mathcal{R}_{2,\Gamma}$ & $\delta Z_{2,\Gamma}$

  $\Rightarrow$ **polynomial** in external momenta and masses (upon subdivergence subtraction)

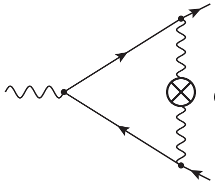
Calculations of rational terms in any renormalisation scheme

For calculation in a generic multiplicative renormalisation scheme Y with scale factor $t_Y^{\varepsilon} = \left(S_Y(\mu_0^2/\mu_R^2)\right)^{\varepsilon}$, for example the QED vertex

$$\delta\mathcal{R}^{(Y)}_{2,\Gamma} = \Bigg[\prod_{i=1}^{3}\mathbf{S}^{(i)}_{X_i}$$

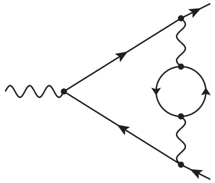


$$+ \quad \mathbf{S}^{(1)}_{X_1}$$

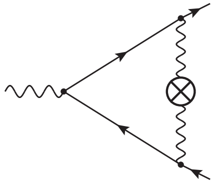


$$\delta Z^{(Y)}_{1,\gamma}\Bigg]_{D_{\mathsf{n}}=D}$$

$$- \Bigg[\prod_{i=1}^{3}\mathbf{S}^{(i)}_{X_i}$$



$$+ \quad \mathbf{S}^{(1)}_{X_1}$$



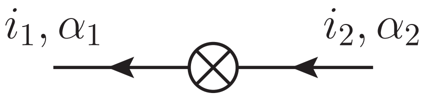
$$\left(\delta Z^{(Y)}_{1,\gamma_i} + t_Y^{\varepsilon}\left(\delta\tilde{Z}^{(\mathrm{MS})}_{1,\gamma_i} + \delta\mathcal{R}^{(\mathrm{MS})}_{1,\gamma}\right)\right)\Bigg]_{D_{\mathsf{n}}=4}$$

- One-loop $\delta\tilde{Z}^{(Y)}_{1,\gamma_i}$ and $\delta\mathcal{R}^{(Y)}_{1,\gamma}$ contain only trivial scheme dependence through scale factor t_Y^{ε}.
- Two-loop $\delta\mathcal{R}^{(Y)}_{2,\Gamma}$ contains non-trivial scheme dependence from the interplay of mass and field renormalisation and ε-dim part of numerator. [Lang, Pozzorini, **HZ**, Zoller, 20']

Two-loop rational terms in SU(N) & U(1) gauge theories

Independent calculations are done within GEFICOM [Chetyrkin, Zoller] and in-house frameworks.

Example: Fermion two-point function in Feynman gauge and renormalisation scheme Y



$$= \delta\mathcal{R}^{(Y)}_{2,\mathrm{ff}} = \mathrm{i}\,\delta_{i_1 i_2} \left(\frac{\alpha\, t^{\varepsilon}_{Y}}{4\pi}\right)^2 \left[\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{2,\mathrm{ff}}\, \not{p}_{\alpha_1\alpha_2} + \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{2,\mathrm{ff}}\, m_f\, \delta_{\alpha_1\alpha_2}\right]$$

$$\begin{aligned} \delta\hat{\mathcal{R}}^{(\mathrm{P})}_{2,\mathrm{ff}} &= \left(\frac{7}{6} C_\mathrm{F}^2 - \frac{61}{36} C_\mathrm{A}\, C_\mathrm{F} + \frac{5}{9} T_\mathrm{F}\, n_\mathrm{f}\, C_\mathrm{F}\right) \varepsilon^{-1} + \frac{43}{36} C_\mathrm{F}^2 - \frac{1087}{216} C_\mathrm{A}\, C_\mathrm{F} + \frac{59}{54} T_\mathrm{F}\, n_\mathrm{f}\, C_\mathrm{F} \\ &\quad - C_\mathrm{F}\left(\delta\hat{\mathcal{Z}}^{(Y)}_{1,\alpha} + \frac{2}{3}\,\delta\hat{\mathcal{Z}}^{(Y)}_{1,f} - \frac{2}{3}\,\delta\hat{\mathcal{Z}}^{(Y)}_{1,\mathrm{gp}}\right) \\ \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{2,\mathrm{ff}} &= \left(-2\, C_\mathrm{F}^2 + \frac{61}{12} C_\mathrm{A}\, C_\mathrm{F} - \frac{5}{3} T_\mathrm{F}\, n_\mathrm{f}\, C_\mathrm{F}\right) \varepsilon^{-1} + C_\mathrm{F}^2 + \frac{199}{24} C_\mathrm{A}\, C_\mathrm{F} - \frac{11}{6} T_\mathrm{F}\, n_\mathrm{f}\, C_\mathrm{F} \\ &\quad + C_\mathrm{F}\left(2\,\delta\hat{\mathcal{Z}}^{(Y)}_{1,\alpha} + 4\,\delta\hat{\mathcal{Z}}^{(Y)}_{1,m_f} - \frac{3}{2}\,\delta\hat{\mathcal{Z}}^{(Y)}_{1,A} - \frac{1}{2}\,\delta\hat{\mathcal{Z}}^{(Y)}_{1,\mathrm{gp}}\right) \end{aligned}$$

- $\delta\mathcal{R}^{(Y)}_{2,\Gamma}$ is **local counterterm**, i.e. polynomial in p and m_f
- $\delta\mathcal{R}^{(Y)}_{2,\Gamma}$ is derived in terms of **generic renormalisation constants** $\Rightarrow$ applicable to any scheme
- Full set of results in Yang-Mills theories are available at [arXiv: 2007.03713]

Summary

- **Renormalised** D-dim two-loop amplitude can be constructed by **amplitude with 4-dim numerator + rational counterterms**

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma_i} (\delta Z_{1,\gamma_i} + \delta \tilde{Z}_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma}) \cdot \mathcal{A}_{1,\Gamma/\gamma_i} + (\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma})$$

  $\Rightarrow$ an important step towards the two-loop automated numerical method.

- We provide a **generic method to compute** $\delta\mathcal{R}_{2,\Gamma}$ from one-scale tadpoles, and show that $\delta\mathcal{R}_{2,\Gamma}$ is **process-independent local counterterm**.

- Full set of SU(N) & U(1) rational terms at two loops in **generic renormalisation schemes**.

Backup

One-loop subdiagram example: photon self-energy

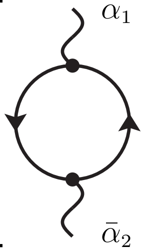
Let $D_{\mathsf{n}} \in \{D, 4\}$ be the dimension of numerator, we have

$$D_{\mathsf{n}} = D \quad \Rightarrow \quad \bar{\mathbf{K}} \int \mathrm{d}\bar{q}_1 \frac{-\mathrm{Tr}\left[\bar{\gamma}^{\bar{\alpha}_1} \bar{\not{q}}_1 \bar{\gamma}^{\bar{\alpha}_2} (\bar{\not{q}}_1 + \bar{\not{q}}_2)\right]}{\bar{q}_1^2\, (\bar{q}_1 + \bar{q}_2)^2} \quad = \quad \frac{1}{\varepsilon}\Bigg(\underbrace{-\frac{4}{3}\left(\bar{q}_2^2\, g^{\bar{\alpha}_1\bar{\alpha}_2} - \bar{q}_2^{\bar{\alpha}_1}\bar{q}_2^{\bar{\alpha}_2}\right)}_{-\delta Z_{1,\gamma}(\bar{q}_2)} \quad \underbrace{+\frac{2\,\varepsilon}{3}\,\bar{q}_2^2\, g^{\bar{\alpha}_1\bar{\alpha}_2}}_{\delta\mathcal{R}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon)} \Bigg)$$

and

$$D_{\mathsf{n}} = 4 \quad \Rightarrow \quad \mathbf{K} \int \mathrm{d}\bar{q}_1 \frac{-\mathrm{Tr}\left[\gamma^{\alpha_1} \not{q}_1 \gamma^{\alpha_2} (\not{q}_1 + \not{q}_2)\right]}{\bar{q}_1^2\, (\bar{q}_1 + q_2 + \tilde{q}_2)^2} \quad = \quad \frac{1}{\varepsilon}\Bigg(\underbrace{-\frac{4}{3}\left(q_2^2\, g^{\alpha_1\alpha_2} - q_2^{\alpha_1}q_2^{\alpha_2}\right)}_{-\delta Z_{1,\gamma}(q_2)} \quad \underbrace{-\frac{2}{3}\,\tilde{q}_2^2\, g^{\alpha_1\alpha_2}}_{-\delta\tilde{Z}_{1,\gamma}(\tilde{q}_2)} \Bigg)$$

$\Rightarrow$ Renormalised photon self-energy insertion:

$$\Bigg[$$ 

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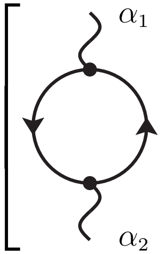
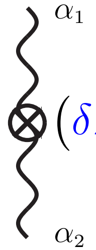
$\delta Z_{1,\gamma}(\bar{q}_2) \Bigg]_{D_{\mathsf{n}}=D} \;=\; \Bigg[$ 

$+$ 

$\left(\delta Z_{1,\gamma}(q_2) + \delta\tilde{Z}_{1,\gamma}(\tilde{q}_2) + \delta\mathcal{R}_{1,\gamma}(q_2)\right) \Bigg]_{D_{\mathsf{n}}=4} + \mathcal{O}(\varepsilon)$

Non-trivial renormalisation scheme dependence

Split the 1st-order multiplicative renormalisation (derivative) operator into finite (ΔY) and MS parts

$$D_1^{(Y)} = t_Y^{\varepsilon}\left(D_1^{(\Delta Y)} + D_1^{(\mathrm{MS})}\right)$$

The two-loop $\delta\mathcal{R}_{2,\Gamma}^{(Y)}$ contains finite renormalisation over one-loop $\delta\mathcal{R}_{1,\Gamma}^{(Y)}$, and non-trivial term $\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$

$$\delta\mathcal{R}_{2,\Gamma}^{(Y)} \;=\; (t_Y^{\varepsilon})^2\,\delta\mathcal{R}_{2,\Gamma}^{(\mathrm{MS})} + (t_Y^{\varepsilon})^2\,D_1^{(\Delta Y)}\,\delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})} + \delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$$

where

$$\begin{aligned}\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)} \;&=\; t_Y^{\varepsilon}\Big(D_1^{(\Delta Y)}\mathcal{A}_{1,\Gamma} - \sum_{\gamma}\delta Z_{1,\gamma}^{(\Delta Y)}\cdot\mathcal{A}_{1,\Gamma/\gamma}\Big) \;\neq\; 0\\ &:=\; t_Y^{\varepsilon}\sum_{\chi}\underbrace{\delta\mathcal{Z}_{1,\chi}^{(\Delta Y)}}_{\mathrm{RCs}}\;\delta\hat{\mathcal{K}}_{1,\Gamma}^{(\chi)}\end{aligned}$$

- $\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$ is due to non-commutativity of multiplicative renormalisation and counterterm insertion in $D=4$, but it can be controlled through a **new kind one-loop counterterm** $\delta\hat{\mathcal{K}}_{1,\Gamma}^{(\chi)}$