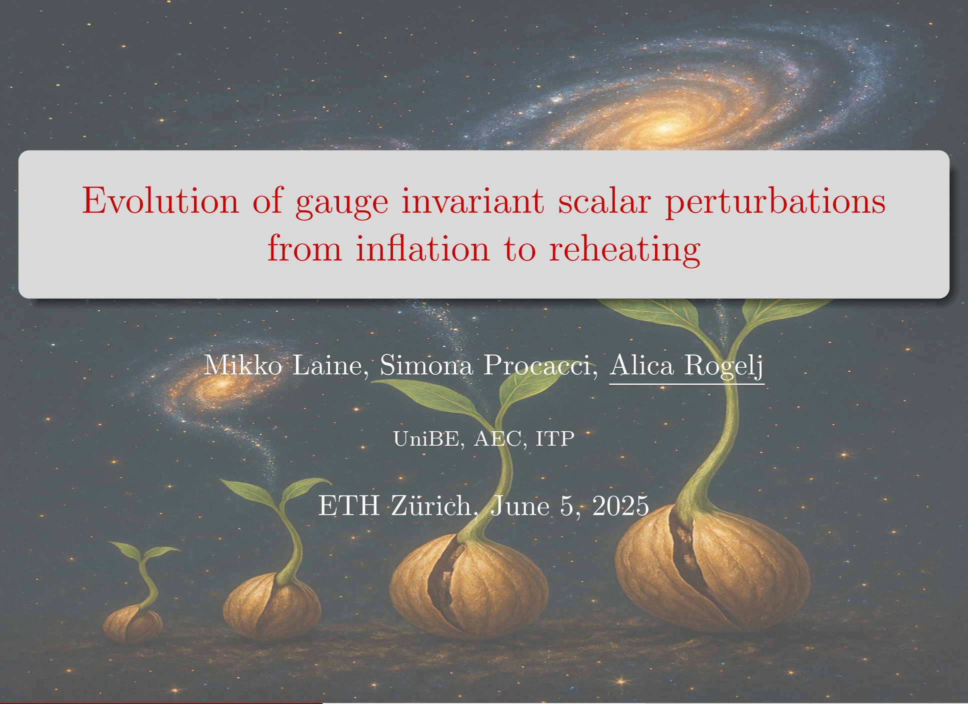

Evolution of gauge invariant scalar perturbations from inflation to reheating

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What this talk is about

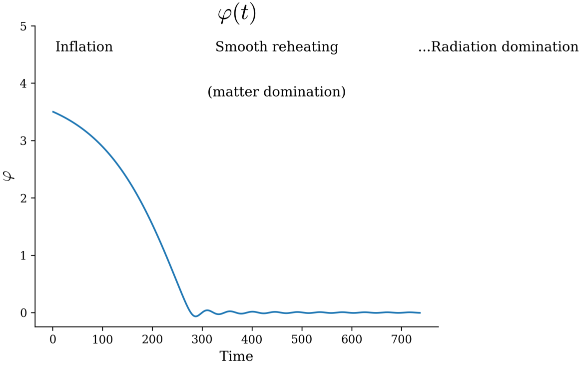
Improve on the standard formalism for generation of primordial perturbations so that:

- it is valid beyond the slow roll regime
- gauge confusions are avoided

Goal: incorporate the influence of a smooth reheating period in predictions of $\mathcal{P}_R(k)$.

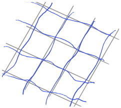
Time frame

- inflation $\xrightarrow[\text{reheating}]{\text{smooth}}$ radiation domination



Main players

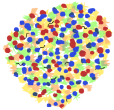
- metric
  - $g_{\mu\nu}(t,x) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,x)$



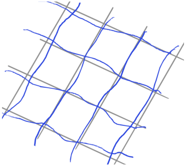
- inflaton
  - $\varphi(t,x) = \bar{\varphi}(t) + \delta\varphi(t,x)$

- plasma
  - $T^{\mu}_{\nu}|_{\mathrm{rad}} = \bar{T}^{\mu}_{\nu}|_{\mathrm{rad}} + \delta T^{\mu}_{\nu}|_{\mathrm{rad}}(\delta T, \delta u \equiv v)$

Written in terms of background values +
scalar perturbations
(scalar-vector-tensor decomposition)



Metric tensor $g_{\mu\nu}$



- FLRW background + perturbations

$$g_{\mu\nu} = a(\tau)^2 \begin{pmatrix} -1 - 2h_0 & \partial_i h \\ \partial_i h & (1 - 2h_{\rm D})\delta_{ij} + 2\left(\partial_i\partial_j - \delta_{ij}\frac{\nabla^2}{3}\right)\vartheta \end{pmatrix}$$

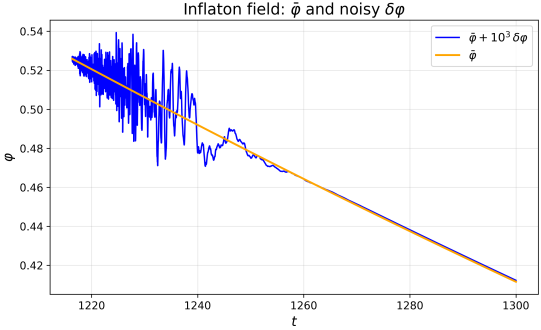
- Einstein equations: $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

$$\rightarrow \bar{G}_{\mu\nu} = 8\pi G\, \bar{T}_{\mu\nu}\ ,\ \delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu}$$

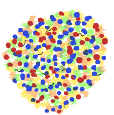
Inflaton φ (interacting with a plasma)

- Langevin equation:

$$\varphi^{;\mu}{}_{;\mu} - \Upsilon(T,\varphi)\, u^{\mu}\varphi_{,\mu} - V_{,\varphi} + \varrho \;=\; 0$$



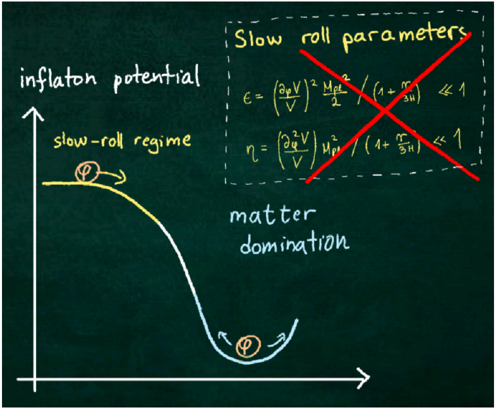
Thermal plasma



- $T^{\mu\nu} \equiv [e(T,\varphi) + p(T,\varphi)]\, u^\mu u^\nu + p(T,\varphi)\, g^{\mu\nu} + \varphi^{,\mu}\varphi^{,\nu} - \frac{1}{2} g^{\mu\nu}\varphi_{,\alpha}\varphi^{,\alpha}$
  - $T \to \bar{T} + \delta T$
  - $u^\mu \to \bar{u}^\mu + \delta u^\mu(v, \delta g^{\alpha\beta})$

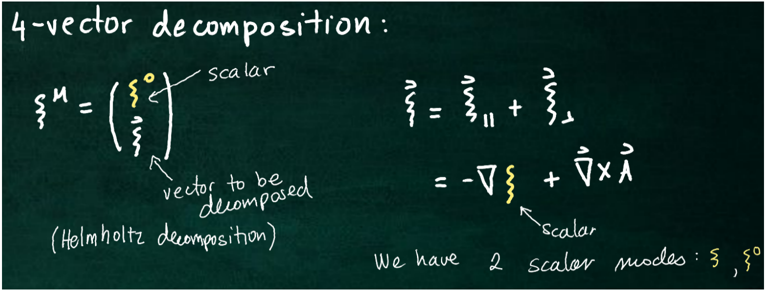
- energy-momentum conservation: $\nabla_\mu T^\mu_\nu = 0$
- assume plasma equilibrates fast: $\Gamma \gg \Upsilon$

- We make no slow roll approximation. Our equations are valid even after the inflation has ended.



Gauge invariance

- gauge invariance in General Relativity = coordinate freedom
- $x^\mu \to x^\mu + \xi^\mu \Rightarrow 2$ scalar gauge variables

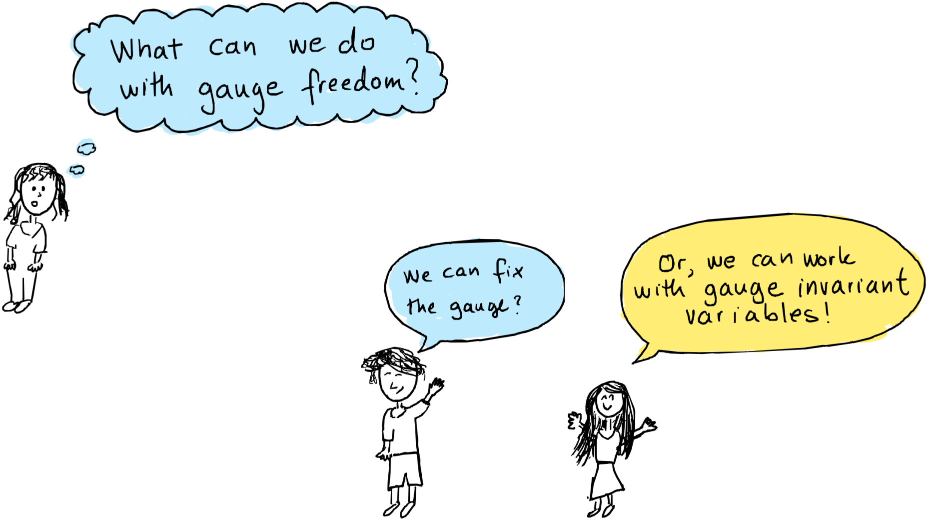


- 7 degrees of freedom $\{\delta\varphi, \delta T, v, h, h_D, h_0, \vartheta\}$
  $\to$ 5 physical variables $\{?\}$

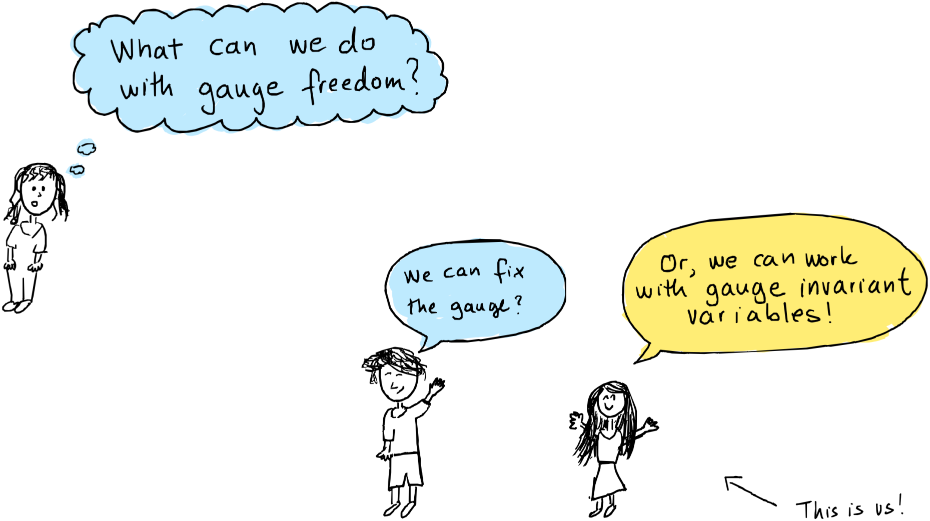
Gauge Freedom

Gauge Freedom

Gauge Freedom



Gauge Freedom



Goal

- Our main players:

- *Continuously* study their behaviour from **inflation** until the onset of **radiation domination** (*smooth reheating*).
- Do it in a **gauge invariant** way & **no** slow roll!

Gauge invariant variables

- curvature perturbations

curvature — gauge invariance

$$\mathcal{R}_\varphi \equiv -\left(h_{\rm D} + \frac{\nabla^2\vartheta}{3}\right) - \mathcal{H}\,\frac{\delta\varphi}{\bar\varphi'}$$

$$\mathcal{R}_v \equiv -\left(h_{\rm D} + \frac{\nabla^2\vartheta}{3}\right) + \mathcal{H}\,(h - v)$$

$$\mathcal{R}_T \equiv -\left(h_{\rm D} + \frac{\nabla^2\vartheta}{3}\right) - \mathcal{H}\,\frac{\delta T}{T'}$$

- isocurvature perturbations

$$\mathcal{S}_v \equiv (\bar e + \bar p)(\mathcal{R}_v - \mathcal{R}_\varphi)\,, \quad \mathcal{S}_T \equiv \bar e_{,T}\,\dot T\,(\mathcal{R}_T - \mathcal{R}_\varphi)$$

Evolution equations

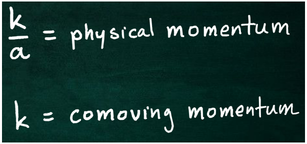
To derive evolution equations:

1. combine the derived Einstein equations, Langevin equation, and energy-momentum conservation equations
2. rewrite them in terms of gauge-invariant variables

$$\begin{bmatrix} \mathcal{R}_\varphi \\ \mathcal{S}_v \\ \mathcal{S}_T \\ \phi \\ \psi \\ -(h_D + \frac{\nabla^2 \vartheta}{3}) \\ h - \vartheta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -\frac{\nabla^2}{3} & -\frac{\mathcal{H}}{\bar{\varphi}'} & 0 & 0 \\ 0 & (\bar{e}+\bar{p})\,\mathcal{H} & 0 & 0 & \frac{(\bar{e}+\bar{p})\,\mathcal{H}}{\bar{\varphi}'} & 0 & -(\bar{e}+\bar{p})\,\mathcal{H} \\ 0 & 0 & 0 & 0 & \frac{\partial_T \bar{e}\,\dot{T}}{\bar{\varphi}'}\mathcal{H} & -H\,\partial_T \bar{e} & 0 \\ 1 & (\mathcal{H}+\partial_\tau) & 0 & -(\mathcal{H}+\partial_\tau)\partial_\tau & 0 & 0 & 0 \\ 0 & -\mathcal{H} & 1 & (\frac{\nabla^2}{3} + \mathcal{H}\partial_\tau) & 0 & 0 & 0 \\ 0 & 0 & -1 & -\frac{\nabla^2}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\partial_\tau & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h \\ h_D \\ \vartheta \\ \delta\varphi \\ \delta T \\ v \end{bmatrix}$$

3. express in terms of comoving momentum k and physical time t
4. simplify

Full Equations



Derived evolution equations:

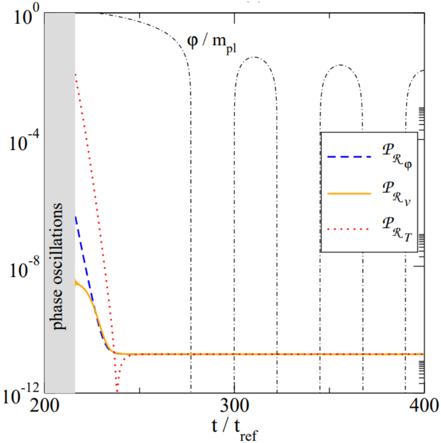
$$
\begin{aligned}
\ddot{\mathcal{R}}_\varphi =& -\frac{\varrho H}{\dot{\bar{\varphi}}} - \dot{\mathcal{R}}_\varphi \left[\Upsilon + 3H + 2\mathcal{F}\right] - \mathcal{R}_\varphi \left[\frac{k^2}{a^2}\right] \\
&+ \mathcal{S}_v \left[\frac{4\pi G(\Upsilon + 2\mathcal{F})}{H}\right] - \mathcal{S}_T \left[\frac{4\pi G}{H}\left(1 - \frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right) + \frac{V_{,\varphi T} + \Upsilon_{,T}\dot{\bar{\varphi}}}{\dot{\bar{\varphi}}\bar{e}_{,T}}\right] ,
\end{aligned}
$$

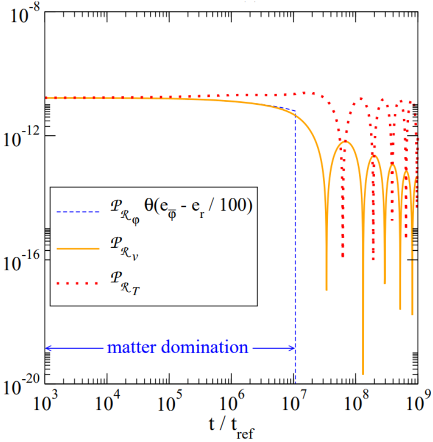
$$
\dot{\mathcal{S}}_v = -\dot{\mathcal{R}}_\varphi \left[\bar{e} + \bar{p}\right] - \mathcal{S}_v \left[3H + \frac{4\pi G \dot{\bar{\varphi}}^2}{H}\right] + \mathcal{S}_T \left[\frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right] ,
$$

$$
\begin{aligned}
\dot{\mathcal{S}}_T =& \varrho \dot{\bar{\varphi}} H + \dot{\mathcal{R}}_\varphi \left[\Upsilon \dot{\bar{\varphi}}^2 + \frac{8\pi G \bar{e}(\bar{e} + \bar{p})}{H}\right] - \mathcal{R}_\varphi \left[(\bar{e} + \bar{p})\frac{k^2}{a^2}\right] \\
&- \mathcal{S}_v \left[\frac{k^2}{a^2} + \frac{4\pi G}{H}\left(\Upsilon \dot{\bar{\varphi}}^2 + \frac{8\pi G \bar{e}(\bar{e} + \bar{p})}{H}\right)\right] \\
&+ \mathcal{S}_T \left[\frac{\dot{H} - 4\pi G(\bar{e} + \bar{p})}{H} - 3H\left(1 + \frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right) + \frac{(V_{,\varphi T} + \Upsilon_{,T}\dot{\bar{\varphi}})\dot{\bar{\varphi}}}{\bar{e}_{,T}}\right] .
\end{aligned}
$$

Solutions

- power spectrum $\mathcal{P}_{\mathcal{R}_\varphi} \equiv \frac{k^3}{2\pi^2}|\mathcal{R}_\varphi|^2$





- regimes: phase oscillations, horizon exit and re-entry (overdamped regime), acoustic oscillations

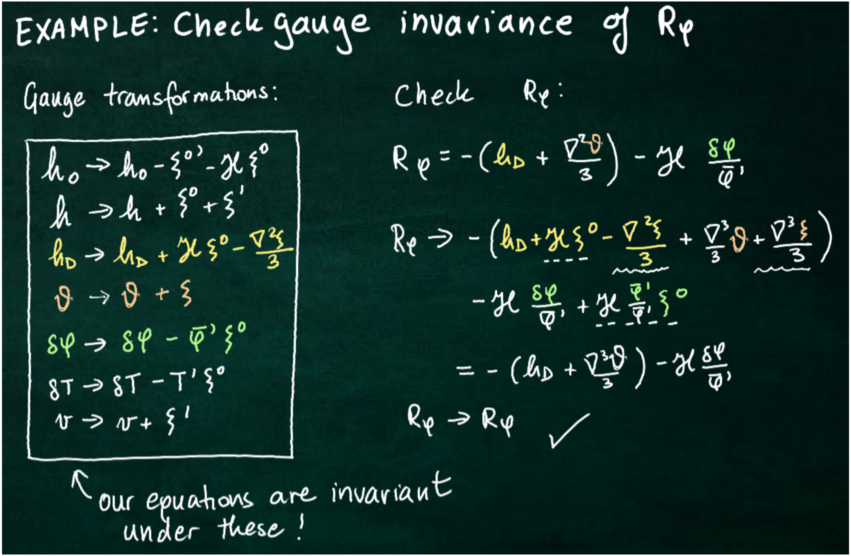
Summary and Outlook

- summary:
  - model-independent, gauge-invariant equations obtained
  - power spectrum obtained smoothly from inflation until radiation domination
- immediate next steps:
  1. add noise ρ, compute observables
  2. compare to literature (WI2easy, WarmSPy,...)
- planned follow up: scalar-induced gravitational waves

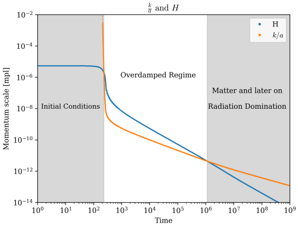
Bonus slides

- gauge invariant variables
- initial conditions
- recap
- background
- analytical simplifications
- overdamped regime
- radiation domination
- singularities
- background averaging
- benchmarks
- changing friction: background
- changing friction: perturbations
- connection to data?

Gauge invariant variables



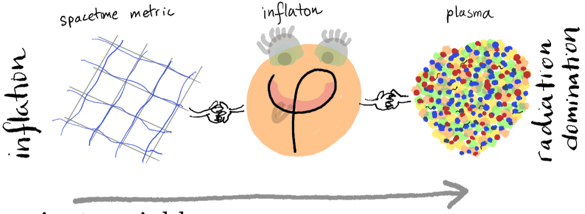
Initial conditions



- very early times: Bunch-Davies vacuum,
  $\delta\varphi'' + (k^2 + \hat{m}^2)\delta\varphi = 0$
- mode expansion solution
  $\delta\varphi(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}} \left[w_{\mathbf{k}}\, \delta\varphi_k(\tau)\, e^{i\mathbf{k}\cdot\mathbf{x}} + w^\dagger_{\mathbf{k}}\, \delta\varphi^*_k(\tau)\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$
- normalize according to canonical commutation relations + choose forward-propagating modes
- complex initial conditions:
  $\mathcal{R}_\varphi(t_1) \approx -\frac{H}{\dot{\bar{\varphi}}}\frac{1}{2\pi}\frac{k}{a_1} \qquad \dot{\mathcal{R}}_\varphi(t_1) \approx \frac{H}{\dot{\bar{\varphi}}}\frac{i}{2\pi}\frac{k^2}{a_1^2}$
- the (only) source of quantum mechanics (evolution classical)

Recap slide

- setting:



- a set of gauge invariant variables

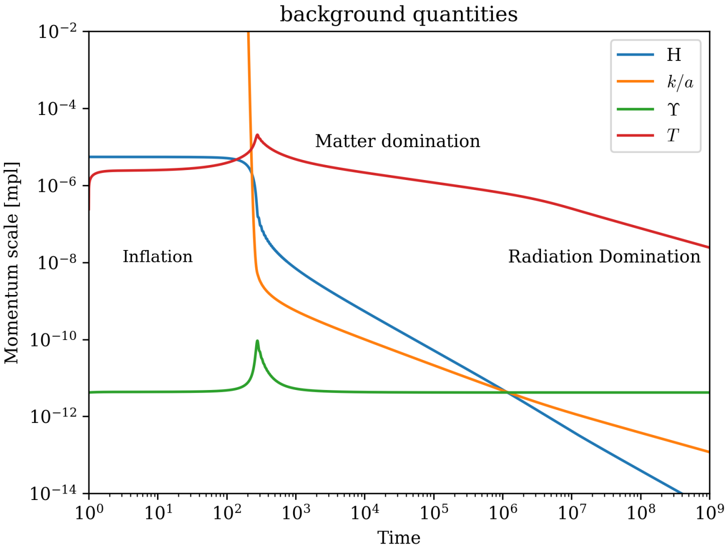
- evolution equations
  - at the beginning, inflaton dominates
  - during radiation domination, plasma evolves on its own

$$\mathcal{R}_\varphi \equiv -\left(h_{\rm D} + \frac{\nabla^2\vartheta}{3}\right) - \mathcal{H}\,\frac{\delta\varphi}{\bar{\varphi}'}$$

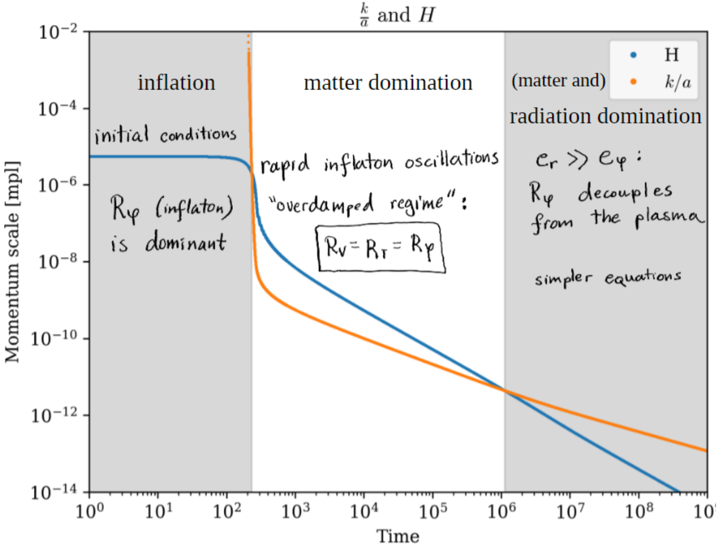
$$\mathcal{R}_v \equiv -\left(h_{\rm D} + \frac{\nabla^2\vartheta}{3}\right) + \mathcal{H}\,(h - v)$$

$$\mathcal{R}_T \equiv -\left(h_{\rm D} + \frac{\nabla^2\vartheta}{3}\right) - \mathcal{H}\,\frac{\delta T}{T'}$$

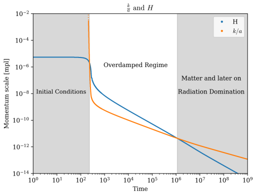
Background


background quantities
H
k/a
Υ
T
Matter domination
Inflation
Radiation Domination
Momentum scale [mpl]
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
Time

Analytical simplifications



"Overdamped regime": $\frac{k}{a} << H$



- simplified: $(\partial_t^2 + 3H\partial_t + \frac{k^2}{a^2})\delta\varphi \approx 0$
- look for a stationary solution
- assume $\rho = 0$ (weak regime)

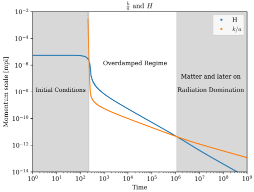
$$\ddot{\mathcal{R}}_\varphi = -\frac{\varrho H}{\dot{\bar{\varphi}}} - \dot{\mathcal{R}}_\varphi \left[\Upsilon + 3H + 2\mathcal{F}\right] - \mathcal{R}_\varphi \left[\frac{k^2}{a^2}\right] + \mathcal{S}_v \left[\frac{4\pi G(\Upsilon + 2\mathcal{F})}{H}\right] - \mathcal{S}_T \left[\frac{4\pi G}{H}\left(1 - \frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right) + \frac{V_{,\varphi T} + \Upsilon_{,T}\dot{\bar{\varphi}}}{\dot{\bar{\varphi}}\bar{e}_{,T}}\right]$$

$$\dot{\mathcal{S}}_v = -\dot{\mathcal{R}}_\varphi \left[\bar{e} + \bar{p}\right] - \mathcal{S}_v \left[3H + \frac{4\pi G\dot{\bar{\varphi}}^2}{H}\right] + \mathcal{S}_T \left[\frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right]$$

$$\dot{\mathcal{S}}_T = \varrho\dot{\bar{\varphi}}H + \dot{\mathcal{R}}_\varphi \left[\Upsilon\dot{\bar{\varphi}}^2 + \frac{8\pi G\bar{e}(\bar{e} + \bar{p})}{H}\right] - \mathcal{R}_\varphi \left[(\bar{e} + \bar{p})\frac{k^2}{a^2}\right] - \mathcal{S}_v \left[\frac{k^2}{a^2} + \frac{4\pi G}{H}\left(\Upsilon\dot{\bar{\varphi}}^2 + \frac{8\pi G\bar{e}(\bar{e} + \bar{p})}{H}\right)\right] + \mathcal{S}_T \left[\frac{\dot{H} - 4\pi G(\bar{e} + \bar{p})}{H} - 3H\left(1 + \frac{\bar{p}_{,T}}{\bar{e}_{,T}}\right) + \frac{(V_{,\varphi T} + \Upsilon_{,T}\dot{\bar{\varphi}})\dot{\bar{\varphi}}}{\bar{e}_{,T}}\right]$$

- overconstrained system: $\mathcal{S}_v = \mathcal{S}_T = 0 \;\Rightarrow\; \mathcal{R}_\varphi = \mathcal{R}_v = \mathcal{R}_T$

Radiation domination

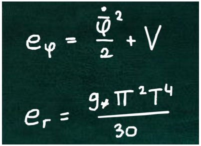


- during radiation domination, $e_r \gg e_\varphi$:

$$\dot{\mathcal{R}}_v = \frac{\dot{\bar{p}}}{\bar{e}+\bar{p}} \left(\mathcal{R}_T - \mathcal{R}_v\right) \ ,$$

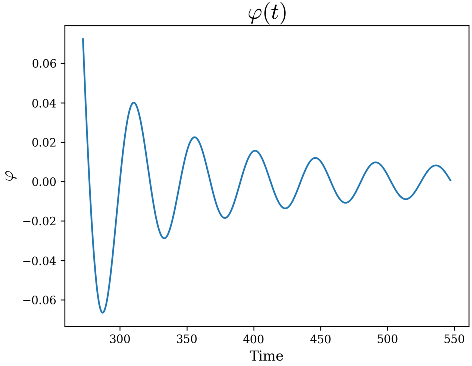
$$\dot{\mathcal{R}}_T = \frac{\mathcal{R}_v}{3H}\frac{k^2}{a^2} - \frac{4\pi G(\bar{e}+\bar{p})}{H}\left(\mathcal{R}_T - \mathcal{R}_v\right) \ .$$

- $\mathcal{R}_\phi$ decouples (plasma evolves independently of the inflaton)



Singularities

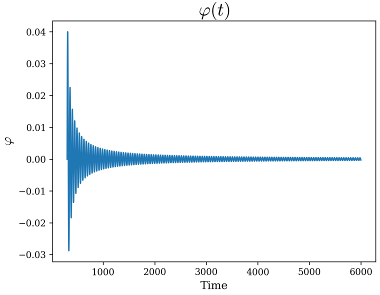
- equations include $\mathcal{F} \supset \frac{\ddot{\varphi}}{\dot{\varphi}}$, but $\dot{\varphi}$ can be 0 $\rightarrow$ problem!

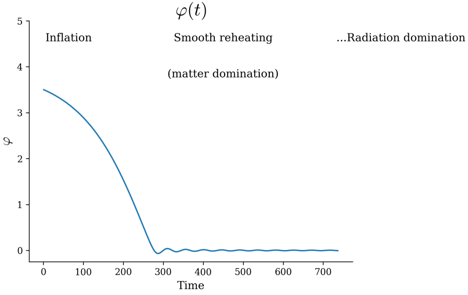


- solution: $\frac{1}{\dot{\varphi}} \longrightarrow \frac{1}{\dot{\varphi}+i\delta}$ (we are in complex field space)

Background averaging

- fast inflaton oscillations in matter domination





- switch to average variable: $e_{\bar{\varphi}} \equiv \frac{\dot{\bar{\varphi}}^2}{2} + V$

Benchmarks

- Our (non-Abelian axion inflation framework inspired) choice:
  - natural inflation[1]: $V = m^2 f_a^2 [1 - \cos(\frac{\bar{\varphi}}{f_a})]$
    $f_a = 1.25\, m_{\rm pl}\ , \quad m = 1.09 \times 10^{-6}\, m_{\rm pl}$
  - weakly coupled Yang-Mills plasma:
    $e_r \ = \ \frac{g_* \pi^2 T^4}{30}\ , \quad p_r \ = \ \frac{g_* \pi^2 T^4}{90}\ , \quad g_* = 16$
  - friction coefficient:[2]

    $$\Upsilon \ \equiv \ \frac{\kappa_T\, (\pi T)^3 + \kappa_m\, m^3}{(4\pi)^3 f_a^2}$$

    κ_T: thermal scattering
    κ_m: vacuum decays

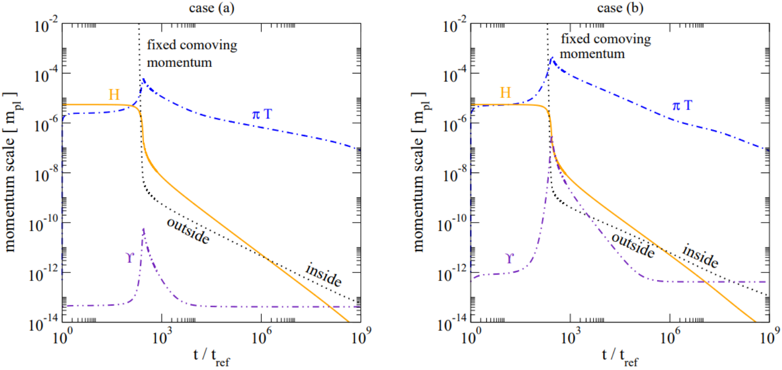
    easy illustration values: $\kappa_T \ = \ 10^6\ , \quad \kappa_m \ = \ 10^{10}$

[1] Freese et al, Phys. Rev. Lett. 65, 3233 (1990)
[2] Laine et al, J. High Energ. Phys. 2022, 126 (2022)

Changing friction coefficient, background

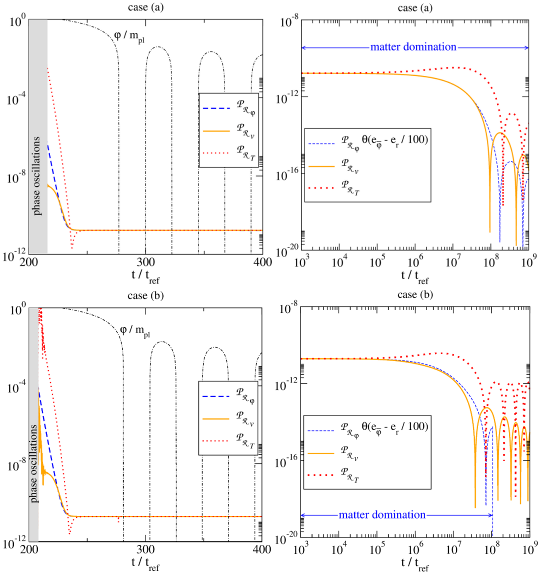
- recall: $\Upsilon \equiv \frac{\kappa_T (\pi T)^3 + \kappa_m m^3}{(4\pi)^3 f_a^2}$
- A) smaller $\kappa_m \to$ longer matter domination

  B) larger $\kappa_T \to$ higher $T_{\max}$, $\Upsilon > H$ twice



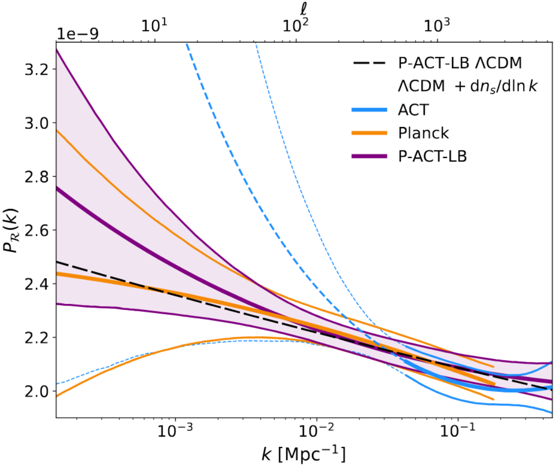
Changing friction coefficient, perturbations

- (A) smaller $\kappa_m \rightarrow$ longer matter domination
- (B) larger $\kappa_T \rightarrow$ higher $T_{\max}$, $\Upsilon > H$ twice



Connection to data?

- One can for example compute n_s, A_s, r (& compare to data[3])



[3] Calabrese et al, *The Atacama Cosmology Telescope: DR6 Constraints on Extended Cosmological Models*. Mar 2025.