The Art of Being Zoltan
oder
Die Kunst, Zoltan zu Sein

Lance Dixon (SLAC)

51st Celebration of Zoltan's 29th Birthday

24 May 2024

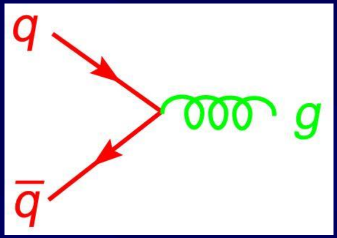
Zoltan’s mentorship

- I was “born” a string theorist
- Around 1991 I decided to learn perturbative QCD
- Of the leading QCD theorists at the time, Zoltan was the one who really took me (and Zvi Bern and David Kosower) under his wing to tell us what we needed to know
- He was the most eager of that generation to learn and develop new methods
- I am very grateful for his mentorship!

In this talk

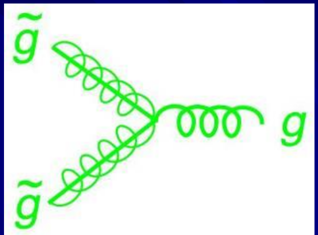
- I would like to trace a line from some of Zoltan’s papers, and especially his many encouragements, from the early days of “amplitudes” to some modern developments
- Since you all remember my talk from Zoltan’s 60th birthday party, I won’t repeat **too** much of that…

Zoltan uses the magic of supersymmetry for QCD



$$-ig_s(T^a)^i_j\gamma_\mu$$

$\sim$



$$-ig_s(T^a)^b_c\gamma_\mu$$

- Supersymmetric helicity amplitudes obey simple relations

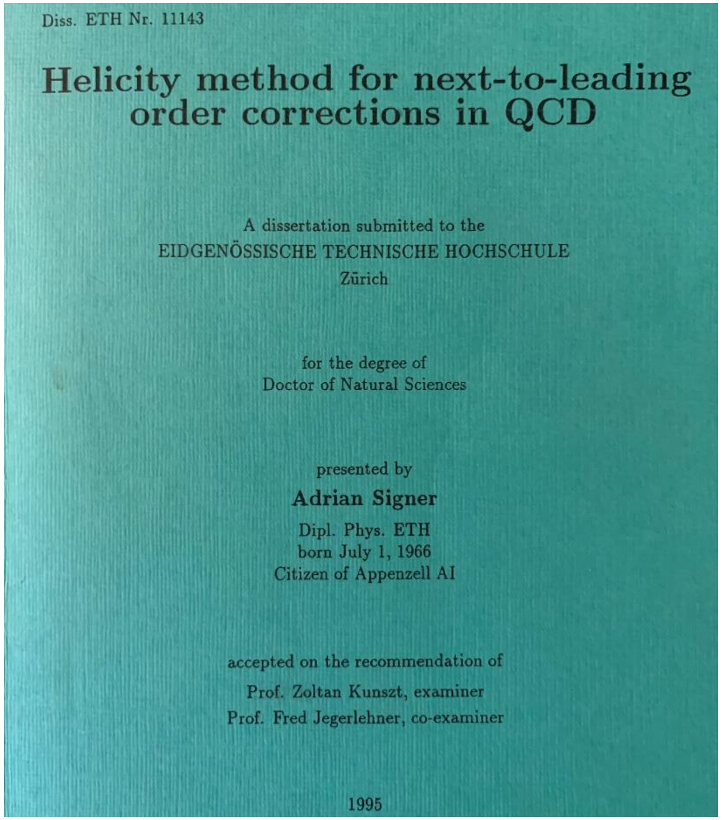
(Grisaru, Pendleton, van Nieuwenhuizen, 1977)

- Exploited in QCD from 1985 on

ZK, Nucl. Phys. B271 (1986) 333 (& Parke, Taylor, Mangano)

Concise expressions are presented for the two independent helicity amplitudes of the subprocess 4g2q. The result has been derived using the improved CALKUL method and is given in terms of spinor inner products in manifestly covariant and crossing symmetric forms. Changing the color factors of the quarks from the fundamental to the adjoint representation we obtain the helicity amplitudes of the four-gluon-two-gluino subprocess. Simple $N = 1$ supersymmetric relations have been found which express the helicity amplitudes of the six gluon parton process in terms of the helicity amplitudes of the 4g2g̃ process. In this way we have avoided the direct calculation of 220 Feynman diagrams. Gauge invariance and the validity of the supersymmetric relations have been tested with an independent numerical calculation.

Zoltan not just a mentor to me


Diss. ETH Nr. 11143

Helicity method for next-to-leading order corrections in QCD

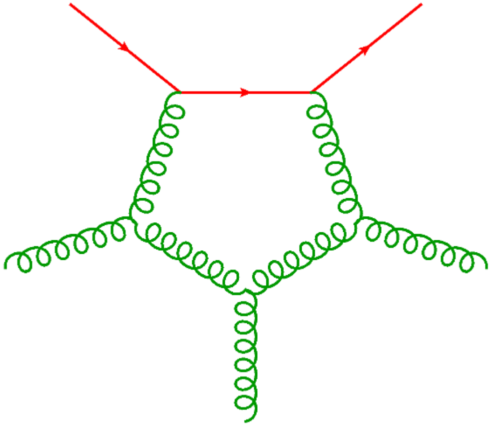
A dissertation submitted to the
EIDGENÖSSISCHE TECHNISCHE HOCHSCHULE
Zürich

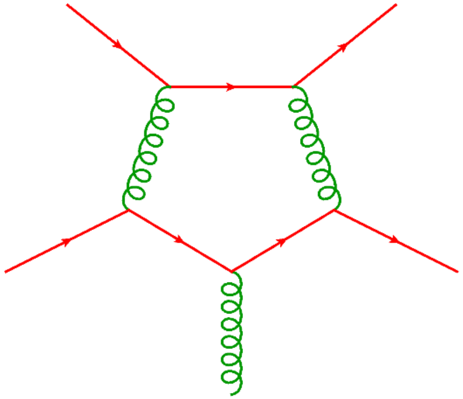
for the degree of
Doctor of Natural Sciences

presented by
Adrian Signer
Dipl. Phys. ETH
born July 1, 1966
Citizen of Appenzell AI

accepted on the recommendation of
Prof. Zoltan Kunszt, examiner
Prof. Fred Jegerlehner, co-examiner

1995





- With Zoltan², Adrian Signer computed fermionic one-loop 5-point amplitudes for his 1995 PhD thesis

- Adrian came to SLAC for a postdoc and we produced the first results for $e^+e^- \to 4\,\mathrm{jets}$ at NLO in QCD, using the Frixione-Kunszt-Signer infrared subtraction algorithm

Many enjoyable visits to Zurich

- The old HPZ tower
- A visit from Oberhofen
- Skiing at Andermatt, …
- Grosser Mythen 2006
- Zoltan and I even collaborated on a couple of papers!

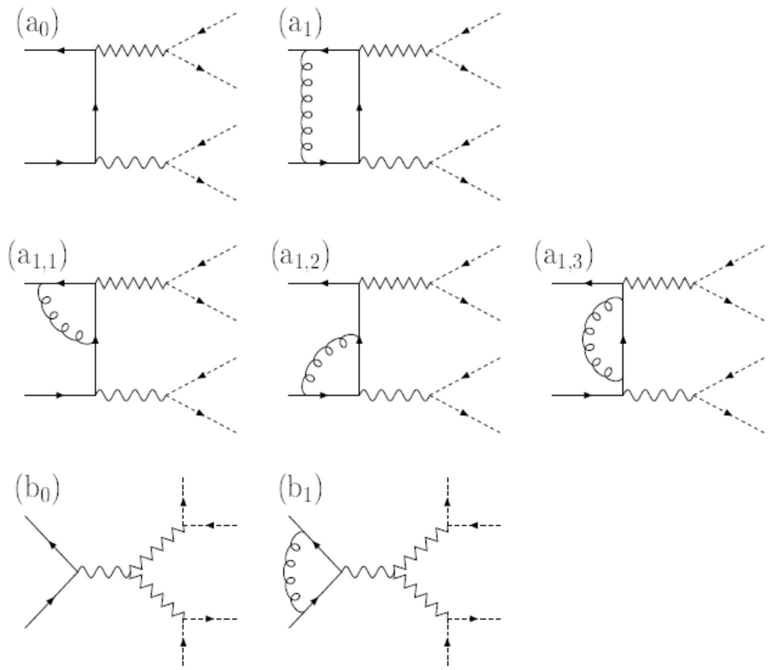
Grosser Mythen - start

Grosser Mythen - summit

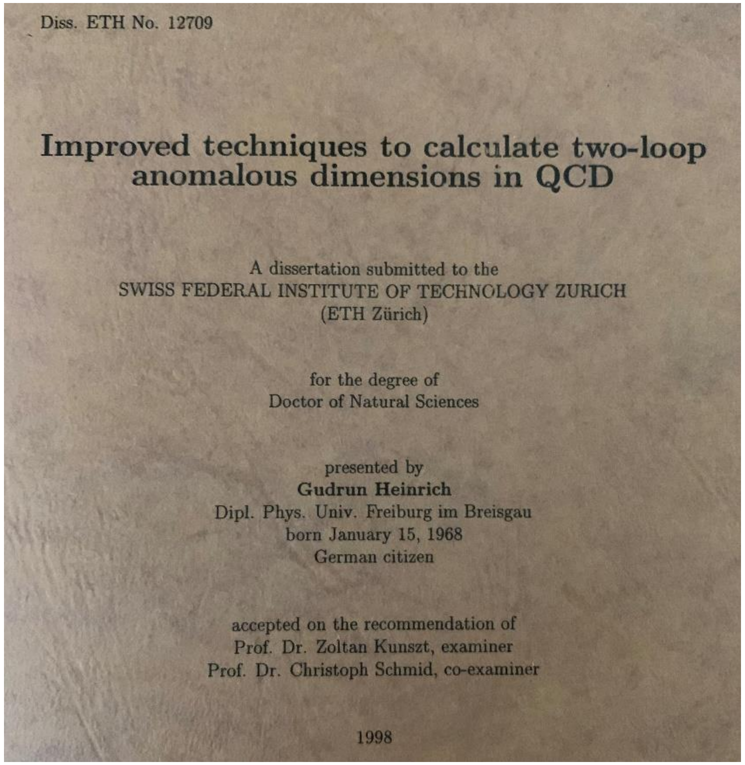
Vector boson pairs at hadron colliders at NLO

LD, Kunszt, Signer, hep-ph/9803250; hep-ph/9907305;
see also Campbell, Ellis, hep-ph/9905386

- We used helicity methods at one-loop to retain all spin correlations in vector boson decays
- Radiation zeros, anomalous couplings,…
- Studied experimentally by CMS, 2110.11231; ATLAS, 2402.16365



Zoltan not just a mentor to me



Diss. ETH No. 12709

Improved techniques to calculate two-loop anomalous dimensions in QCD

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH
(ETH Zürich)

for the degree of
Doctor of Natural Sciences

presented by
Gudrun Heinrich
Dipl. Phys. Univ. Freiburg im Breisgau
born January 15, 1968
German citizen

accepted on the recommendation of
Prof. Dr. Zoltan Kunszt, examiner
Prof. Dr. Christoph Schmid, co-examiner

1998

- On one visit to ETH (Nov. 1998?) Zoltan said "here's how one can compute higher-loop splitting amplitudes/functions" [or words to that effect] and handed me Gudrun Heinrich's PhD thesis, as well as Curci, Furmanski, Petronzio, NPB175 (1980) 27

- Zoltan taught me all about light-cone gauge issues

More about light-cone gauge

From bern@physics.ucla.edu Fri Sep 19 02:24:19 2003
Date: Thu, 18 Sep 2003 16:05:46 -0700 (PDT)
From: Zvi Bern <bern@physics.ucla.edu>
To: lance@SLAC.Stanford.EDU, babis@SLAC.Stanford.EDU
Cc: kosower@spht.saclay.cea.fr, bern@gluon.physics.ucla.edu
Subject: light cone singularities

Hi Lance and Babis,

We believe we now understand the basic issue in regulating potential extra divergences arising from the `lightcone' denominators. As we know, in all the integrals we've encountered in the cut calculation, these singularities are happily regulated by dimensional regularization.

On the other hand, we know that there is a huge background of noise and commotion in the literature on light-cone gauge integrals, which look to be very much the same as the integrals we're evaluating. So we wanted to look for a `simple' reason to quiet the instinctive reaction of anyone reading the (future) papers. The commotion (partially formal, partially practical) is echoed in the following papers,

1) Evolution Of Parton Densities Beyond Leading Order: The Nonsinglet Case,
G.~Curci, W.~Furmanski and R.~Petronzio,
Nucl.\ Phys.\ B {\bf 175}, 27 (1980). especially page 47.

2) INTRODUCTION TO NONCOVARIANT GAUGES.
By George Leibbrandt (Guelph U.). GUELPH-1986-108, Apr 1987. 174pp.
Published in Rev.Mod.Phys.59:1067,1987

3) The Light Cone Gauge In Yang-Mills Theory,''
By G.~Leibbrandt, Phys.\ Rev.\ D {\bf 29}, 1699 (1984).

4) G.~Heinrich and Z.~Kunszt,
Two-loop anomalous dimension in light-cone gauge with Mandelstam-Leibbrandt
prescription, Nucl.\ Phys.\ B {\bf 519}, 405 (1998) [arXiv:hep-ph/9708334].

For example, on page 47 of Curci et al there is a statement about
inconsistencies with formal manipulations (which Zoltan Kunszt

#################### Zoltan's e-mail ############################

Dear Zvi,

I see that you can go with formal manipuliation quite far in
your paper.

We have a paper with Basetto, Heinrich and Vogelsang where we
calculated the x=1 value of the splitting function without using
the sum rules. We carried out loop calculations for self energy diagrams
using ML and PV prescription
and got the same answer what you
expect from the sum rule. There is claim in the literature that
if three gluon couplings are involved we shall get trouble. Unfortunately
We did carry out
the calculation also for the gluon self energy but in the case of the ML
prescription we do not know what is the expected unswer since
one get new type of contribution from the x is not equal 1 part of
the calculation as well.

I expect if we carry out the remaining part of the calculation
we get the correct answer since I do not see how can we make
a mistake if all mathematical manipulations are well defined.

Hopefully we shall go back to this issue in the future.

To my opinion the Curci et.al. footnote is too strong and very likely
is not correct.
So you can take the optimistic view that if the answer is independent from
n (since it is gauge invariant) it should not matter how
do you regularize the 1/nk factor. Our paper is the only two loop
calculation in the literature.

I hope I have answered your question

Best regards Zoltan

On Mon, 15 Sep 2003, Zvi Bern wrote:

> Hi Zoltan,
>
> While I've got you, would you happen to know off hand a specific
> example of the type of ambiguity that Curci, Furmanski and Petronzio
> mention on page 47 of their seminal paper Nucl. Phys B175, 27 (1980).
> They say that it one is not careful with light-cone denominator
> singularities one can obtain different answer by making different
> formal manipulations. Are there any one-loop examples of this or
> does one have to go to at least 2 loops? Thanks.

Zoltan assured us that light-cone gauge could be made to work, despite the “commotion” in the literature

With that encouragement

- Babis, Zvi, David and I computed 2-loop collinear splitting amplitudes in N=4 super-Yang-Mills theory

hep-th/0309040

Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory

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Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

Z. Bern
Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095-1547

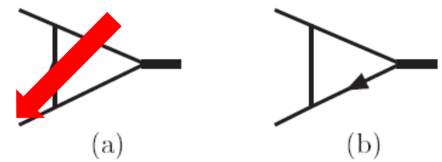
L. Dixon
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

D. A. Kosower
Service de Physique, Centre d'Etudes de Saclay, F–91191 Gif-sur-Yvette cedex, France

(Dated: September, 2003)

The collinear factorization properties of two-loop scattering amplitudes in dimensionally-regulated $N = 4$ super-Yang-Mills theory suggest that, in the planar ('t Hooft) limit, higher-loop contributions can be expressed entirely in terms of one-loop amplitudes. We demonstrate this relation explicitly for the two-loop four-point amplitude and, based on the collinear limits, conjecture an analogous relation for n-point amplitudes. The simplicity of the relation is consistent with intuition based on the AdS/CFT correspondence that the form of the large-N_c L-loop amplitudes should be simple enough to allow a resummation to all orders.

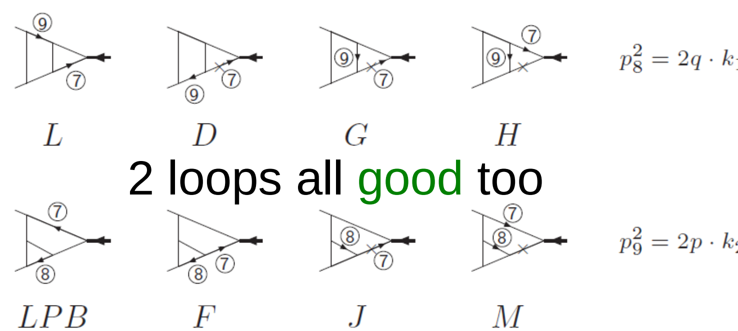
possible light-cone denominators



doesn't appear!

evil

good



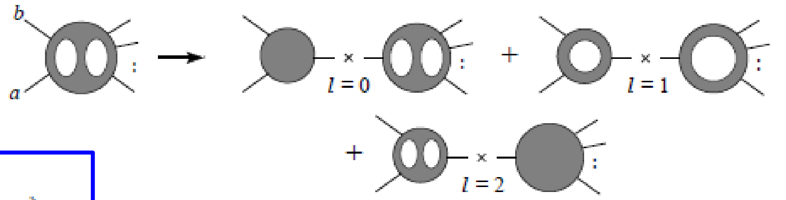
2 loops all good too

Figure 30: List of required light-cone denominator insertions for the $g \to gg$ splitting amplitude.

Also BDK, hep-ph/0404293

Remarkably simple collinear behavior!

$$\mathrm{Split}^{(L)}_{-\lambda_P}(1^{\lambda_1},2^{\lambda_2}) \; = \; r_S^{(L)}\,\mathrm{Split}^{(0)}_{-\lambda_P}(1^{\lambda_1},2^{\lambda_2})$$



$$r_S^{(2)}(\epsilon;z,s) = \frac{1}{2}\Big(r_S^{(1)}(\epsilon;z,s)\Big)^2 + f(\epsilon) r_S^{(1)}(2\epsilon;z,s)\,,$$

where

$$f(\epsilon) \equiv (\psi(1-\epsilon)-\psi(1))/\epsilon = -(\zeta_2+\zeta_3\epsilon+\zeta_4\epsilon^2+\cdots)$$

- Suggested to us the "ABDK ansatz"

We now present evidence that through $\mathcal{O}(\epsilon^0)$ the two-loop planar amplitudes are related to one-loop ones via,

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\Big(M_n^{(1)}(\epsilon)\Big)^2 + f(\epsilon)\,M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4\,. \qquad (13)$$

- Might not have been found without Zoltan's encouragement!

Later → all-orders "BDS ansatz"

Bern, LD, Smirnov, hep-th/0505205

- After absorbing the QCD literature on IR divergences, including Magnea, Sterman (1990), Sterman, Tejeda-Yeomans, hep-ph/0210310, Kunszt, Signer, Trocsanyi, hep-ph/9401294, …
- We realized that everything simplified in the planar (large N_c) limit of SYM and we proposed (for MHV)

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon)\right)\right]$$

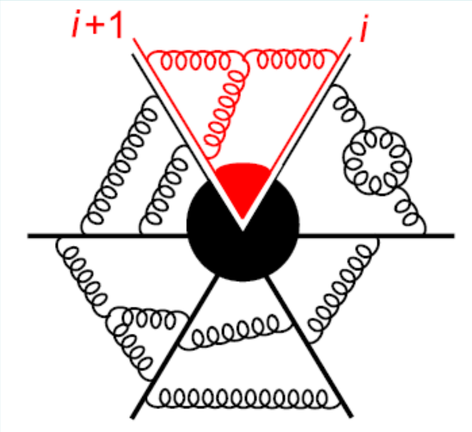
with

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)} \quad \text{cusp}$$

$$f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)} \quad \text{collinear}$$

← anomalous dimensions



- Main conjecture was that remainder is constant: $h_n^{(l)}(k_i, \epsilon) = C^{(l)}$

ABDK/BDS ansatz failure

- Failed for amplitudes with $n \geq 6$

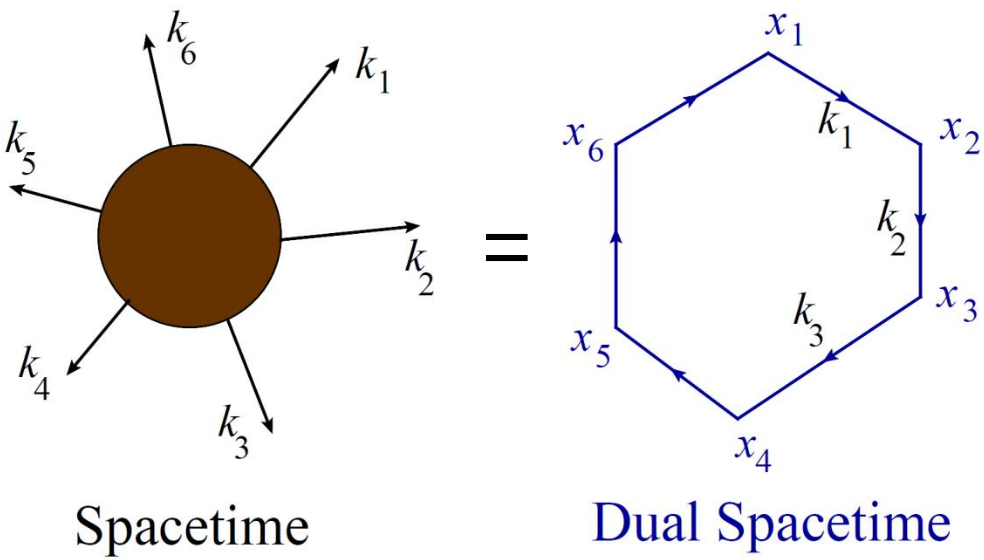
  Alday, Maldacena, 0705.0303; Bartels, Lipatov, Sabio Vera, 0807.0894; Drummond, Henn, Korchemsky, Sokatchev, 0712.4138, 0803.1466; Bern, LD, Kosower, Roiban, Spradlin, Volovich, Vergu, 0803.1465

- Reason it worked for $n = 4{,}5$ was due to a hidden symmetry, dual conformal invariance → $h_n = R_n(u_{ijkl})$

dual conformal cross ratios:
4 fewer variables than expected!

- Related to duality with polygonal Wilson loops

Amplitudes = Wilson loops



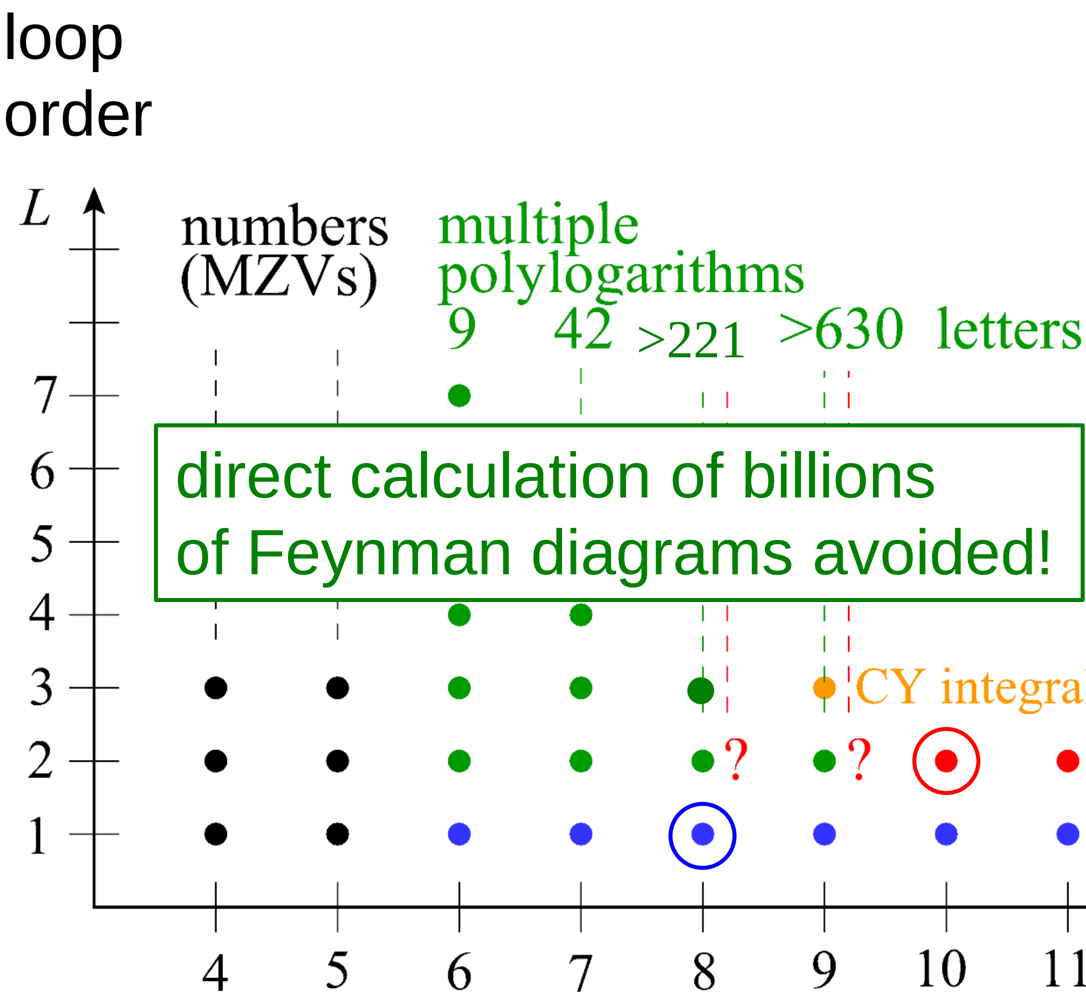
- Polygon vertices x_i are not positions but dual momenta,
  $$x_i - x_{i+1} = k_i$$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

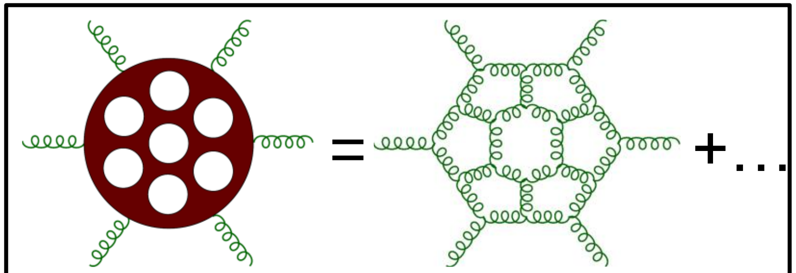
Duality verified to hold at weak coupling too

weak-weak duality, holds order-by-order

Planar N=4 SYM Amplitudes Known Today



direct calculation of billions
of Feynman diagrams avoided!



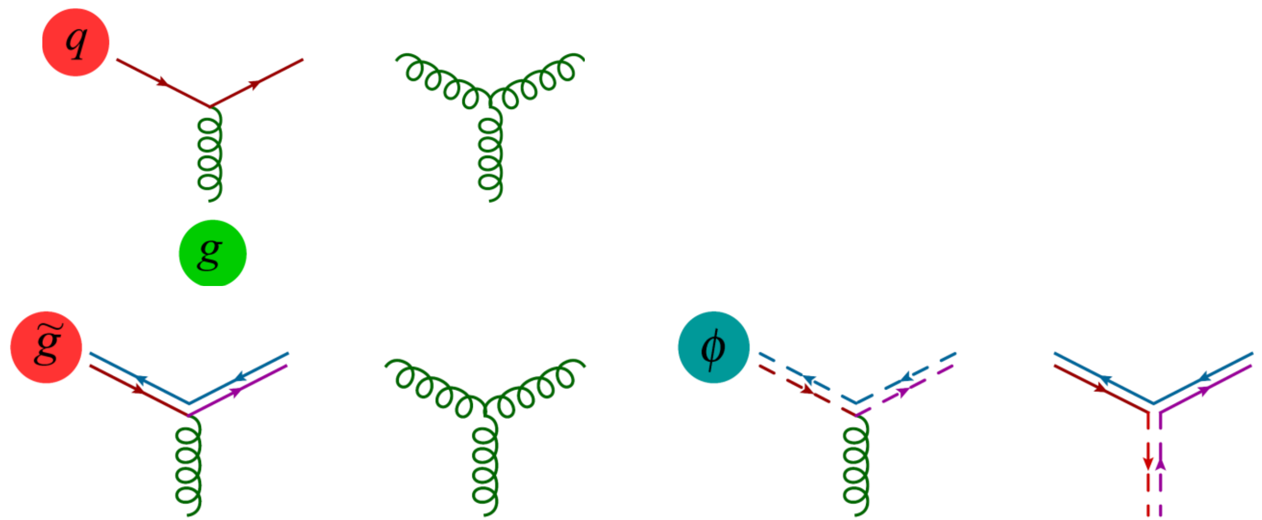
sum all planar Feynman graphs with
L loops and *n* external lines

QCD vs. N=4 SYM

- QCD: gluons, plus quarks in fundamental rep. of $SU(N_c)$
- N=4: Replace quarks with 4 copies of fermions in adjoint rep. (gluinos) and add 6 real adjoint rep. scalars
- All in same supermultiplet (like one particle)
- Feynman vertices:

QCD

N=4 SYM



Planar N=4 SYM

- QCD's maximally supersymmetric relative*, gauge group $SU(N_c)$, in large N_c (planar) limit
- Structure very rigid:

  Amplitudes = $\sum_i rational_i \times transcendental_i$
- For planar N=4 SYM, rational structure well understood (related to same SUSY Ward identities exploited by Zoltan at tree level)
- Focus now on transcendental functions.
- At least three dualities hold:

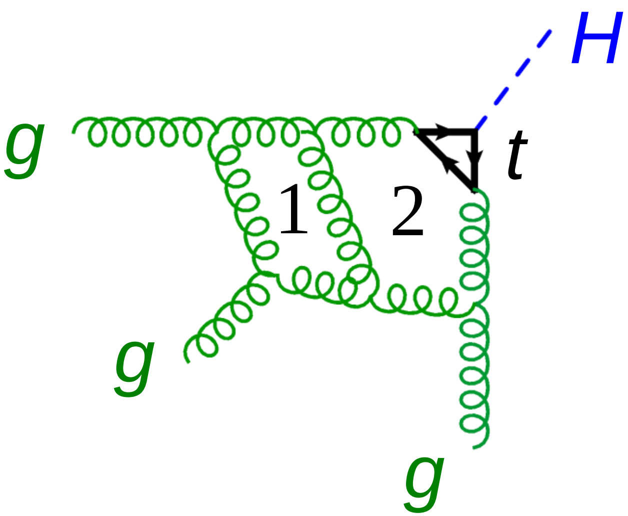
1. AdS/CFT
2. Amplitudes dual to Wilson loops
3. New "antipodal" (self-)duality involving form factors

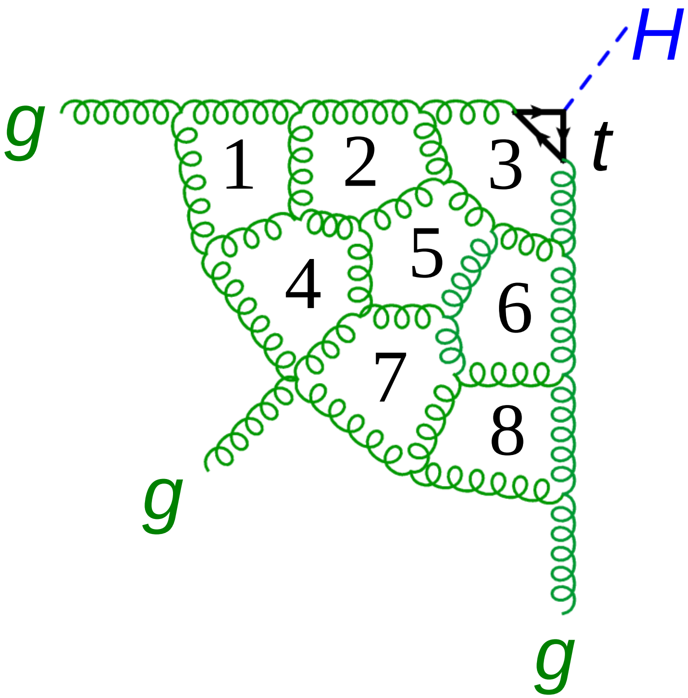
*Fairy godmother of QCD – grants all your analytic wishes

“Goldilocks Process”: $gg \rightarrow Hg$

QCD state of art is two loops (not counting top quark loop)
Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554

We can get to **eight** loops in planar N=4 SYM !!!
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm
2012.12286, 2112.06243, 2204.11901





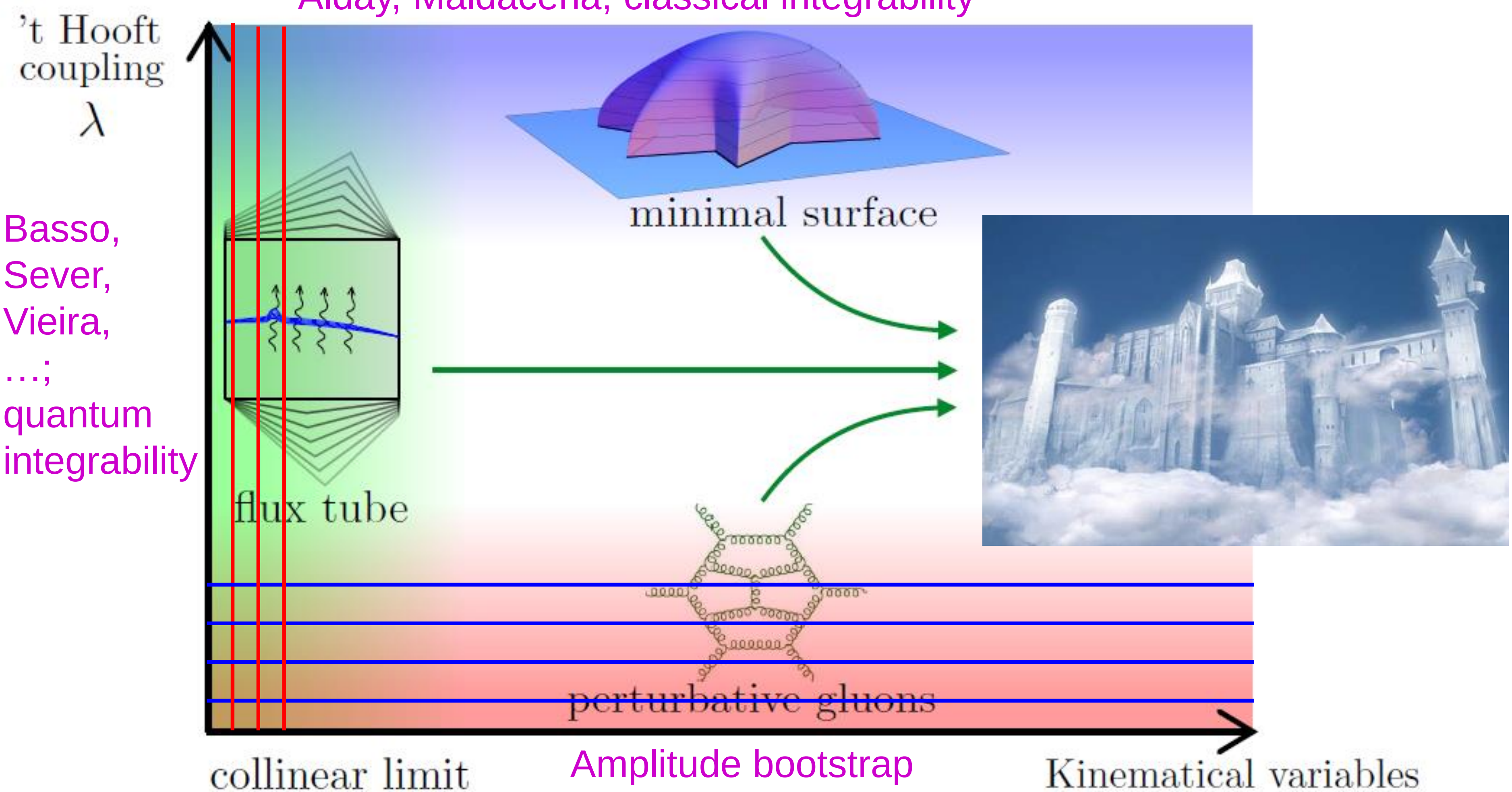
Planar N=4 SYM Dream:

Solve for amplitudes nonperturbatively for any kinematics

Images: A. Sever, N. Arkani-Hamed

Alday, Maldacena; classical integrability

Basso, Sever, Vieira, …; quantum integrability

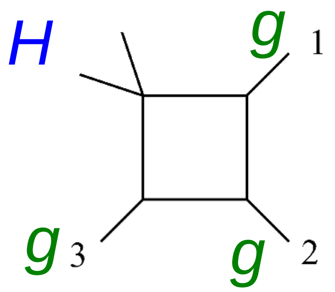


Amplitude bootstrap

Beyond tree level

- Loop level Feynman diagrams have to be integrated over all loop momenta

- For example, at one loop the amplitude for *gg* → *Hg* involves the “scalar box” integral



$$= \int \frac{d^4p}{p^2(p-p_1)^2(p-p_1-p_2)^2(p-p_1-p_2-p_3)^2}$$

$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

where $s_{ij} = (p_i + p_j)^2$, $s_{123} = (p_1+p_2+p_3)^2 = m_H^2$
and the dilogarithm is $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t}\ln(1-t)$

One loop not too bad

- For any number of external particles, all one-loop integrals (even in dimensional regularization, $D = 4 - 2\epsilon$) can be reduced to scalar box integrals + simpler

Brown-Feynman (1952), Melrose (1965), ‘t Hooft-Veltman (1974), Passarino-Veltman (1979), van Neerven-Vermaseren (1984), Bern, LD, Kosower (1992)

→ combinations of $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t)$

where x is (many different) functions of the kinematic variables (Mandelstam invariants), plus logarithms

Multi-loop much more complex

- At L loops, instead of just Li_2's, get special functions with up to $2L$ integrations
- Weight $2L$ "iterated integrals"
- **Best case**: generalized polylogarithms G, defined iteratively by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

- Very intricate multi-variate functions

Symbol ~ DNA Code for Amplitudes

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

- Complexity **encoded** in words written in an alphabet defined by iterative differentiation

- Code is analog of ATGC code for DNA.
- Characterizes answer fairly completely
- We can learn to read it
- We can use that understanding to go to ever higher loop orders (at least in planar N=4 SYM)

Symbol from iterated derivatives

- Let $F(x_i)$ be a linear combination of generalized polylogarithms $G(a_1, a_2, \dots, a_n, x)$ of weight n
- Partial derivatives of F with respect to underlying coordinates x_i are given by

$$\frac{\partial F}{\partial x_i} = \sum_{s_k \in \mathcal{L}} F^{s_k} \frac{\partial \ln s_k}{\partial x_i}$$

- F^{s_k} belong to same space of functions, but have weight $n-1$, and s_k are letters in some finite symbol alphabet $\mathcal{L}$
- Iterative definition of symbol $\mathcal{S}$ as tensor product:

$$\mathcal{S}[F] = \sum_{s_k \in \mathcal{L}} \mathcal{S}[F^{s_k}] \otimes \ln s_k$$

Now Iterate

- Define F^{s_j,s_k} via derivatives of F^{s_k}:

$$\frac{\partial F^{s_k}}{\partial x_i} \equiv \sum_{s_j \in \mathcal{S}} F^{s_j,s_k} \frac{\partial \ln s_j}{\partial x_i}$$

$$\Leftrightarrow \quad \mathcal{S}[F] = \sum_{s_j,s_k \in \mathcal{L}} \mathcal{S}[F^{s_j,s_k}] \otimes \ln s_j \otimes \ln s_k$$

- Iterating, n times for weight n function, gives symbol $\mathcal{S}[F]$ as n–fold tensor,

$$\mathcal{S}[F] = \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}} \, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

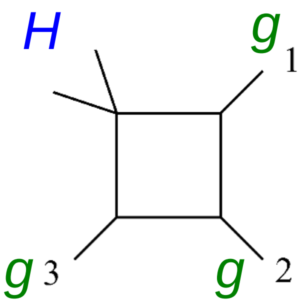
where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers (drop the “ln”)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Symbol as scaffolding for function

- Symbol resolves all identities among generalized polylogarithms (because all log identities are trivial)
- Taking n derivatives drops constants like $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$
- Restore by tracking values at some boundary point(s)

One-loop $gg \to Hg$ example



$$F = \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

$$= \mathrm{Li}_2\left(1 - \frac{1}{u}\right) + \mathrm{Li}_2\left(1 - \frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

where $u = \frac{s_{12}}{s_{123}}$ and $v = \frac{s_{23}}{s_{123}}$ are only 2 dimensionless variables
$\left(w = \frac{s_{13}}{s_{123}} = 1 - u - v\right)$

symbol $\mathcal{S}[F] = u \otimes (1-u) + v \otimes (1-v) - u \otimes v - v \otimes u$

Six letter alphabet: $\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$

Two loops, same alphabet: Gehrmann, Remiddi, hep-ph/0008287, hep-ph/0101124

Better alphabet for $gg \rightarrow Hg$

These 6 letters are equivalent to $\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$
but "diagonalize" things better: $\mathcal{L}' = \{a, b, c, d, e, f\}$
where $a = \sqrt{\frac{u}{vw}},\ b = \sqrt{\frac{v}{wu}},\ c = \sqrt{\frac{w}{uv}},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w}$

- Symbols of $gg \rightarrow Hg$ amplitude $F_3^{(L)}$ simplify (remarkably) at $L = 1$ and 2 loops, to just 6 and 12 terms:

$$\mathcal{S}\left[F_3^{(1)}\right] = (-2)[\, b\otimes d + c\otimes e + a\otimes f + b\otimes f + c\otimes d + a\otimes e]$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 8[\, b\otimes d\otimes d\otimes d + c\otimes e\otimes e\otimes e + a\otimes f\otimes f\otimes f + b\otimes f\otimes f\otimes f + c\otimes d\otimes d\otimes d + a\otimes e\otimes e\otimes e]$$
$$+16\,[b\otimes b\otimes b\otimes d + c\otimes c\otimes c\otimes e + a\otimes a\otimes a\otimes f + b\otimes b\otimes b\otimes f + c\otimes c\otimes c\otimes d + a\otimes a\otimes a\otimes e]$$

(really only 1 and 2 terms, plus images under dihedral symmetry)

$gg \rightarrow Hg$ symbol terms per loop

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

Not advisable to look at full symbol directly!

However, there is a lot of data to be mined from it

All coefficients are integers! Many, many correlations
→ Ripe area for machine learning with large language models
(Cai, Cranmer, Charton, LD, Merz, Nolte, Wilhelm, 2405.06107)

Symbol alphabets for n-gluon amplitudes in planar N=4 SYM

$n = 4{,}5$ trivial in this theory

parity-odd letters, algebraic in $\hat{u}, \hat{v}, \hat{w}$

$n = 6$ has 9 letters: $\mathcal{L}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \underbrace{\hat{y}_u, \hat{y}_v, \hat{y}_w}\}$

(3 kinematic variables)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$n = 7$ has 42 letters

(6 var's)

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

(9 var's)

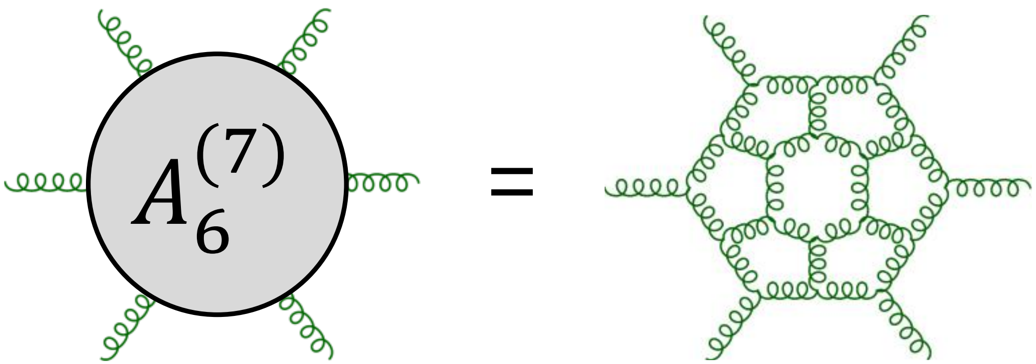
Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Gürdoğan, Kalousios, 1912.08217
Henke, Papathanasiou 1912.08254; Z. Li, C. Zhang, 2110.00350, …

Structures closely tied to mathematics:

cluster algebras Fomin, Zelevinsky (2001), tropical geometry & polytopes

We knew $n = 6$ amplitude to 7 loops

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890



+ ~10^9 more

L	A_6 no. of terms
1	6
2	84
3	5,034
4	243,000
5	15,534,750

There's a "parity-preserving" surface, $\Delta(\hat{u},\hat{v},\hat{w}) = 0$, where

$$\hat{y}_u = \hat{y}_v = \hat{y}_w = 1$$

→ any word containing them is 0,

→ 9 letters drops to 6 letters

L	$A_6\|_{\Delta=0}$ no. of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568

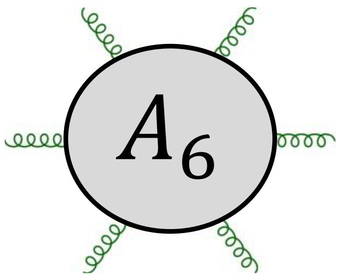
Q: Where have we seen these numbers before? A: In *gg*→*Hg* !!!

Bizarre new "antipodal" duality

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

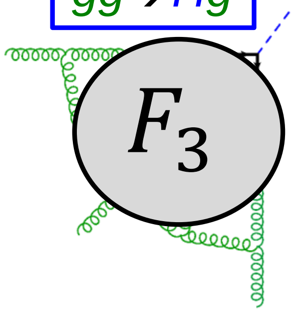
Relates two seemingly unrelated processes!

(MHV) $gg \to gggg$



$$A_6^{(L)}(\hat{u}, \hat{v}, \hat{w})|_{\Delta=0} = S(F_3^{(L)}(u, v, w))$$

$gg \to Hg$



"Antipode map" S
reverses order of all entries in symbol!!!
$$S[s_1 \otimes s_2 \otimes \ldots \otimes s_{2L}] = s_{2L} \otimes s_{2L-1} \otimes \ldots \otimes s_1$$

L	$A_6\vert_{\Delta=0}$ no. of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568

$$\sqrt{\hat{a}} \Leftrightarrow d$$
$$\sqrt{\hat{b}} \Leftrightarrow e$$
$$\sqrt{\hat{c}} \Leftrightarrow f$$
$$\hat{d} \Leftrightarrow a$$
$$\hat{e} \Leftrightarrow b$$
$$\hat{f} \Leftrightarrow c$$

L	F_3 no. of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

8 loops now also checked;
LD, Liu, 2308.08199

Antipodal Self Duality
for a 4-point form factor in planar N=4 SYM

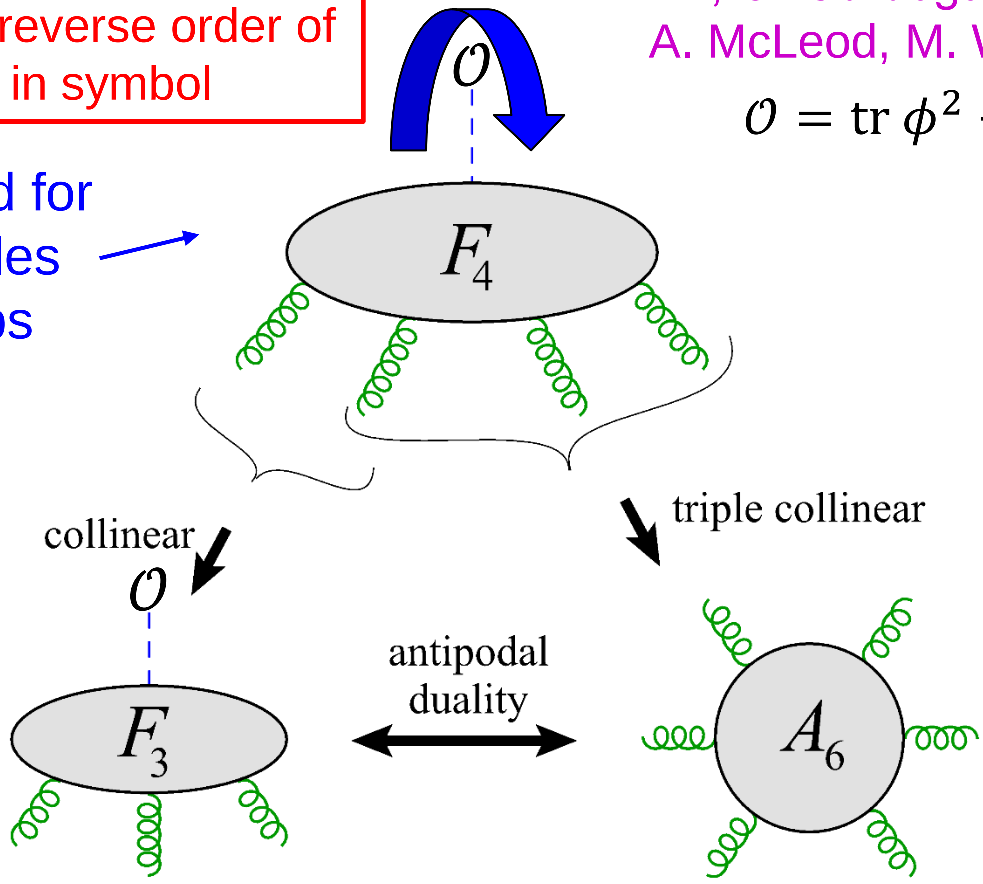
LD, Ö. Gürdoğan, Y.-T. Liu,
A. McLeod, M. Wilhelm, 2212.02410

again reverse order of letters in symbol

$$\mathcal{O} = \mathrm{tr}\,\phi^2 + \cdots + \mathrm{tr}\,G_{\mu\nu}^{\mathrm{SD}} G^{\mu\nu\,\mathrm{SD}}$$

validated for 4 variables to 3 loops →

validated for 2 variables to 8 loops →



kinematic map found for F_4 reduces to $F_3 \leftrightarrow A_6$ kinematic map in the (triple) collinear limit

→ ASD "explains" AD

but "who ordered ASD"?

In conclusion

- Zoltan saw virtue of new methods for QCD before most people: supersymmetry, helicity methods, generalized unitarity, …
- His insights and interest in collinear splitting encouraged Babis, Zvi, David and me to compute 2-loop collinear behavior in planar N=4 SYM
- Led directly to ABDK and BDS ansätze, which in turn led to an explosion of interest in amplitudes in planar N=4 SYM
- Can now get to 8 loops for "*gg*→*Hg*" and 6-point amplitudes; 3-4 loops in other cases; many limits to explore
- Symbol of amplitudes uses code words of maximal length $2L$ at L loops, and is a general tool for disentangling structure of scattering amplitudes in QCD as well as planar N=4
- Bizarre new antipodal duality relates 3-gluon form factor to 6-gluon amplitude, swapping role of branch cuts and derivatives
- "Explained" by antipodal self-duality for 4-gluon form factor

Champagne & caviar

Happy Birthday Zoltan!

Extra Slides

Example: Classical polylogarithms

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

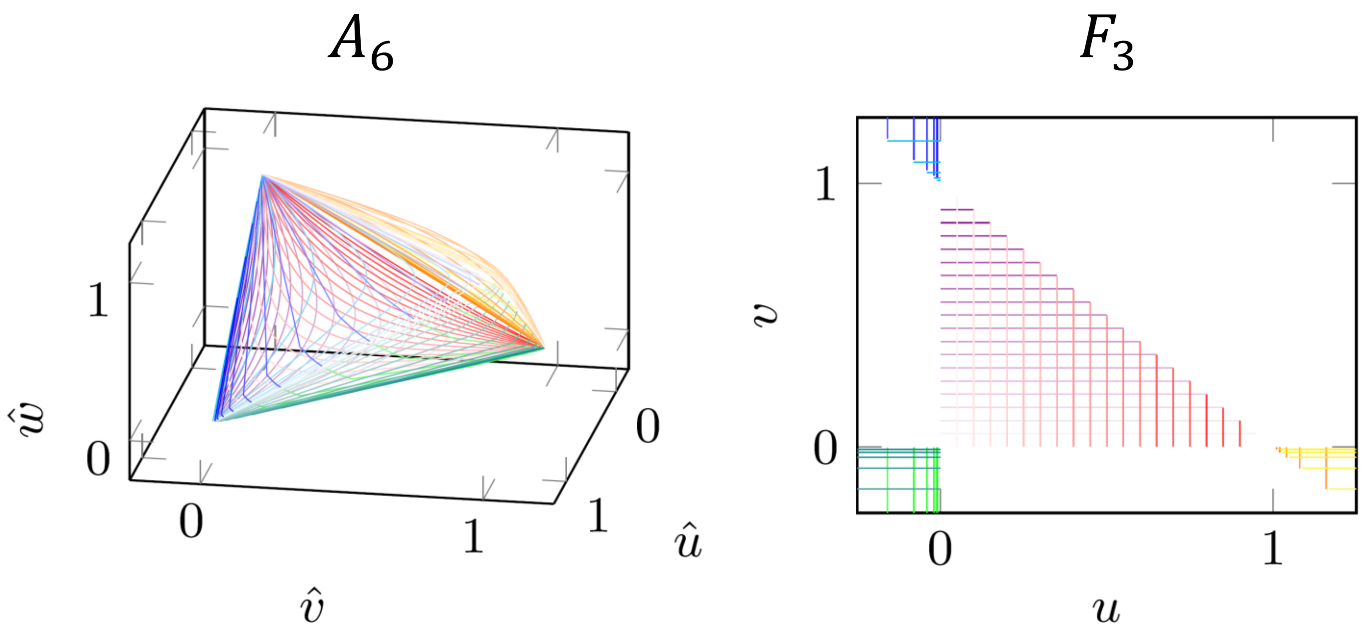
$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

- Regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives the symbol:

$$\mathcal{S}[\mathrm{Li}_n(x)] = \mathcal{S}[\mathrm{Li}_{n-1}(x)] \otimes x$$
$$= \cdots = -(1-x) \otimes x \otimes x \; \ldots \otimes x$$

- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading $(1-x)$ to compute discontinuity at $x = 1$.
- Derivatives computed from symbol by clipping last entry, multiplying by that $d\ln(\ldots)$. **Alphabet** $\mathcal{L} = \{x, 1-x\}$

AD Kinematic map covers entire phase space for 3-gluon form factor



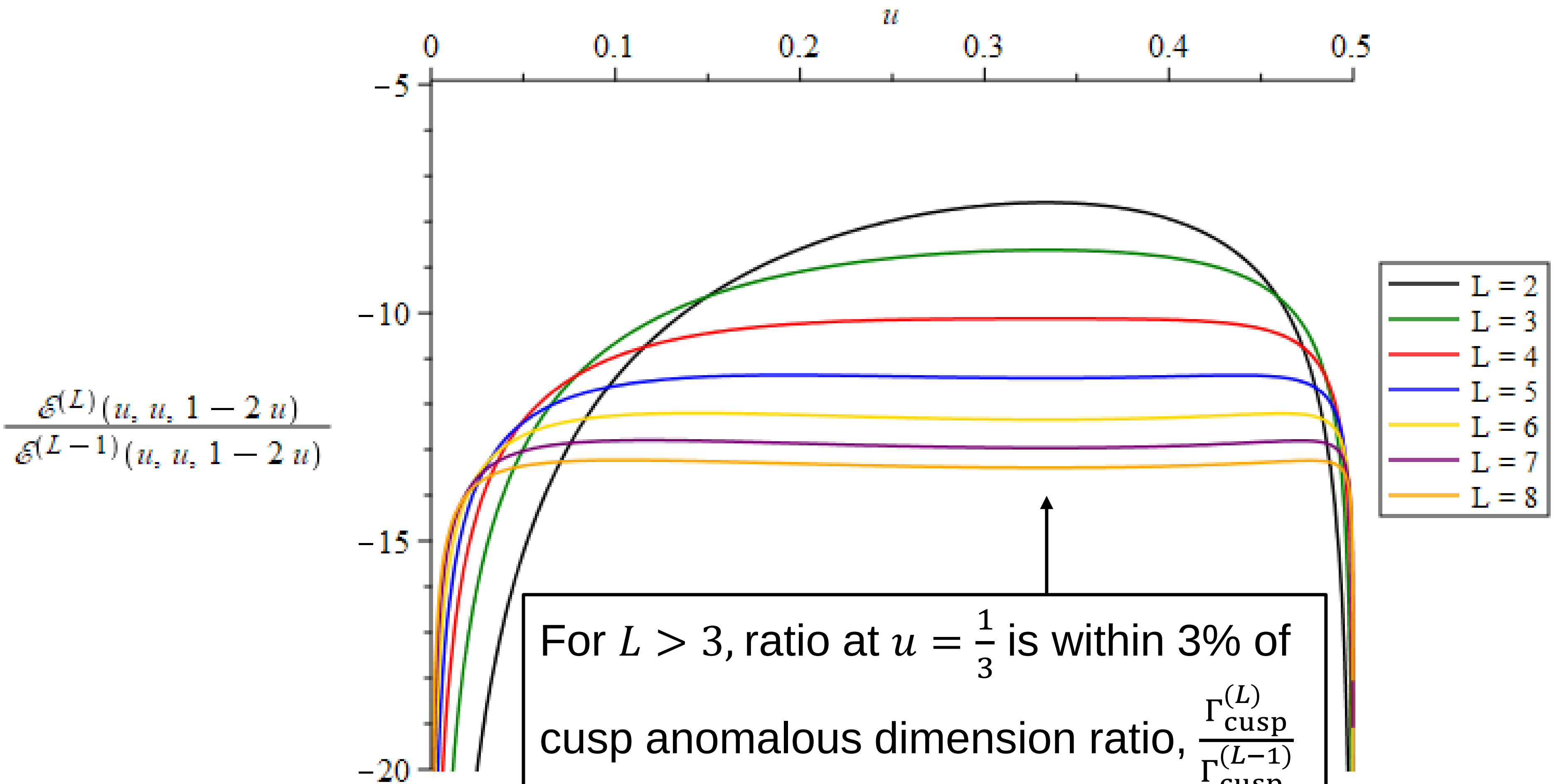
- Soft regions dual to collinear regions and vice versa
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Eight loop test of Antipodal Duality

$$
\begin{aligned}
A_6^{(8)}(1,1,1) \; = \; & 9122624\, f_{9,7} + 11543472\, f_{7,9} + 5153280\, f_{11,5} + 19603536\, f_{5,11} + 23915376\, f_{3,13} \\
& + 371520\, f_{5,3,3,5} + 400320\, f_{3,3,5,5} + 400320\, f_{3,5,3,5} + 825216\, f_{3,3,3,7} \\
& - \zeta_2\,(701856\, f_{7,7} + 1303232\, f_{9,5} + 430656\, f_{5,9} + 2061312\, f_{11,3} - 309696\, f_{3,11} \\
& \qquad + 160128\, f_{3,5,3,3} + 160128\, f_{3,3,5,3} + 117888\, f_{3,3,3,5} + 148608\, f_{5,3,3,3}) \\
& - \zeta_4\,(3243888\, f_{5,7} + 3475296\, f_{7,5} + 3909696\, f_{9,3} + 3215472\, f_{3,9} + 353664\, f_{3,3,3,3}) \\
& - \zeta_6(3612804\, f_{5,5} + 3791520\, f_{7,3} + 3409152\, f_{3,7}) - \zeta_8(3720664\, f_{5,3} + 3456614\, f_{3,5}) \\
& - \frac{19560489}{5}\,\zeta_{10}\, f_{3,3} - \frac{512193667550809}{7639104}\,\zeta_{16}
\end{aligned}
$$

All 9 coefficients in blue match those in $F_3^{(8)}(\cdot)$,
after reversing letter-ordering in f-alphabet

Symbology enables plots to 8 loops



For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\text{cusp}}^{(L)}}{\Gamma_{\text{cusp}}^{(L-1)}}$

→ same finite radius of convergence

Finite radius of convergence

- Planar N=4 SYM has no renormalons ($\beta(g) = 0$) and no instantons ($e^{-1/g_{\mathrm{YM}}^2} = e^{-N_c/\lambda}$)
- Perturbative expansion can have finite radius of convergence, unlike QCD, QED, whose perturbative series are asymptotic.
- For cusp anomalous dimension, using coupling $g^2 \equiv \frac{N_c g_{\mathrm{YM}}^2}{16\pi^2} = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$

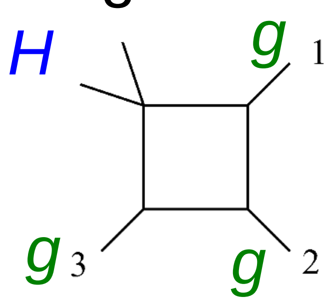
  Beisert, Eden, Staudacher (BES), 0610251

- Ratio of successive loop orders $\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{\Gamma_{\mathrm{cusp}}^{(L-1)}} \to -16$
- Find same radius of convergence in high-loop-order behavior of amplitudes and form factors, in most kinematic regions.

N=4 SYM very special

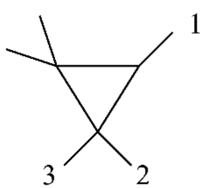
- At one loop, cancellation of loop momenta in numerator
  → only scalar box integrals
  Bern, LD, Dunbar, Kosower, hep-ph/9403226

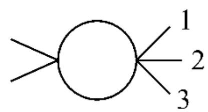
- Weight 2 functions – dilogs. E.g., *gg* → *Hg* @ 1 loop ⊃



$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right)+\cdots$$

- QCD results also contain single log's and rational parts from (tensor) triangle + bubble integrals





$$= \frac{1}{\epsilon} - \ln(s_{123})$$

Higher loops

- N=4 SYM amplitudes have "uniform weight" (transcendentality) $2L$ at loop order L
- Weight ~ number of integrations, e.g.

$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

- QCD amps typically all weights from 0 to $2L$

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

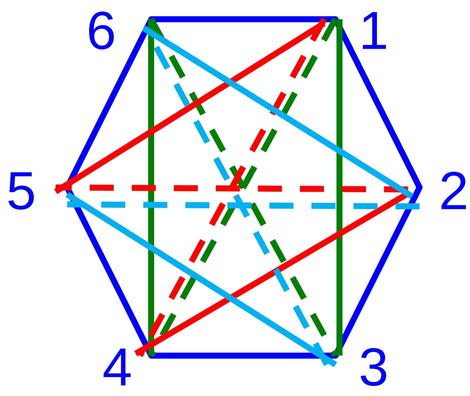
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

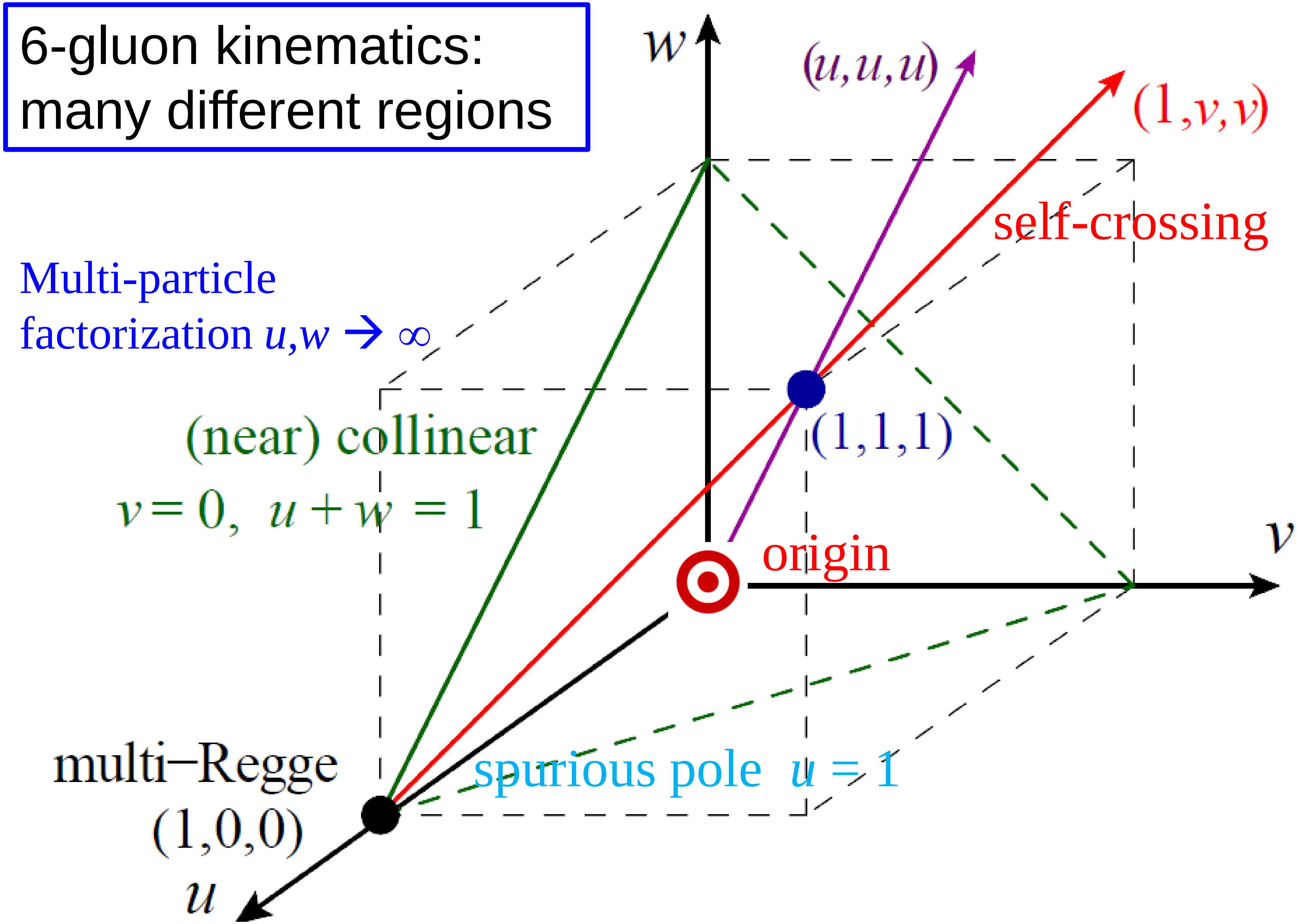
$$v = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



$n = 7$ → 6 ratios.
In general, $3n$-15 ratios.

6-gluon kinematics: many different regions



Hggg kinematics is two-dimensional

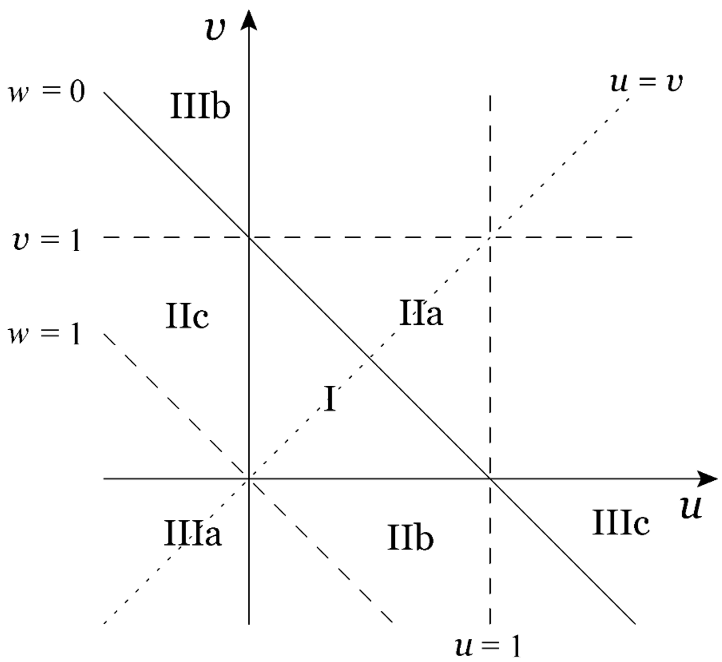
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

$$w = 1 - u - v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To **bootstrap** *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight

Example: Harmonic Polylogarithms in one variable (HPL{0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize classical polylogs
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- Symbol alphabet: $\mathcal{S} = \{x, 1-x\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$ is number of binary strings: 2^{2L}
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

Heuristic view of function space

weight

…

$\int\int\int\int\int\int$

derivatives

$\int\int\int\int\int$

4

$\int\int\int\int$

3

$\mathrm{Li}_3(1\text{-}1/u_i)$, true 2D HPLs, …

2

$\mathrm{Li}_2(1\text{-}1/u_i)$ $\ln^2 u_i$ $\ln u_i \ln u_{i+1}$ - ζ_2

1

$\ln u$ $\ln v$ $\ln w$

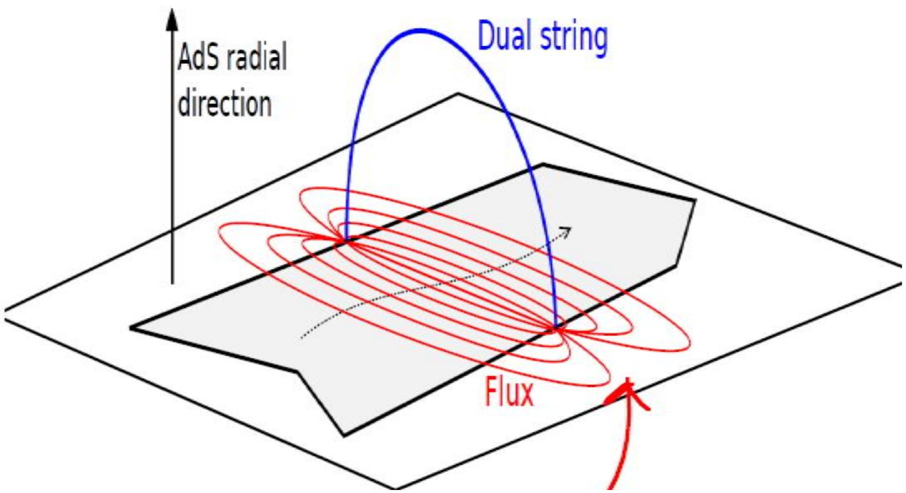
0

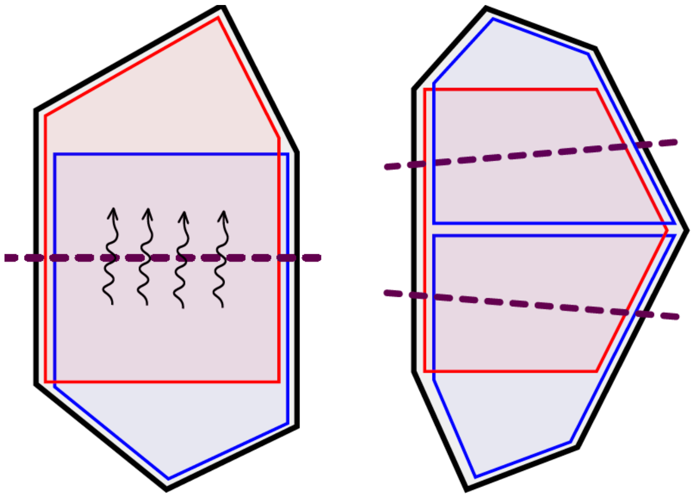
1

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

Bootstrap boundary data:
Flux tubes at finite coupling

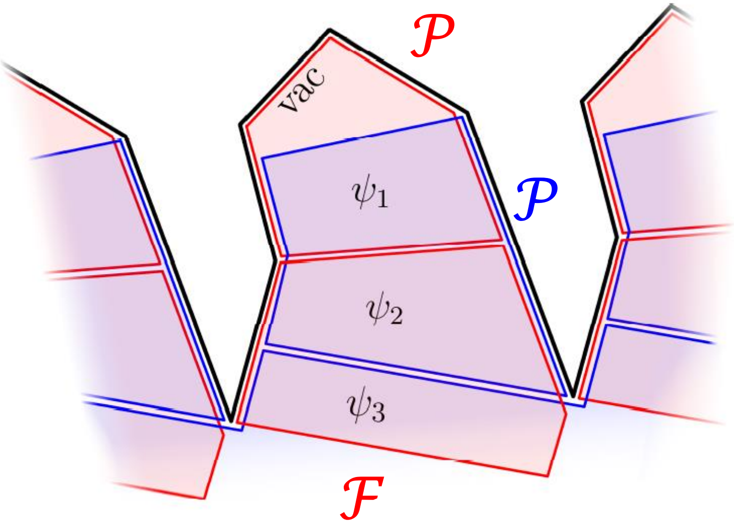
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

A New Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

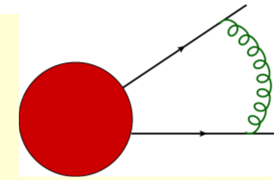
Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

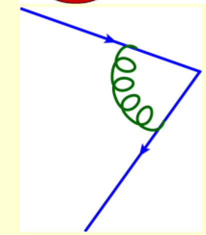
Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension Γ_{cusp}



– known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\text{BDS-like}}(s_{i,i+1},\epsilon)} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\mathcal{E}^{(1)} + R_6\right]$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\rm MHV,\,tree}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\rm cusp}^{(L)}}{4} + \mathcal{O}(\epsilon)\right) M^{1-{\rm loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w)\right]\right\}$$

BDS ansatz

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjac... y constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-{\rm loop}}}{\mathcal{F}_3^{\rm MHV,\,tree}} \equiv M^{1-{\rm loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \left(1-\frac{1}{v}\right) + {\rm Li}_2\left(1-\frac{1}{w}\right)\Big]$$

$$\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys “adjacency constraints”

Now div... by:

$$\frac{\mathcal{F}_3^{\rm BDS-like}}{\mathcal{F}_3^{\rm MHV,\,tree}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\rm cusp}}{4} + \mathcal{O}(\epsilon)\right) M(L\epsilon) + C^{(L)}\right]\right\} \Rightarrow$$

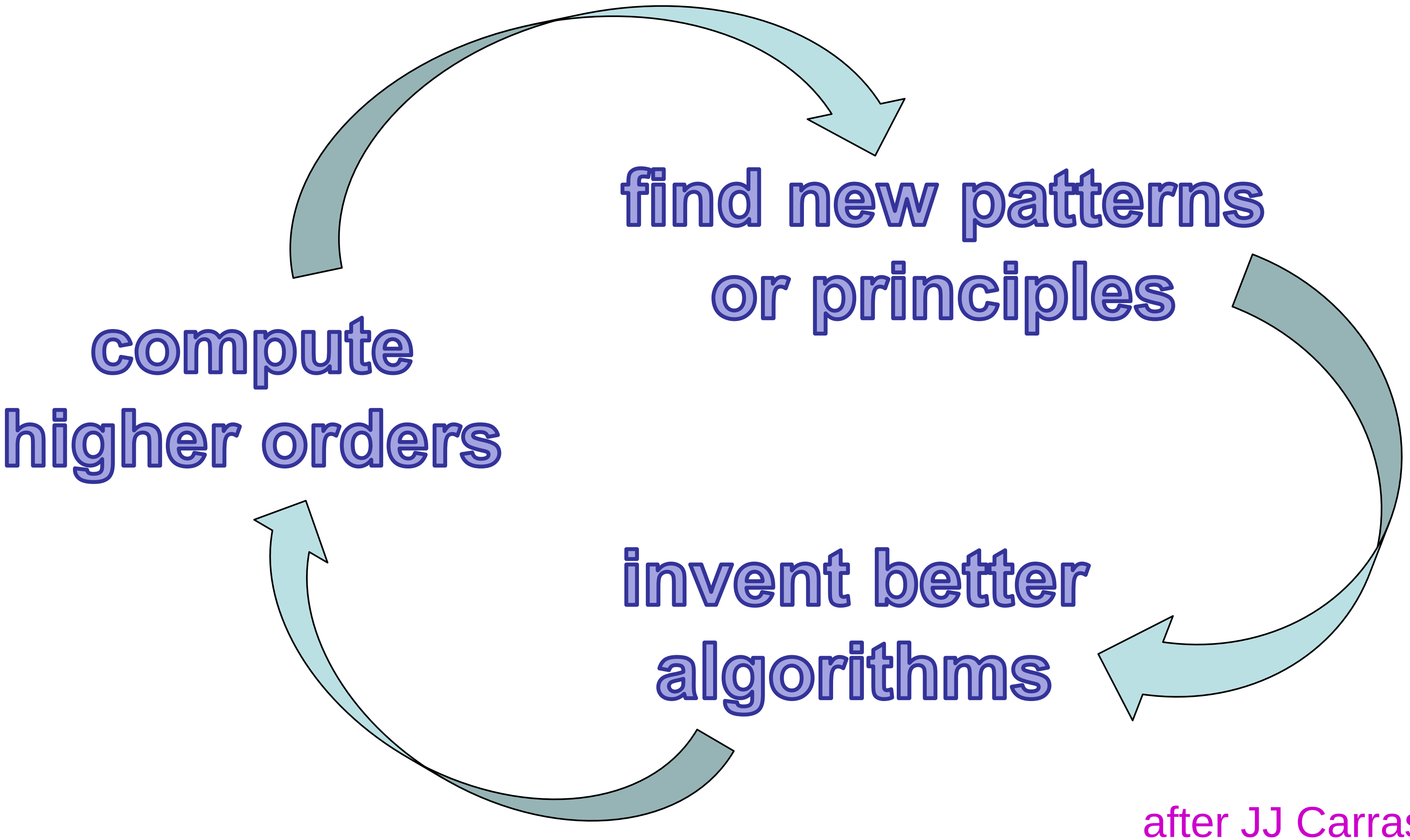
$$\mathcal{E} = \exp\left[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}^{(1)} + R\right]$$

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	3	1														
$L = 2$	1	3	6	3	1												
$L = 3$	1	3	9	12	6	3	1										
$L = 4$	1	3	9	21	24	12	6	3	1								
$L = 5$	1	3	9	21	46	45	24	12	6	3	1						
$L = 6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L = 7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L = 8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

Amplitudes virtuous circle



after JJ Carrasco